The Higher-Order Unscented Estimator

A-Exam Presentation

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New imaging missions require increasingly accurate state estimation

- Driven by high-resolution optical instruments
- Need to know position, attitude, camera focus, etc.
- Required for controls, image processing, etc.

How can we improve state estimation for future missions?
Cross-Calibration using Image Decomposition

The Higher-Order Unscented Estimator

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D. Savransky (2020)
Spacecraft State Estimation

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D. Savransky (2020)
Challenges: Nonlinear Dynamics & Non-Gaussian Noise

- Optimal filter does not exist in general case
- Approximations must be used
- Filter must be tractable in real time without sacrificing accuracy
Non-Gaussian Distributions: Image Feature Location

J. Shapiro (2020)
Non-Gaussian Distributions: Image Feature Orientation & Scale

J. Shapiro (2020)
The Nonlinear Filtering Problem

Estimate system state $x(k)$ based on measurements $z(k)$

**Dynamics**

$$x(k+1) = f(x(k), u(k), w(k), k)$$

**Measurement**

$$z(k) = h(x(k), u(k), n(k), k)$$

- $w(k)$ — Process noise
- $f(k)$ — State function
- $n(k)$ — Measurement noise
- $h(k)$ — Measurement function
- $u(k)$ — Control
Filter Operation

**Prediction**
- Given distribution of $\mathbf{x}(k - 1)$
- Find distribution of predicted state $\mathbf{x}(k|k - 1)$

**Update**
- Given measurement $\mathbf{z}(k)$ and distribution of $\mathbf{x}(k|k - 1)$
- Find distribution of updated state $\mathbf{x}(k)$
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The Unscented Kalman Filter (UKF)

- First proposed by Julier and Uhlmann (1997)
- Evaluates $f$ and $h$ at deterministic set of points
- Approximates distribution rather than functions
- Based on the *unscented transform*
The Unscented Transform

- Given a random variable $\mathbf{x}$ on $\mathbb{R}^n$ with mean $\bar{\mathbf{x}}$ and covariance $\mathbf{P}_{xx}$
- How to approximate distribution of random variable $\mathbf{y} = \mathbf{g}(\mathbf{x})$?

One way is to take a set of points $\mathbf{x}^{(j)}$ and compute $\mathbf{y}^{(j)} = \mathbf{g}(\mathbf{x}^{(j)})$

Julier & Uhlmann (1997)
The Unscented Transform

Sigma Points

\[
\begin{align*}
\mathbf{x}^{(0)} &= \bar{\mathbf{x}} \\
\mathbf{x}^{(i)} &= \bar{\mathbf{x}} + \sqrt{n + \kappa} \mathbf{c}^{(i)} \\
\mathbf{x}^{(i+n)} &= \bar{\mathbf{x}} - \sqrt{n + \kappa} \mathbf{c}^{(i)}
\end{align*}
\]

- \( \mathbf{c}^{(i)} \) are columns of \( \sqrt{\mathbf{P}_{xx}} \)
- Matrix square root usually taken to be Cholesky decomposition
- \( \kappa \) is a parameter for fine-tuning the filter

Weights

\[
\begin{align*}
\omega_0 &= \frac{\kappa}{n + \kappa} \\
\omega_i &= \frac{1}{2(n + \kappa)} \\
\omega_{i+n} &= \frac{1}{2(n + \kappa)}
\end{align*}
\]
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The Unscented Transform (UT)

- Evaluate $y$ at sigma points for $j = 0, 1, \ldots, 2n$:

  $$y^{(j)} = g(x^{(j)})$$

- Approximate mean and covariance of $y$:

  $$\bar{y} = \sum_{j=0}^{2n} w_j y^{(j)}$$
  $$P_{yy} = \sum_{j=0}^{2n} w_j (y^{(j)} - \bar{y})(y^{(j)} - \bar{y})^T$$
  $$P_{yx} = \sum_{j=0}^{2n} w_j (y^{(j)} - \bar{y})(x^{(j)} - \bar{x})^T$$
Advantages of the UKF

