The Higher-Order Unscented Estimator

A-Exam Presentation

Zvonimir Stojanovski

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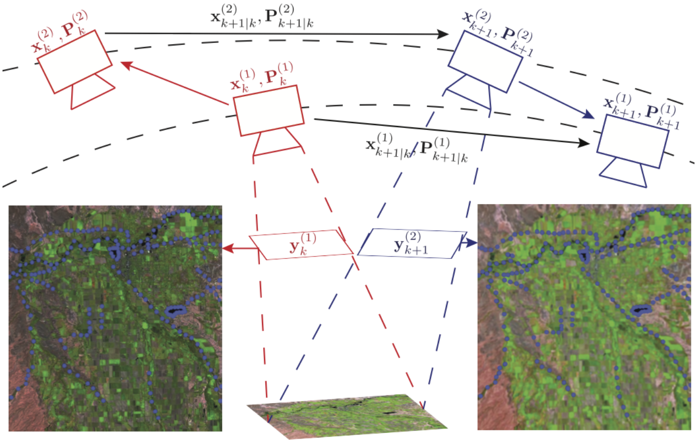
Motivation

New imaging missions require increasingly accurate state estimation

- Driven by high-resolution optical instruments
- Need to know position, attitude, camera focus, etc.
- Required for controls, image processing, etc.

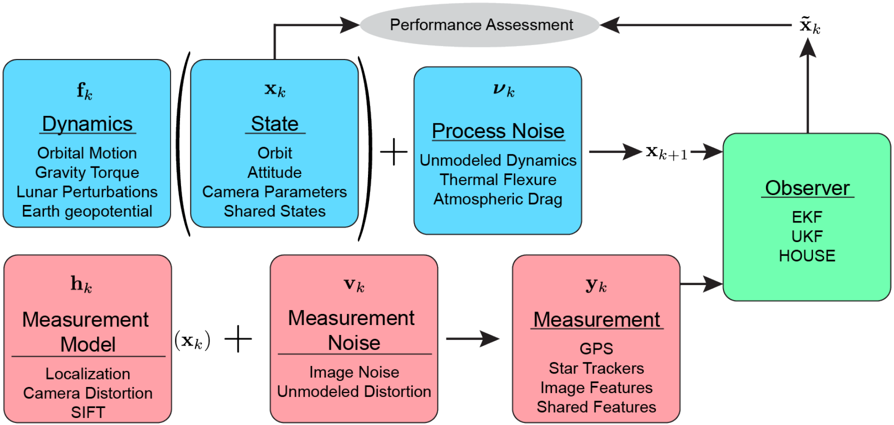
How can we improve state estimation for future missions?

Cross-Calibration using Image Decomposition



D. Savransky (2020)

Spacecraft State Estimation

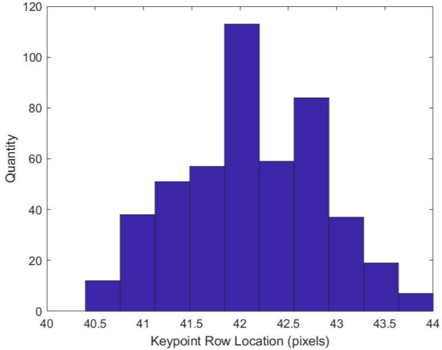


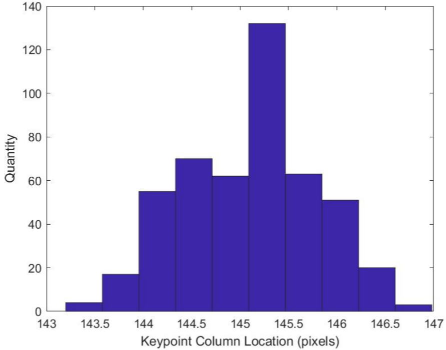
D. Savransky (2020)

Challenges: Nonlinear Dynamics & Non-Gaussian Noise

- Optimal filter does not exist in general case
- Approximations must be used
- Filter must be tractable in real time without sacrificing accuracy

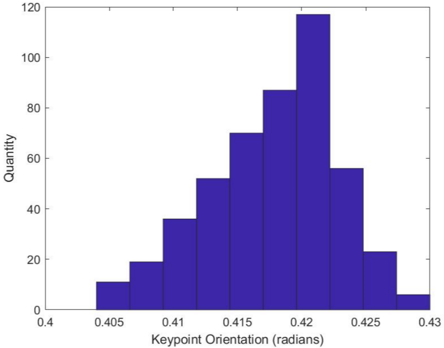
Non-Gaussian Distributions: Image Feature Location

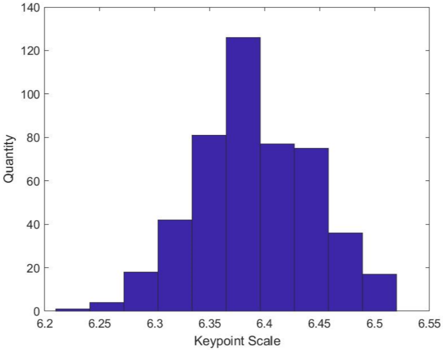




J. Shapiro (2020)

Non-Gaussian Distributions: Image Feature Orientation & Scale





J. Shapiro (2020)

The Nonlinear Filtering Problem

Estimate system state $\mathbf{x}(k)$ based on measurements $\mathbf{z}(k)$

Dynamics

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k), \mathbf{w}(k), k)$$

Measurement

$$\mathbf{z}(k) = \mathbf{h}(\mathbf{x}(k), \mathbf{u}(k), \mathbf{n}(k), k)$$

$\mathbf{w}(k)$ — Process noise

$\mathbf{n}(k)$ — Measurement noise

$\mathbf{u}(k)$ — Control

$\mathbf{f}(k)$ — State function

$\mathbf{h}(k)$ — Measurement function

Filter Operation

Prediction

- Given distribution of $\mathbf{x}(k-1)$
- Find distribution of predicted state $\mathbf{x}(k|k-1)$

Update

- Given measurement $\mathbf{z}(k)$ and distribution of $\mathbf{x}(k|k-1)$
- Find distribution of updated state $\mathbf{x}(k)$

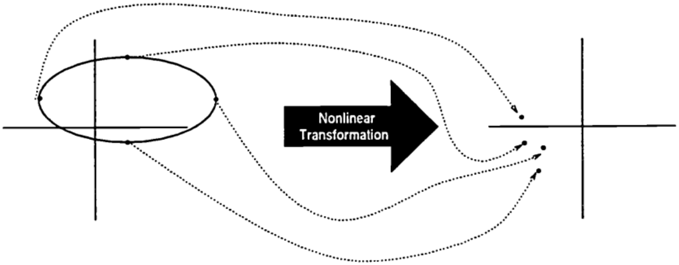
The Unscented Kalman Filter (UKF)

- First proposed by Julier and Uhlmann (1997)
- Evaluates $\mathbf{f}$ and $\mathbf{h}$ at deterministic set of points
- Approximates distribution rather than functions
- Based on the *unscented transform*

The Unscented Transform

- Given a random variable $\mathbf{x}$ on $\mathbb{R}^n$ with mean $\bar{\mathbf{x}}$ and covariance $\mathbf{P}_{xx}$
- How to approximate distribution of random variable $\mathbf{y} = \mathbf{g}(\mathbf{x})$?

One way is to take a set of points $\mathbf{x}^{(j)}$ and compute $\mathbf{y}^{(j)} = \mathbf{g}(\mathbf{x}^{(j)})$



Julier & Uhlmann (1997)

The Unscented Transform

Sigma Points

$$\mathbf{x}^{(0)} = \bar{\mathbf{x}}$$
$$\mathbf{x}^{(i)} = \bar{\mathbf{x}} + \sqrt{n+\kappa}\,\mathbf{c}^{(i)}$$
$$\mathbf{x}^{(i+n)} = \bar{\mathbf{x}} - \sqrt{n+\kappa}\,\mathbf{c}^{(i)}$$

Weights

$$w_0 = \frac{\kappa}{n+\kappa}$$
$$w_i = \frac{1}{2(n+\kappa)}$$
$$w_{i+n} = \frac{1}{2(n+\kappa)}$$

- $\mathbf{c}^{(i)}$ are columns of $\sqrt{\mathbf{P}_{xx}}$
- Matrix square root usually taken to be Cholesky decomposition
- κ is a parameter for fine-tuning the filter

The Unscented Transform (UT)

- Evaluate $\mathbf{y}$ at sigma points for $j = 0, 1, \ldots, 2n$:

$$\mathbf{y}^{(j)} = \mathbf{g}(\mathbf{x}^{(j)})$$

- Approximate mean and covariance of $\mathbf{y}$:

$$\bar{\mathbf{y}} = \sum_{j=0}^{2n} w_j \mathbf{y}^{(j)} \qquad \mathbf{P}_{yy} = \sum_{j=0}^{2n} w_j (\mathbf{y}^{(j)} - \bar{\mathbf{y}})(\mathbf{y}^{(j)} - \bar{\mathbf{y}})^{\mathrm{T}}$$

$$\mathbf{P}_{yx} = \sum_{j=0}^{2n} w_j (\mathbf{y}^{(j)} - \bar{\mathbf{y}})(\mathbf{x}^{(j)} - \bar{\mathbf{x}})^{\mathrm{T}}$$

Advantages of the UKF

- No Jacobians required, unlike extended Kalman filter (EKF)
- Fewer points required than particle filter (PF)

Limitations of the UKF

- Determines only mean and covariance of state
- Sufficient for Gaussian distributions, but might not work with non-Gaussian effects

How can the UKF be generalized to represent a broader family of distributions?

Higher-Order Moments

For a random variable x with mean μ and standard deviation σ:

- Skewness — measure of asymmetry

$$\gamma = \frac{\mathrm{E}[(x-\mu)^3]}{\sigma^3}$$

- Kurtosis — measure of "tailedness"

$$\kappa = \frac{\mathrm{E}[(x-\mu)^4]}{\sigma^4}$$

- Satisfy Pearson's inequality

$$\kappa \geq \gamma^2 + 1$$

Generalizing the UKF

Ponomareva, Date, and Wang (2010) introduce two coefficients α and β to account for higher moments

Sigma Points

$$\mathbf{x}^{(0)} = \bar{\mathbf{x}}$$
$$\mathbf{x}^{(i)} = \bar{\mathbf{x}} + \alpha\sqrt{n}\mathbf{c}^{(i)}$$
$$\mathbf{x}^{(i+n)} = \bar{\mathbf{x}} - \beta\sqrt{n}\mathbf{c}^{(i)}$$

Weights

$$w_0 = \frac{1}{2n}$$
$$w_i = \frac{1}{\alpha(\alpha+\beta)n}$$
$$w_{i+n} = \frac{1}{\beta(\alpha+\beta)n}$$

- $\mathbf{c}^{(i)}$ are columns of $\sqrt{\mathbf{P}_{xx}}$
- α and β chosen to preserve marginal skewness and kurtosis *averaged* over components

Contributions

A new extension of the unscented Kalman filter:
Higher-Order Unscented Estimator (HOUSE)

- Developed methodology and mathematical formalism
- Implemented and tested with simulations of dynamical systems

Publication: Z. Stojanovski and D. Savransky, "The Higher-Order Unscented Estimator," submitted to the *Journal of Guidance, Control and Dynamics.*

Other Relevant Work

- Courses
  - Intermediate Dynamics & Vibrations (MAE 5730)
  - Feedback Control Systems (MAE 5780)
  - Methods of Applied Mathematics I (MAE 6810)
  - Advanced Dynamics (MAE 6700)
  - Statistics (MATH 4720)
  - Intelligent Sensor Planning & Control (MAE 6790)
  - Advanced Astrodynamics (MAE 6720)
  - Model-Based Estimation (MAE 6760) — *in progress*
- Teaching Assistant for System Dynamics (MAE 3260), Spring 2020

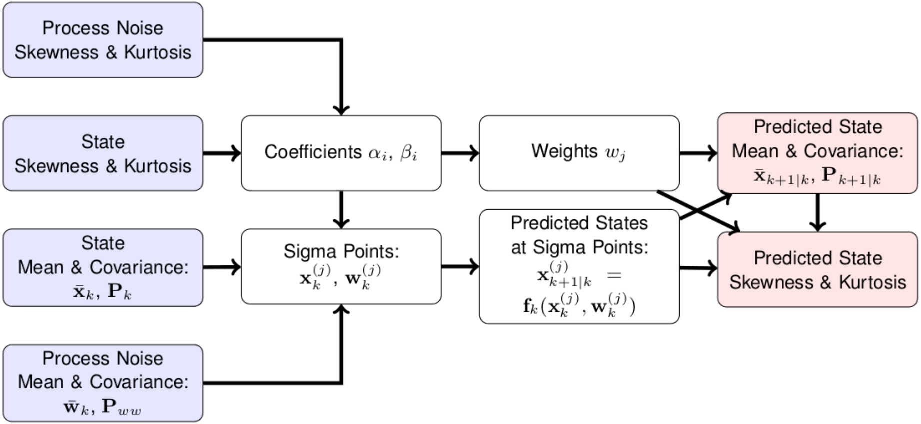
The Higher-Order Unscented Estimator (HOUSE)

- A new extension of the UKF that captures third and fourth moments in n directions
- Builds on the method of Ponomareva et al.
- Instead of one value of α and β, we have n of each
- Sigma points are given by

$$\mathbf{x}^{(j)} = \begin{cases} \bar{\mathbf{x}} + \alpha_j \mathbf{c}^{(j)}, & 1 \leq j \leq n \\ \bar{\mathbf{x}} - \beta_j \mathbf{c}^{(j)} & n+1 \leq j \leq 2n \\ \bar{\mathbf{x}} & j = 2n+1 \end{cases}$$

HOUSE Prediction

For brevity, we denote $\mathbf{P}_{xx}$ by $\mathbf{P}$



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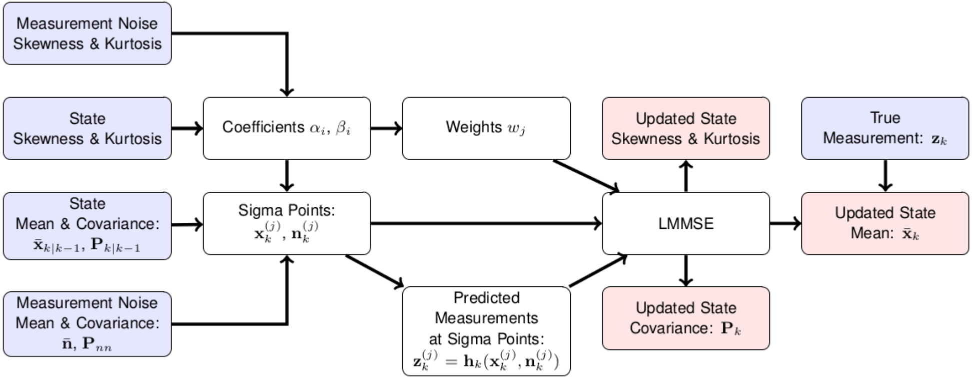
HOUSE Update