- No Jacobians required, unlike extended Kalman filter (EKF)
- Fewer points required than particle filter (PF)
Limitations of the UKF

- Determines only mean and covariance of state
- Sufficient for Gaussian distributions, but might not work with non-Gaussian effects

How can the UKF be generalized to represent a broader family of distributions?
Higher-Order Moments

For a random variable $x$ with mean $\mu$ and standard deviation $\sigma$:

- **Skewness** — measure of asymmetry
  \[
  \gamma = \frac{\mathbb{E}[(x - \mu)^3]}{\sigma^3}
  \]

- **Kurtosis** — measure of “tailedness”
  \[
  \kappa = \frac{\mathbb{E}[(x - \mu)^4]}{\sigma^4}
  \]

- **Satisfy Pearson’s inequality**
  \[
  \kappa \geq \gamma^2 + 1
  \]
Generalizing the UKF

Ponomareva, Date, and Wang (2010) introduce two coefficients $\alpha$ and $\beta$ to account for higher moments.

**Sigma Points**

\[
\begin{align*}
\mathbf{x}^{(0)} &= \bar{\mathbf{x}} \\
\mathbf{x}^{(i)} &= \bar{\mathbf{x}} + \alpha \sqrt{n} \mathbf{c}^{(i)} \\
\mathbf{x}^{(i+n)} &= \bar{\mathbf{x}} - \beta \sqrt{n} \mathbf{c}^{(i)}
\end{align*}
\]

- $\mathbf{c}^{(i)}$ are columns of $\sqrt{\mathbf{P}_{xx}}$
- $\alpha$ and $\beta$ chosen to preserve marginal skewness and kurtosis *averaged* over components.

**Weights**

\[
\begin{align*}
\omega_0 &= \frac{1}{2n} \\
\omega_i &= \frac{1}{\alpha(\alpha+\beta)n} \\
\omega_{i+n} &= \frac{1}{\beta(\alpha+\beta)n}
\end{align*}
\]
Contributions

A new extension of the unscented Kalman filter: Higher-Order Unscented Estimator (HOUSE)

- Developed methodology and mathematical formalism
- Implemented and tested with simulations of dynamical systems

Other Relevant Work

- **Courses**
  - Intermediate Dynamics & Vibrations (MAE 5730)
  - Feedback Control Systems (MAE 5780)
  - Methods of Applied Mathematics I (MAE 6810)
  - Advanced Dynamics (MAE 6700)
  - Statistics (MATH 4720)
  - Intelligent Sensor Planning & Control (MAE 6790)
  - Advanced Astrodynamics (MAE 6720)
  - Model-Based Estimation (MAE 6760) — *in progress*

- **Teaching Assistant for System Dynamics (MAE 3260), Spring 2020**
The Higher-Order Unscented Estimator (HOUSE)

- A new extension of the UKF that captures third and fourth moments in \( n \) directions
- Builds on the method of Ponomareva et al.
- Instead of one value of \( \alpha \) and \( \beta \), we have \( n \) of each
- Sigma points are given by

\[
\mathbf{x}^{(j)} = \begin{cases} 
\bar{\mathbf{x}} + \alpha_j \mathbf{c}^{(j)}, & 1 \leq j \leq n \\
\bar{\mathbf{x}} - \beta_j \mathbf{c}^{(j)}, & n + 1 \leq j \leq 2n \\
\bar{\mathbf{x}}, & j = 2n + 1
\end{cases}
\]
For brevity, we denote $P_{xx}$ by $P$
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For brevity, we denote $P_{xx}$ by $P$

Measurement Noise
Skewness & Kurtosis

State
Skewness & Kurtosis

Coefficients $\alpha_i, \beta_i$

Weights $w_j$

Updated State
Skewness & Kurtosis

True Measurement: $z_k$

State Mean & Covariance:
$\bar{x}_k|k-1, P_k|k-1$

Sigma Points:
$x_k^{(j)}, n_k^{(j)}$

LMMSE

Updated State Mean: $\bar{x}_k$

Updated State Covariance: $P_k$

Predicted Measurements at Sigma Points:
$z_k^{(j)} = h_k(x_k^{(j)}, n_k^{(j)})$
Determining the Coefficients & Weights