For brevity, we denote $\mathbf{P}_{xx}$ by $\mathbf{P}$



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Determining the Coefficients & Weights

- Find coefficients α_j, β_j and weights w_j such that

$$\underbrace{\int_\Omega \phi(\mathbf{x})p(\mathbf{x})\,d\mathbf{x}}_{\text{Exact } \mathrm{E}[\phi(\mathbf{x})]} = \underbrace{w_{2n+1}\phi(\bar{\mathbf{x}}) + \sum_{j=1}^{n}\left(w_j\phi(\bar{\mathbf{x}} + \alpha_j\mathbf{c}^{(j)}) + w_{n+j}\phi(\bar{\mathbf{x}} - \beta_j\mathbf{c}^{(j)})\right)}_{\text{Weighted sum approximation of } \mathrm{E}[\phi(\mathbf{x})]}$$

  for all functions ϕ in some family

- Effectively finding a cubature (multivariate quadrature) rule

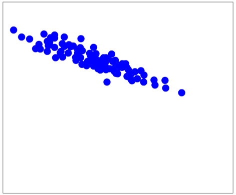
Standardization

For any random variable $\mathbf{x}$ with mean $\bar{\mathbf{x}}$ and covariance $\mathbf{P}_{xx}$,

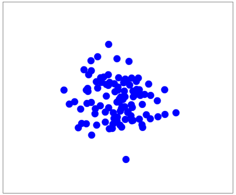
$$\tilde{\mathbf{x}} = \left(\sqrt{\mathbf{P}_{xx}}\right)^{-1}(\mathbf{x} - \bar{\mathbf{x}})$$

has zero mean and identity covariance

$\mathbf{x}$



$\tilde{\mathbf{x}}$

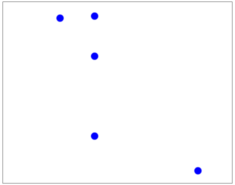


Sigma Point Standardization

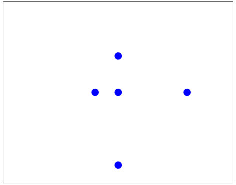
$$\tilde{\mathbf{x}}^{(j)} = \begin{cases} \alpha_j \mathbf{e}^{(j)}, & 1 \leq j \leq n \\ -\beta_j \mathbf{e}^{(j)} & n+1 \leq j \leq 2n \\ \mathbf{0} & j = 2n+1 \end{cases}$$

- $\mathbf{e}^{(j)}$ are the standard basis vectors (orthonormal)
- α_j and β_j are the scaling coefficients (same as before)

$\mathbf{x}^{(j)}$



$\tilde{\mathbf{x}}^{(j)}$



Cubature Standardization

Cubature rule with coefficients α_j, β_j and weights w_j

$$\int_{\Omega} \phi(\mathbf{x}) p(\mathbf{x})\, d\mathbf{x} = w_{2n+1}\phi(\bar{\mathbf{x}}) + \sum_{j=1}^{n} \left(w_j \phi(\bar{\mathbf{x}} + \alpha_j \mathbf{c}^{(j)}) + w_{n+j}\phi(\bar{\mathbf{x}} - \beta_j \mathbf{c}^{(j)}) \right)$$

holds for function ϕ if and only if standardized cubature rule

$$\int_{\tilde{\Omega}} \tilde{\phi}(\tilde{\mathbf{x}}) \tilde{p}(\tilde{\mathbf{x}})\, d\tilde{\mathbf{x}} = w_{2n+1}\tilde{\phi}(\mathbf{0}) + \sum_{j=1}^{n} \left(w_j \tilde{\phi}(\alpha_j \mathbf{e}^{(j)}) + w_{n+j}\tilde{\phi}(-\beta_j \mathbf{e}^{(j)}) \right)$$

holds for

$$\tilde{\phi}(\tilde{\mathbf{x}}) = \phi\left((\sqrt{\mathbf{P}_{xx}})\tilde{\mathbf{x}} + \bar{\mathbf{x}} \right)$$

More

Determining the Coefficients & Weights

Want the standardized cubature rule

$$\int_{\tilde{\Omega}} \tilde{\phi}(\tilde{\mathbf{x}})\tilde{p}(\tilde{\mathbf{x}})\, d\tilde{\mathbf{x}} = w_{2n+1}\tilde{\phi}(\mathbf{0}) + \sum_{j=1}^{n}\left(w_j\tilde{\phi}(\alpha_j\mathbf{e}^{(j)}) + w_{n+j}\tilde{\phi}(-\beta_j\mathbf{e}^{(j)})\right)$$

to hold exactly for

$$\tilde{\phi}(\tilde{\mathbf{x}}) = 1,\ \tilde{x}_i,\ \tilde{x}_i^2,\ \tilde{x}_i^3,\ \tilde{x}_i^4 \qquad \text{for } i = 1, \ldots, n$$

Determining the Coefficients & Weights

System of equations for unknown coefficients α_i, β_i and weights w_i

$$\text{For } i = 1, \dots, n \quad \begin{cases} w_i\alpha_i - w_{n+i}\beta_i = & 0 \\ w_i\alpha_i^2 + w_{n+i}\beta_i^2 = & 1 \\ w_i\alpha_i^3 - w_{n+i}\beta_i^3 = & \gamma_i \\ w_i\alpha_i^4 + w_{n+i}\beta_i^4 = & \kappa_i \end{cases}$$

$$\sum_{j=1}^{2n+1} w_j = 1$$

where γ_i and κ_i are the skewness and kurtosis of the components of $\tilde{\mathbf{x}}$

Determining the Coefficients & Weights

Solution

$$\alpha_i = \frac{\gamma_i + \sqrt{4\kappa_i - 3\gamma_i^2}}{2}$$

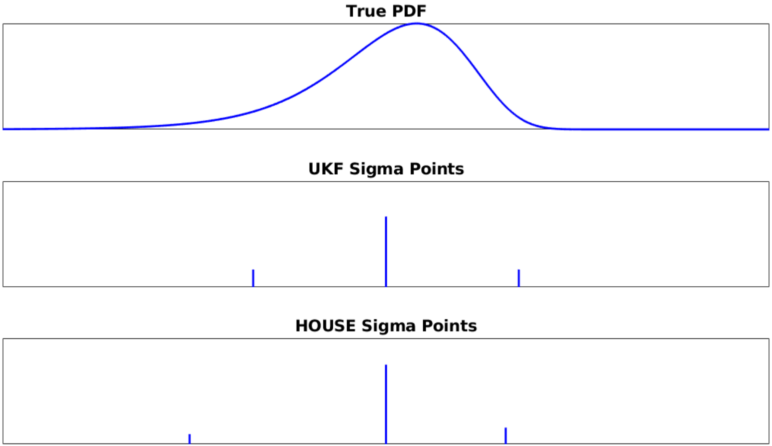
$$\beta_i = \frac{-\gamma_i + \sqrt{4\kappa_i - 3\gamma_i^2}}{2}$$

$$w_i = \frac{1}{\alpha_i^2 + \alpha_i\beta_i}$$

$$w_{n+i} = \frac{1}{\beta_i^2 + \alpha_i\beta_i}$$

$$w_{2n+1} = 1 - \sum_{j=1}^{2n} w_j$$

Comparison of Sigma Points



Properties of the Sigma Points

- Due to Pearson's inequality, coefficients α_i and β_i must be real
- We have chosen the signs in the solution such that $\alpha_i, \beta_i > 0$
- All weights except w_{2n+1} guaranteed positive

Mixed Terms

- With our quadrature, all mixed monomials map to zero
- Exact for second-order terms (covariance)
- Effectively "drops" higher-order mixed terms in Taylor expansion of a function

The Weight at the Mean

- w_{2n+1} is the only weight that might not be positive
- Not a valid probability distribution if negative — can be problematic
- Regardless of sign chosen for coefficients α_i and β_i,

$$w_{2n+1} = 1 - \sum_{i=1}^{n} \frac{1}{\kappa_i - \gamma_i^2}$$

Staying Positive

- One solution is to scale γ_i and κ_i:

$$\gamma_i' = \gamma_i \sqrt{1+\theta}$$
$$\kappa_i' = \kappa_i (1+\theta)$$

  where $\theta \geq 0$ is small
- Minimizes effects on lower-order moments
- Satisfies Pearson's inequality

$$\kappa_i' \geq \gamma_i'^2 + 1$$

Selecting θ

- To attain $w'_{2n+1} \geq \delta$, where $\delta \geq 0$, we can set

$$\theta = \begin{cases} 0, & S \leq 1-\delta \\ \frac{S}{1-\delta} - 1 & S > 1-\delta \end{cases}$$

  where

$$S = \sum_{i=1}^{n} \frac{1}{\kappa_i - \gamma_i^2}$$

- Minimal case $\delta = 0$ successfully used in simulations

HOUSE Constraints

- For large values of kurtosis, sigma points may grow without bound
- To avoid this, impose the constraint, for some radius R

$$\|\mathbf{x}^{(i)} - \bar{\mathbf{x}}\| < R$$

- Other constraint — to ensure valid probability distribution

$$w_{2n+1} \geq 0$$

- Must hold for either original or tuned skewness and kurtosis

HOUSE Constraints

- Assume that $\mathbf{x}$ has a radially symmetric distribution about $\bar{\mathbf{x}}$
- Then constraint inequalities reduce to

$$\sigma\sqrt{\kappa} < R$$

and

$$\frac{n}{\kappa} \leq 1$$

σ — Standard deviation
κ — Kurtosis
n — State dimension

More

HOUSE Constraints

- The range of κ where HOUSE can reliably operate is

$$n \leq \kappa < \frac{R^2}{\sigma^2}$$

- Expect the filter to perform better when this range is wider, i.e.,

$$n\sigma^2 \ll R^2$$

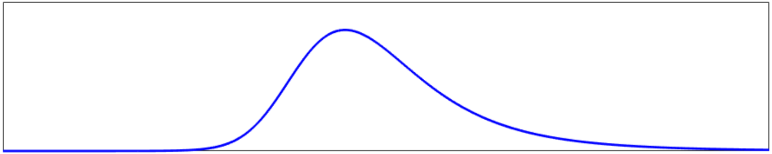
- Suggests that this filter is better for lower-dimensional systems

The Pearson Type IV Distribution

Probability density function on $(-\infty, \infty)$

$$p(x) \propto \left(1 + \left(\frac{x-\lambda}{a}\right)^2\right)^{-m} \exp\left(-\nu \arctan\left(\frac{x-\lambda}{a}\right)\right)$$

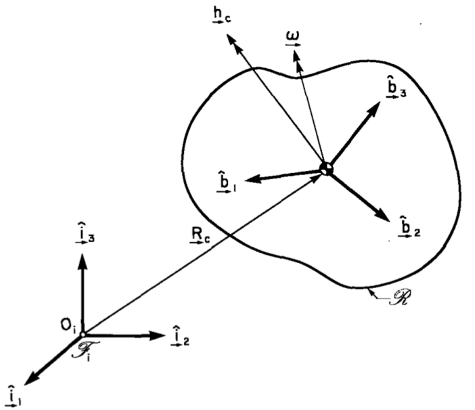
- Free parameters $a > 0$, $m > 1/2$, λ, and ν
- Uniquely determined by first four moments



Applications of Pearson Type IV

- Wind shear fluctuations (Ramsdell, 1978)
- Fluctuating pressure on aircraft skin panels (Steinwolf & Rizzi, 2012)
- Solar wind intensity (Krafft, Gauthier & Volotkin, 2019)

Rigid Body Example



"Rigid body $\mathscr{R}$ with body-fixed frame $\mathscr{F}_b$ and inertial frame $\mathscr{F}_i$." (Hughes, 2004)

Rigid Body Example: Dynamics & Measurements

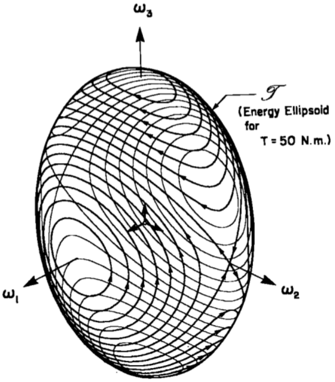
- Equations of motion

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + \tau_1$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 + \tau_2$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + \tau_3$$

- τ_1, τ_2, τ_3 are random external disturbance torques (process noise)
- $I_1 = 400\,\text{kg}\cdot\text{m}^2$, $I_2 = 200\,\text{kg}\cdot\text{m}^2$, and $I_3 = 100\,\text{kg}\cdot\text{m}^2$
- Only ω_1 is measured directly, with 10 Hz sampling rate

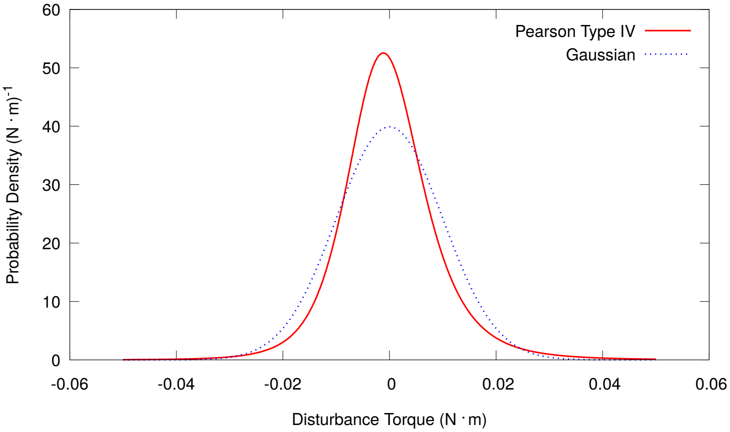
Rigid Body Example: Initial Conditions

- Test two sets of initial conditions
  - Major axis spinner: $\omega_1 = 0.01$ rad/s, $\omega_2 = \omega_3 = 0$
  - Intermediate axis spinner: $\omega_2 = 0.01$ rad/s, $\omega_1 = \omega_3 = 0$
- Both are stationary points
- First is stable in Lyapunov sense; second is unstable