- Find coefficients \( \alpha_j, \beta_j \) and weights \( w_j \) such that

\[
\int_{\Omega} \phi(x) \rho(x) \, dx = \omega_{2n+1} \phi(\bar{x}) + \sum_{j=1}^{n} \left( w_j \phi(\bar{x} + \alpha_j c^{(j)}) + w_{n+j} \phi(\bar{x} - \beta_j c^{(j)}) \right)
\]

for all functions \( \phi \) in some family

- Effectively finding a cubature (multivariate quadrature) rule
Standardization

For any random variable $x$ with mean $\bar{x}$ and covariance $P_{xx}$,

$$\tilde{x} = (\sqrt{P_{xx}})^{-1}(x - \bar{x})$$

has zero mean and identity covariance.
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Sigma Point Standardization

\[ \tilde{x}(j) = \begin{cases} 
\alpha_j e^{(j)}, & 1 \leq j \leq n \\
-\beta_j e^{(j)}, & n + 1 \leq j \leq 2n \\
0, & j = 2n + 1 
\end{cases} \]

- \( e^{(j)} \) are the standard basis vectors (orthonormal)
- \( \alpha_j \) and \( \beta_j \) are the scaling coefficients (same as before)
Cubature Standardization

Cubature rule with coefficients $\alpha_j, \beta_j$ and weights $w_j$

$$\int_{\Omega} \phi(x)p(x)\,dx = w_{2n+1}\phi(\bar{x}) + \sum_{j=1}^{n} (w_j\phi(\bar{x} + \alpha_j c^{(j)}) + w_{n+j}\phi(\bar{x} - \beta_j c^{(j)}))$$

holds for function $\phi$ if and only if standardized cubature rule

$$\int_{\tilde{\Omega}} \tilde{\phi}(\tilde{x})\tilde{p}(\tilde{x})\,d\tilde{x} = w_{2n+1}\tilde{\phi}(0) + \sum_{j=1}^{n} (w_j\tilde{\phi}(\alpha_j e^{(j)}) + w_{n+j}\tilde{\phi}(-\beta_j e^{(j)}))$$

holds for

$$\tilde{\phi}(\tilde{x}) = \phi\left((\sqrt{P_{xx}})\tilde{x} + \bar{x}\right)$$

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Determining the Coefficients & Weights

Want the standardized cubature rule

\[
\int_{\tilde{\Omega}} \tilde{\phi}(\tilde{x}) \tilde{p}(\tilde{x}) d\tilde{x} = \omega_{2n+1} \tilde{\phi}(0) + \sum_{j=1}^{n} (\omega_j \tilde{\phi}(\alpha_j e^{(j)}) + \omega_{n+j} \tilde{\phi}(-\beta_j e^{(j)}))
\]

to hold exactly for

\[
\tilde{\phi}(\tilde{x}) = 1, \tilde{x}_i, \tilde{x}_i^2, \tilde{x}_i^3, \tilde{x}_i^4 \quad \text{for } i = 1, \ldots, n
\]
Determining the Coefficients & Weights

System of equations for unknown coefficients $\alpha_i$, $\beta_i$ and weights $w_i$

$$\begin{cases}
   w_i \alpha_i - w_{n+i} \beta_i = 0 \\
   w_i \alpha_i^2 + w_{n+i} \beta_i^2 = 1 \\
   w_i \alpha_i^3 - w_{n+i} \beta_i^3 = \gamma_i \\
   w_i \alpha_i^4 + w_{n+i} \beta_i^4 = \kappa_i \\
\end{cases}$$

For $i = 1, \ldots, n$

\[ \sum_{j=1}^{2n+1} w_j = 1 \]

where $\gamma_i$ and $\kappa_i$ are the skewness and kurtosis of the components of $\tilde{x}$
Determining the Coefficients & Weights