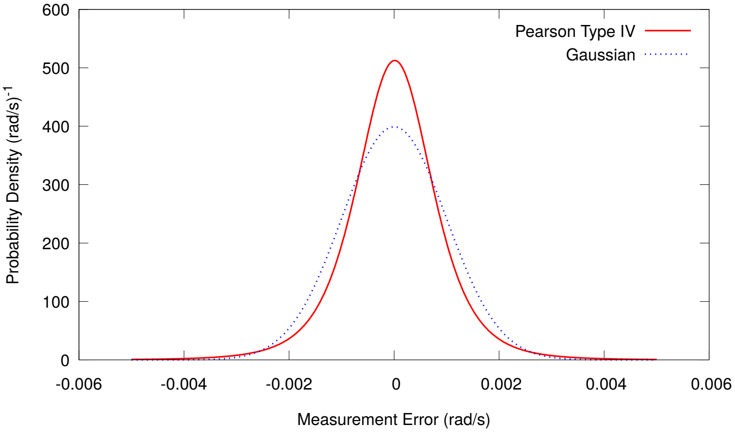


Hughes (2004)

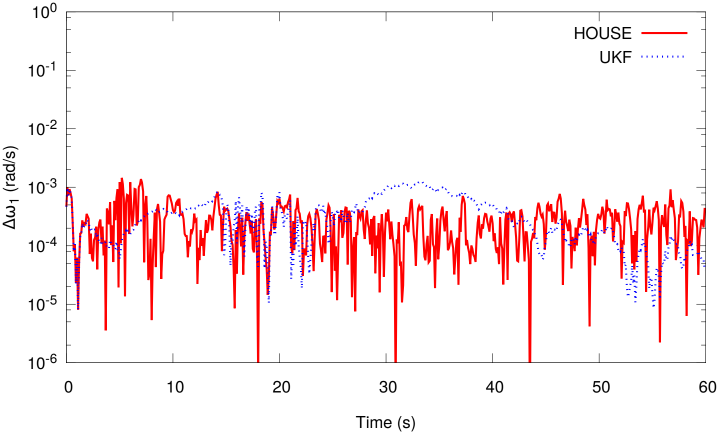
Rigid Body Example: Process Noise



Rigid Body Example: Measurement Noise



Rigid Body Example: Estimation Error
Stable Spinner; Gaussian Noise



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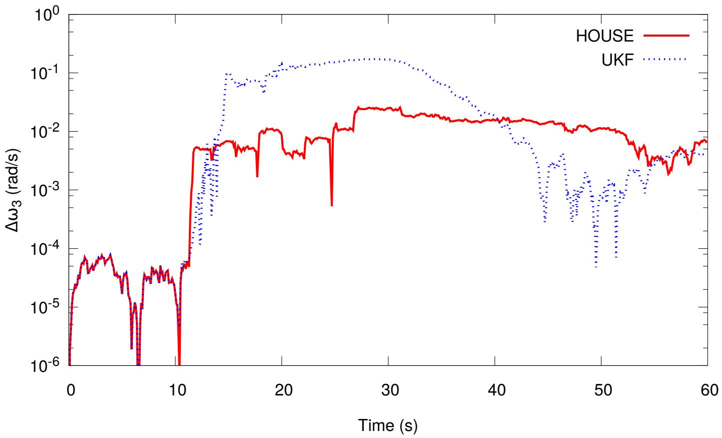
Rigid Body Example: Estimation Error
Stable Spinner; Gaussian Noise



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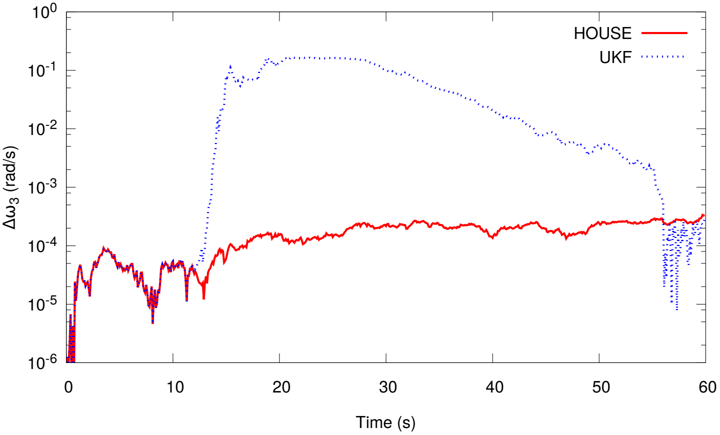
Rigid Body Example: Estimation Error
Stable Spinner; Pearson Type IV Noise



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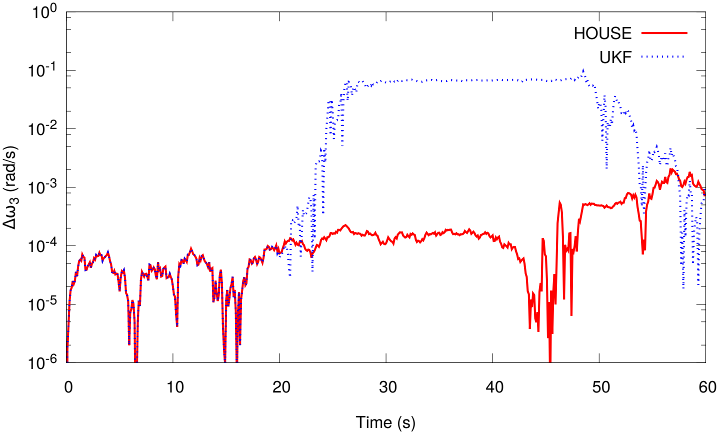
Rigid Body Example: Estimation Error

Unstable Spinner; Gaussian Noise



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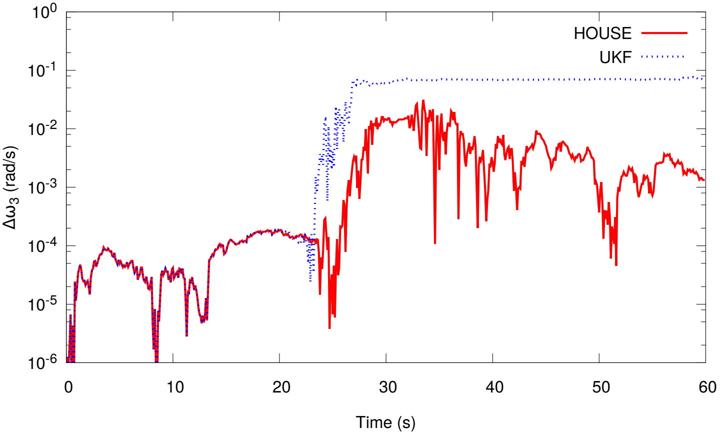
Rigid Body Example: Estimation Error
Unstable Spinner; Pearson Type IV Noise



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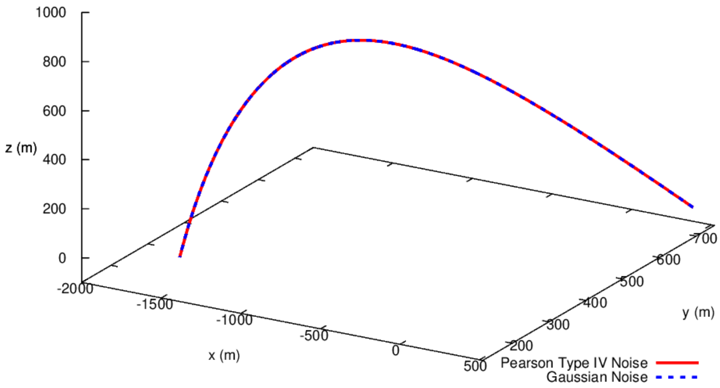
Projectile Example: True Trajectories



Projectile Example: Dynamics

- State consists of position (x, y, z) and velocity $(\dot{x}, \dot{y}, \dot{z})$
- Ballistic motion with drag

$$\ddot{x} = -bv\dot{x} + f_x$$
$$\ddot{y} = -bv\dot{y} + f_y$$
$$\ddot{z} = -bv\dot{z} + f_z - g$$

where

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \qquad b = \frac{AC_D\rho}{2m} = 0.001\,\mathrm{m}^{-1}$$

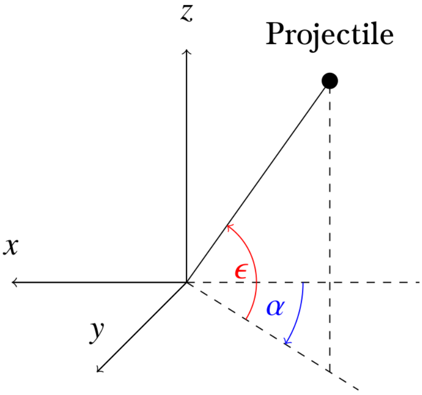
Projectile Example: Measurements

- Line-of-sight only: Azimuth & Elevation

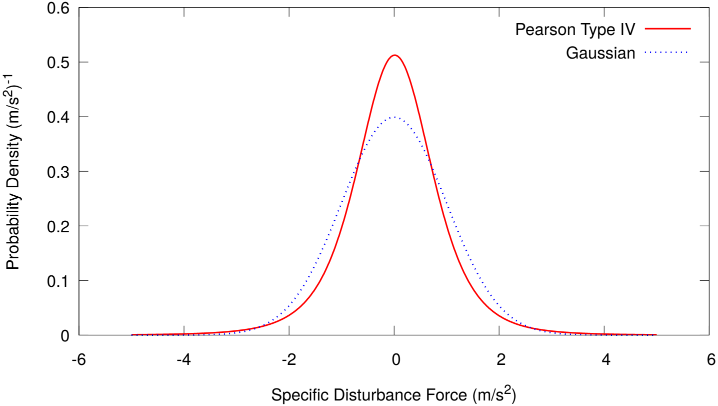
$$\alpha = \text{atan2}(y, -x) + n_\alpha$$

$$\epsilon = \text{atan2}\left(z, \sqrt{x^2 + y^2}\right) + n_\epsilon$$

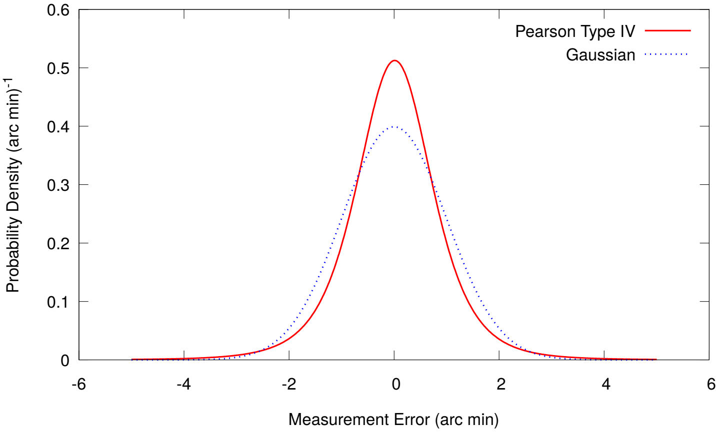
- Measurements taken at 100 Hz
- Noise n_α, n_ϵ has standard deviation of 1 arc-minute



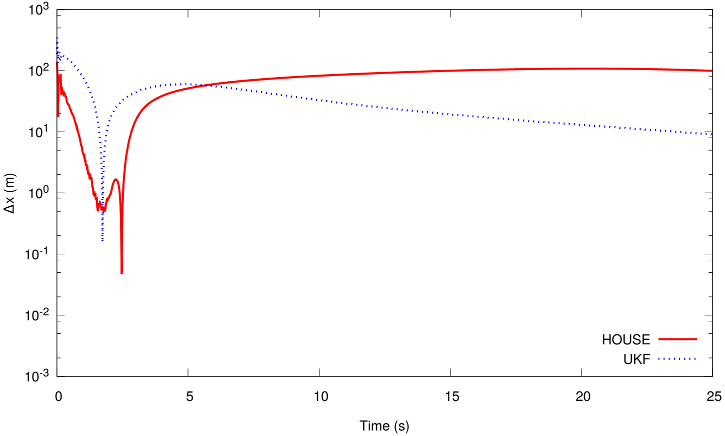
Projectile Example: Process Noise



Projectile Example: Measurement Noise



Projectile Example: Estimation Error
Gaussian Noise



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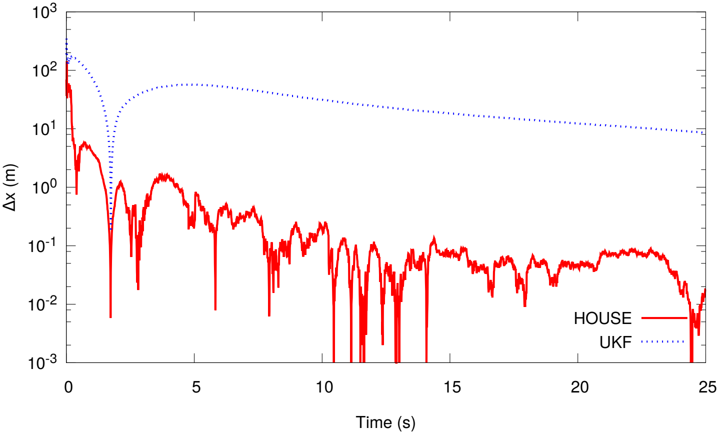
Projectile Example: Estimation Error

Pearson Type IV Noise



More

Conclusions on HOUSE Performance

- Provides significantly more accurate estimates than conventional UKF for heavy-tailed noise distributions
- Also appears to be more accurate for unstable systems
- Not as accurate for higher-dimensional systems with Gaussian noise

Future Work

- Refining HOUSE—goals:
  - Make accuracy independent of system dimension
  - Higher accuracy for systems with Gaussian noise
- Analyzing computational complexity vs. accuracy trade-off
- Testing
  - High-fidelity spacecraft model
  - Key points extracted from images

Future Work

- Next paper
  - HOUSE refinements
  - Deeper analysis and testing
- Exploring other filtering methods
  - Continuous-time prediction
  - Approximate Bayesian correction
- Spacecraft jitter estimation (NSTGRO proposal)