Solution

\[ \alpha_i = \frac{\gamma_i + \sqrt{4\kappa_i - 3\gamma_i^2}}{2} \]

\[ \beta_i = \frac{-\gamma_i + \sqrt{4\kappa_i - 3\gamma_i^2}}{2} \]

\[ \omega_i = \frac{1}{\alpha_i^2 + \alpha_i\beta_i} \]

\[ \omega_{n+i} = \frac{1}{\beta_i^2 + \alpha_i\beta_i} \]

\[ \omega_{2n+1} = 1 - \sum_{j=1}^{2n} w_j \]
Comparison of Sigma Points

True PDF

UKF Sigma Points

HOUSE Sigma Points
Properties of the Sigma Points

- Due to Pearson’s inequality, coefficients $\alpha_i$ and $\beta_i$ must be real
- We have chosen the signs in the solution such that $\alpha_i, \beta_i > 0$
- All weights except $w_{2n+1}$ guaranteed positive
Mixed Terms

- With our quadrature, all mixed monomials map to zero
- Exact for second-order terms (covariance)
- Effectively “drops” higher-order mixed terms in Taylor expansion of a function
The Weight at the Mean

- $w_{2n+1}$ is the only weight that might not be positive.
- Not a valid probability distribution if negative — can be problematic.
- Regardless of sign chosen for coefficients $\alpha_i$ and $\beta_i$,

$$w_{2n+1} = 1 - \sum_{i=1}^{n} \frac{1}{\kappa_i - \gamma_i^2}$$
Staying Positive

- One solution is to scale $\gamma_i$ and $\kappa_i$:

  $\gamma'_i = \gamma_i \sqrt{1 + \theta}$

  $\kappa'_i = \kappa_i (1 + \theta)$

  where $\theta \geq 0$ is small

- Minimizes effects on lower-order moments

- Satisfies Pearson’s inequality

  $\kappa'_i \geq \gamma'_i^2 + 1$
Selecting $\theta$

- To attain $w'_{2n+1} \geq \delta$, where $\delta \geq 0$, we can set

$$\theta = \begin{cases} 
0, & S \leq 1 - \delta \\
\frac{S}{1-\delta} - 1, & S > 1 - \delta 
\end{cases}$$

where

$$S = \sum_{i=1}^{n} \frac{1}{\kappa_i - \gamma_i^2}$$

- Minimal case $\delta = 0$ successfully used in simulations
HOUSE Constraints

- For large values of kurtosis, sigma points may grow without bound
- To avoid this, impose the constraint, for some radius $R$
  \[
  \| \mathbf{x}^{(i)} - \bar{\mathbf{x}} \| < R
  \]
- Other constraint — to ensure valid probability distribution
  \[
  \omega_{2n+1} \geq 0
  \]
- Must hold for either original or tuned skewness and kurtosis
HOUSE Constraints

- Assume that $\mathbf{x}$ has a radially symmetric distribution about $\bar{\mathbf{x}}$
- Then constraint inequalities reduce to

$$\sigma \sqrt{\kappa} < R$$

and

$$\frac{n}{\kappa} \leq 1$$

- $\sigma$ — Standard deviation
- $\kappa$ — Kurtosis
- $n$ — State dimension

More
HOUSE Constraints

- The range of $\kappa$ where HOUSE can reliably operate is

$$n \leq \kappa < \frac{R^2}{\sigma^2}$$

- Expect the filter to perform better when this range is wider, i.e.,

$$n\sigma^2 \ll R^2$$

- Suggests that this filter is better for lower-dimensional systems
The Pearson Type IV Distribution

Probability density function on $(-\infty, \infty)$

$$p(x) \propto \left(1 + \left(\frac{x - \lambda}{a}\right)^2\right)^{-m} \exp\left(-\nu \arctan\left(\frac{x - \lambda}{a}\right)\right)$$

- Free parameters $a > 0$, $m > 1/2$, $\lambda$, and $\nu$
- Uniquely determined by first four moments
Applications of Pearson Type IV