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Acknowledgements

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Appendix

HOUSE Operation: Prediction

1. For the augmented state $\mathbf{y}_{\mathrm{P}}(k) = [\mathbf{x}(k)^{\mathrm{T}} \quad \mathbf{w}(k)^{\mathrm{T}}]^{\mathrm{T}}$, generate the modified sigma points $\mathbf{y}_{\mathrm{P}}^{(j)}(k) = [\mathbf{x}^{(j)}(k)^{\mathrm{T}} \quad \mathbf{w}^{(j)}(k)^{\mathrm{T}}]^{\mathrm{T}}$ and weights $w_j(k)$
2. Propagate the state for each sigma point:

$$\mathbf{x}^{(j)}(k+1|k) = \mathbf{f}(\mathbf{x}^{(j)}(k), \mathbf{w}^{(j)}(k), k)$$

Back

HOUSE Operation: Prediction

The Higher-Order Unscented Estimator

Zvonimir Stojanovski

Motivation

Background

Z.S.'s Work

Methodology

Quadratures

Sigma Points & Weights

Tuning

Applicability

Examples of Application

Rigid Body

Projectile

Appendix

3. Compute the predicted mean and covariance:

$$\bar{\mathbf{x}}(k+1|k) = \sum_{j=1}^{N} w_j \mathbf{x}^{(j)}(k+1|k)$$

$$\boldsymbol{\eta}^{(j)}(k+1|k) = \mathbf{x}^{(j)}(k+1|k) - \bar{\mathbf{x}}(k+1|k)$$

$$\mathbf{P}_{xx}(k+1|k) = \sum_{j=1}^{N} w_j \boldsymbol{\eta}^{(j)}(k+1|k)\boldsymbol{\eta}^{(j)}(k+1|k)^{\mathrm{T}}$$

Back

4. Compute the standardized states at the sigma points:

$$\tilde{\mathbf{x}}^{(j)}(k+1|k) = \left(\sqrt{\mathbf{P}_{xx}(k+1|k)}\right)^{-1} \boldsymbol{\eta}^{(j)}(k+1|k)$$

5. Compute the skewness and kurtosis of the standardized state:

$$\gamma_i(k+1|k) = \sum_{i=1}^{N} w_j \tilde{x}_i^{(j)}(k+1|k)^3$$

$$\kappa_i(k+1|k) = \sum_{i=1}^{N} w_j \tilde{x}_i^{(j)}(k+1|k)^4$$

Back

HOUSE Operation: Update

Here, $\mathbf{z}(k)$ denotes the true measurement.

1. For the augmented state $\mathbf{y}_\mathrm{C}(k) = [\mathbf{x}(k)^\mathrm{T} \quad \mathbf{n}(k)^\mathrm{T}]^\mathrm{T}$, generate the modified sigma points $\mathbf{y}_\mathrm{C}^{(j)}(k) = [\mathbf{x}^{(j)}(k)^\mathrm{T} \quad \mathbf{n}^{(j)}(k)^\mathrm{T}]^\mathrm{T}$ and weights $w_j(k)$
2. Compute the measurement for each sigma point:

$$\mathbf{z}^{(j)}(k) = \mathbf{h}(\mathbf{x}^{(j)}(k), \mathbf{w}^{(j)}(k), k)$$

Back

HOUSE Operation: Update

3. Compute the measurement mean and covariance:

$$\bar{\mathbf{z}}(k) = \sum_{j=1}^{N} w_j \mathbf{z}^{(j)}(k)$$

$$\mathbf{P}_{zz}(k) = \sum_{j=1}^{N} w_j (\mathbf{z}^{(j)}(k) - \bar{\mathbf{z}}(k))(\mathbf{z}^{(j)}(k) - \bar{\mathbf{z}}(k))^{\mathrm{T}}$$

$$\mathbf{P}_{xz}(k) = \sum_{j=1}^{N} w_j (\mathbf{x}^{(j)}(k|k-1) - \bar{\mathbf{x}}(k|k-1))(\mathbf{z}^{(j)}(k) - \bar{\mathbf{z}}(k))^{\mathrm{T}}$$

Back

HOUSE Operation: Update

4. Compute the LMMSE error and standardized state at the sigma points:

$$\epsilon^{(j)}(k) = \mathbf{x}^{(j)}(k) - \bar{\mathbf{x}}(k|k-1) - \mathbf{P}_{xz}(k)\mathbf{P}_{zz}(k)^{-1}(\mathbf{z}^{(j)}(k) - \bar{\mathbf{z}}(k))$$

$$\tilde{\mathbf{x}}^{(j)} = \left(\sqrt{\mathbf{P}_{xx}(k)}\right)^{-1}\epsilon^{(j)}(k)$$

5. Compute the skewness and kurtosis of the standardized state:

$$\gamma_i(k) = \sum_{i=1}^{N} w_j \tilde{X}_i^{(j)}(k)^3 \qquad \kappa_i(k) = \sum_{i=1}^{N} w_j \tilde{X}_i^{(j)}(k)^4$$

Back

HOUSE Constraints

The constraint

$$\|\mathbf{x}^{(i)} - \bar{\mathbf{x}}\| < R$$

is equivalent to

$$\max_{i \in \{1,\ldots,n\}} \left(\|\mathbf{c}^{(i)}\| \max(\alpha_i, \beta_i) \right) < R$$

Back

HOUSE Constraints

- Since

$$\max(\alpha_i, \beta_i) = \frac{|\gamma_i| + \sqrt{4\kappa_i - 3\gamma_i^2}}{2}$$

the constraint reduces to

$$\max_{i \in \{1,\dots,n\}} \left(\frac{|\gamma_i| + \sqrt{4\kappa_i - 3\gamma_i^2}}{2} \|\mathbf{c}^{(i)}\| \right) < R$$

- Needs further simplification

Back

HOUSE Constraints

- To have valid weights, the original or tuned γ_i and κ_i must satisfy

$$\sum_{i=1}^{n} \frac{1}{\kappa_i - \gamma_i^2} \leq 1$$

- This imposes another bound on the moments
- Again, needs simplification

Back

HOUSE Constraints

- Assume that $\mathbf{x}$ has a radially symmetric distribution about $\bar{\mathbf{x}}$
- Then, we have $\kappa_i = \kappa$, $\gamma_i = 0$, and $\|\mathbf{c}^{(i)}\| = \sigma$ for $i = 1, \ldots, n$
- Constraint inequalities reduce to

$$\sigma\sqrt{\kappa} < R$$

and

$$\frac{n}{\kappa} \leq 1$$

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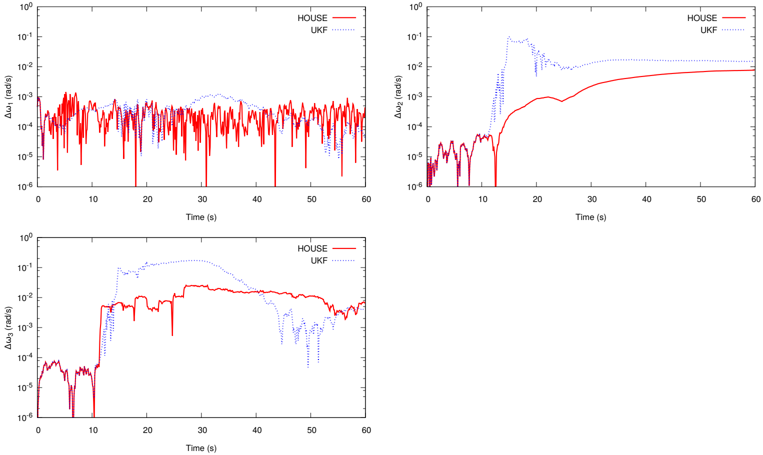
Rigid Body Example: Estimation Error
Stable Spinner; Gaussian Noise