- Wind shear fluctuations (Ramsdell, 1978)
- Fluctuating pressure on aircraft skin panels (Steinwolf & Rizzi, 2012)
- Solar wind intensity (Krafft, Gauthier & Volotkin, 2019)
“Rigid body $\mathcal{R}$ with body-fixed frame $\mathcal{F}_b$ and inertial frame $\mathcal{F}_i$.” (Hughes, 2004)
Rigid Body Example: Dynamics & Measurements

- Equations of motion

\[
\begin{align*}
I_1 \dot{\omega}_1 &= (I_2 - I_3) \omega_2 \omega_3 + \tau_1 \\
I_2 \dot{\omega}_2 &= (I_3 - I_1) \omega_3 \omega_1 + \tau_2 \\
I_3 \dot{\omega}_3 &= (I_1 - I_2) \omega_1 \omega_2 + \tau_3
\end{align*}
\]

- \(\tau_1, \tau_2, \tau_3\) are random external disturbance torques (process noise)
- \(I_1 = 400\) kg \(\cdot\) m\(^2\), \(I_2 = 200\) kg \(\cdot\) m\(^2\), and \(I_3 = 100\) kg \(\cdot\) m\(^2\)
- Only \(\omega_1\) is measured directly, with 10 Hz sampling rate
Rigid Body Example: Initial Conditions

- Test two sets of initial conditions
  - Major axis spinner:
    \[ \omega_1 = 0.01 \text{ rad/s}, \quad \omega_2 = \omega_3 = 0 \]
  - Intermediate axis spinner:
    \[ \omega_2 = 0.01 \text{ rad/s}, \quad \omega_1 = \omega_3 = 0 \]
- Both are stationary points
- First is stable in Lyapunov sense; second is unstable

Hughes (2004)
Rigid Body Example: Process Noise
Rigid Body Example: Measurement Noise

![Probability Density vs Measurement Error graph](image)
Rigid Body Example: Estimation Error

Stable Spinner; Gaussian Noise
Rigid Body Example: Estimation Error

Stable Spinner; Gaussian Noise
Rigid Body Example: Estimation Error

Stable Spinner; Pearson Type IV Noise
Rigid Body Example: Estimation Error

Unstable Spinner; Gaussian Noise

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Rigid Body Example: Estimation Error

Unstable Spinner; Pearson Type IV Noise
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Projectile Example: True Trajectories
Projectile Example: Dynamics

- State consists of position \((x, y, z)\) and velocity \((\dot{x}, \dot{y}, \dot{z})\)
- Ballistic motion with drag

\[
\begin{align*}
\ddot{x} &= -bv\dot{x} + f_x \\
\ddot{y} &= -bv\dot{y} + f_y \\
\ddot{z} &= -bv\dot{z} + f_z - g
\end{align*}
\]

where

\[
v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \quad b = \frac{AC_D\rho}{2m} = 0.001 \text{ m}^{-1}
\]
Projectile Example: Measurements

- Line-of-sight only: Azimuth & Elevation

\[ \alpha = \text{atan2}(y, -x) + n_{\alpha} \]

\[ \epsilon = \text{atan2}\left(z, \sqrt{x^2 + y^2}\right) + n_{\epsilon} \]

- Measurements taken at 100 Hz
- Noise \( n_{\alpha}, n_{\epsilon} \) has standard deviation of 1 arc-minute
Projectile Example: Process Noise
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Pearson Type IV
Gaussian

Measurement Error (arc min)
Probability Density (arc min)^{-1}

-6  -4  -2  0  2  4  6

0  0.1  0.2  0.3  0.4  0.5  0.6

Probability Density (arc min)
Measurement Error (arc min)
Projectile Example: Estimation Error

Gaussian Noise

![Graph showing estimation error over time with HOUSE and UKF lines]

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Projectile Example: Estimation Error
Pearson Type IV Noise

\[ \Delta x \text{ (m)} \]

\[ \text{Time (s)} \]

\[ 10^{-3} \]

\[ 10^{-2} \]

\[ 10^{-1} \]

\[ 10^0 \]

\[ 10^1 \]

\[ 10^2 \]

\[ 10^3 \]

\[ 0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \]

\[ \Delta x \text{ (m)} \]

\[ \text{Time (s)} \]

HOUSE
UKF

More
Conclusions on HOUSE Performance

- Provides significantly more accurate estimates than conventional UKF for heavy-tailed noise distributions
- Also appears to be more accurate for unstable systems
- Not as accurate for higher-dimensional systems with Gaussian noise
Future Work

- Refining HOUSE—goals:
  - Make accuracy independent of system dimension
  - Higher accuracy for systems with Gaussian noise
- Analyzing computational complexity vs. accuracy trade-off
- Testing
  - High-fidelity spacecraft model
  - Key points extracted from images
Future Work

- Next paper
  - HOUSE refinements
  - Deeper analysis and testing
- Exploring other filtering methods
  - Continuous-time prediction
  - Approximate Bayesian correction
- Spacecraft jitter estimation (NSTGRO proposal)
References I


References II


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Appendix
**HOUSE Operation: Prediction**

1. For the augmented state $y_P(k) = [x(k)^T \ w(k)^T]^T$, generate the modified sigma points $y_p^{(j)}(k) = [x^{(j)}(k)^T \ w^{(j)}(k)^T]^T$ and weights $w_j(k)$

2. Propagate the state for each sigma point:

$$x^{(j)}(k + 1|k) = f(x^{(j)}(k), w^{(j)}(k), k)$$
HOUSE Operation: Prediction

Compute the predicted mean and covariance:

\[
\bar{x}(k + 1|k) = \sum_{j=1}^{N} w_j x^{(j)}(k + 1|k)
\]

\[
\eta^{(j)}(k + 1|k) = x^{(j)}(k + 1|k) - \bar{x}(k + 1|k)
\]

\[
P_{xx}(k + 1|k) = \sum_{j=1}^{N} w_j \eta^{(j)}(k + 1|k) \eta^{(j)}(k + 1|k)^T
\]
Compute the standardized states at the sigma points:

\[
\tilde{x}^{(j)}(k+1|k) = \left(\sqrt{P_{xx}(k+1|k)}\right)^{-1} \eta^{(j)}(k+1|k)
\]

Compute the skewness and kurtosis of the standardized state:

\[
\gamma_i(k+1|k) = \sum_{i=1}^{N} w_j \tilde{x}_i^{(j)}(k+1|k)^3
\]

\[
\kappa_i(k+1|k) = \sum_{i=1}^{N} w_j \tilde{x}_i^{(j)}(k+1|k)^4
\]
HOUSE Operation: Update

Here, $z(k)$ denotes the true measurement.

1. For the augmented state $y_C(k) = [x(k)^T \ n(k)^T]^T$, generate the modified sigma points $y^{(j)}_C(k) = [x^{(j)}(k)^T \ n^{(j)}(k)^T]^T$ and weights $w_j(k)$

2. Compute the measurement for each sigma point:

$$z^{(j)}(k) = h(x^{(j)}(k), w^{(j)}(k), k)$$
HOUSE Operation: Update

3. Compute the measurement mean and covariance:

\[
\bar{z}(k) = \sum_{j=1}^{N} \omega_j z^{(j)}(k)
\]

\[
P_{zz}(k) = \sum_{j=1}^{N} \omega_j (z^{(j)}(k) - \bar{z}(k))(z^{(j)}(k) - \bar{z}(k))^T
\]

\[
P_{xz}(k) = \sum_{j=1}^{N} \omega_j (x^{(j)}(k|k-1) - \bar{x}(k|k-1))(z^{(j)}(k) - \bar{z}(k))^T
\]
HOUSE Operation: Update

4. Compute the LMMSE error and standardized state at the sigma points:

\[
\epsilon^{(j)}(k) = x^{(j)}(k) - \bar{x}(k|k-1) - P_{xz}(k) P_{zz}(k)^{-1}(z^{(j)}(k) - \bar{z}(k))
\]

\[
\tilde{x}^{(j)} = \left(\sqrt{P_{xx}(k)}\right)^{-1} \epsilon^{(j)}(k)
\]

5. Compute the skewness and kurtosis of the standardized state:

\[
\gamma_i(k) = \sum_{i=1}^{N} \omega_j \tilde{X}_i^{(j)}(k)^3 \quad \kappa_i(k) = \sum_{i=1}^{N} \omega_j \tilde{X}_i^{(j)}(k)^4
\]
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HOUSE Constraints

The constraint

\[ \| \mathbf{x}^{(i)} - \bar{x} \| < R \]

is equivalent to

\[ \max_{i \in \{1, \ldots, n\}} \left( \| \mathbf{c}^{(i)} \| \max(\alpha_i, \beta_i) \right) < R \]

Back
HOUSE Constraints

- Since

\[ \max(\alpha_i, \beta_i) = \frac{|\gamma_i| + \sqrt{4\kappa_i - 3\gamma_i^2}}{2} \]

the constraint reduces to

\[ \max_{i \in \{1, \ldots, n\}} \left( \frac{|\gamma_i| + \sqrt{4\kappa_i - 3\gamma_i^2}}{2} \|c(i)\| \right) < R \]

- Needs further simplification
HOUSE Constraints

- To have valid weights, the original or tuned $\gamma_i$ and $\kappa_i$ must satisfy
  \[ \sum_{i=1}^{n} \frac{1}{\kappa_i - \gamma_i^2} \leq 1 \]

- This imposes another bound on the moments
- Again, needs simplification
HOUSE Constraints

- Assume that $\mathbf{x}$ has a radially symmetric distribution about $\bar{\mathbf{x}}$
- Then, we have $\kappa_i = \kappa$, $\gamma_i = 0$, and $\|c^{(i)}\| = \sigma$ for $i = 1, \ldots, n$
- Constraint inequalities reduce to

$$\sigma \sqrt{\kappa} < R$$

and

$$\frac{n}{\kappa} \leq 1$$
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Equivalence of Quadrature Rules

$$\int_{\Omega} \phi(x)p(x)dx = \int_{\tilde{\Omega}} \phi((\sqrt{P_{xx}})\tilde{x} + \bar{x}) \tilde{p}(\tilde{x})d\tilde{x} = \int_{\tilde{\Omega}} \phi(\tilde{x})\tilde{p}(\tilde{x})d\tilde{x}$$

$$= w_{2n+1}\tilde{\phi}(0) + \sum_{j=1}^{n} (w_{j}\tilde{\phi}(\alpha_{j}e^{(j)}) + w_{n+j}\tilde{\phi}(-\beta_{j}e^{(j)}))$$

$$= w_{2n+1}\phi(\bar{x}) + \sum_{j=1}^{n} (w_{j}\phi(\bar{x} + \alpha_{j}c^{(j)}) + w_{n+j}\phi(\bar{x} - \beta_{j}c^{(j)}))$$
Equivalence of Quadrature Rules

\[ \int_{\tilde{\Omega}} \tilde{\phi}(\tilde{x})\tilde{p}(\tilde{x})d\tilde{x} = \int_{\tilde{\Omega}} \phi \left( \sqrt{P_{xx}} \tilde{x} + \bar{x} \right) \tilde{p}(\tilde{x})d\tilde{x} = \int_{\Omega} \phi(x)p(x)dx \]

\[ = \omega_{2n+1}\phi(\bar{x}) + \sum_{j=1}^{n} (\omega_j \phi(\bar{x} + \alpha_j c^{(j)}) + \omega_{n+j}\phi(\bar{x} - \beta_j c^{(j)})) \]

\[ = \omega_{2n+1}\tilde{\phi}(0) + \sum_{j=1}^{n} (\omega_j \tilde{\phi}(\alpha_j e^{(j)}) + \omega_{n+j}\tilde{\phi}(\beta_j e^{(j)})) \]