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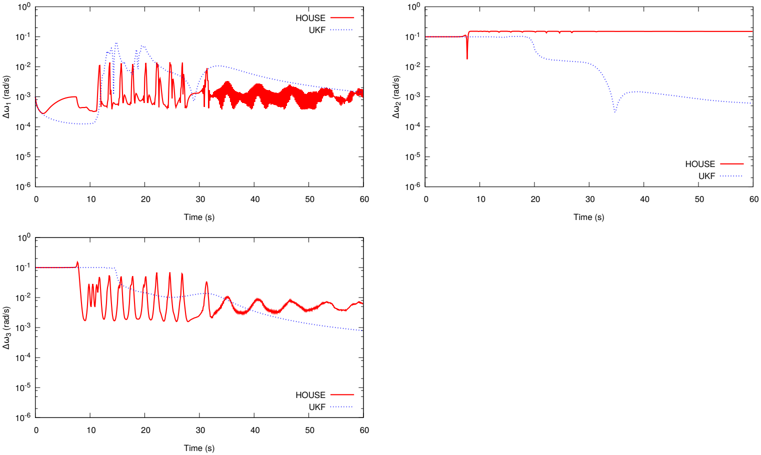
Rigid Body Example: Standard Deviation
Stable Spinner; Gaussian Noise



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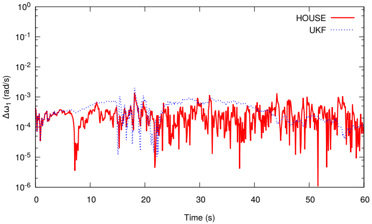
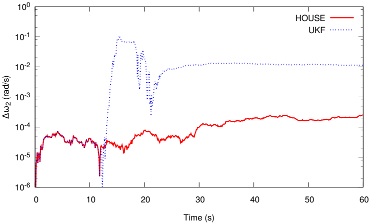
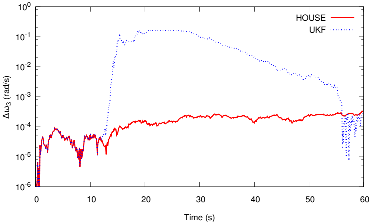
Rigid Body Example: Estimation Error
Stable Spinner; Pearson Type IV Noise







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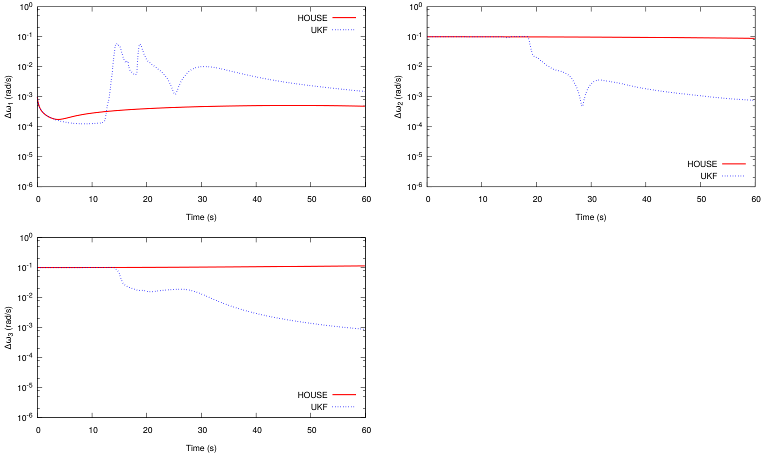
Rigid Body Example: Standard Deviation
Stable Spinner; Pearson Type IV Noise



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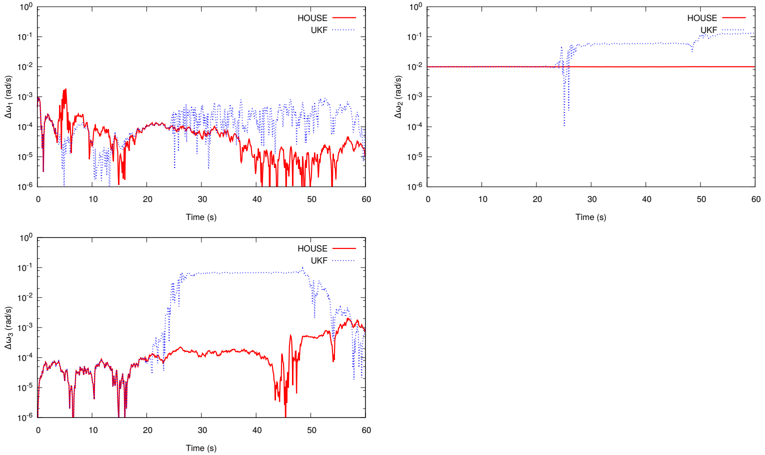
Rigid Body Example: Estimation Error
Unstable Spinner; Gaussian Noise



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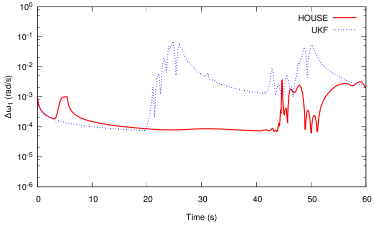
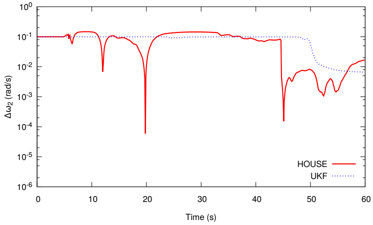
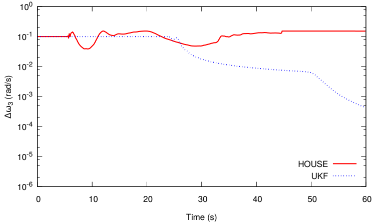
Rigid Body Example: Standard Deviation
Unstable Spinner; Gaussian Noise







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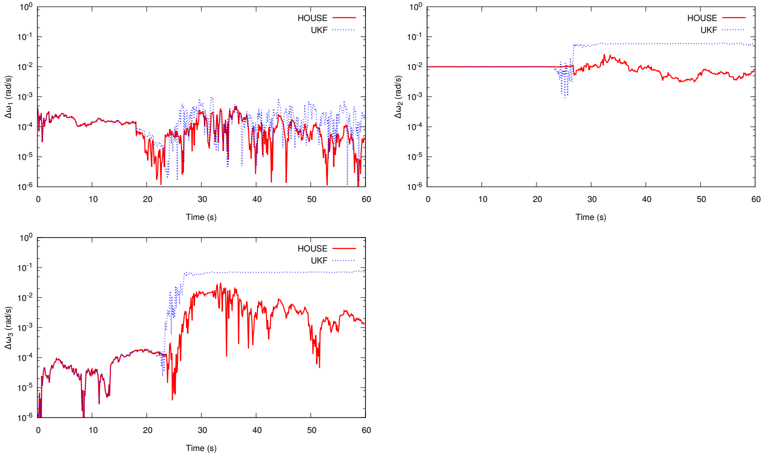
Rigid Body Example: Estimation Error
Unstable Spinner; Pearson Type IV Noise



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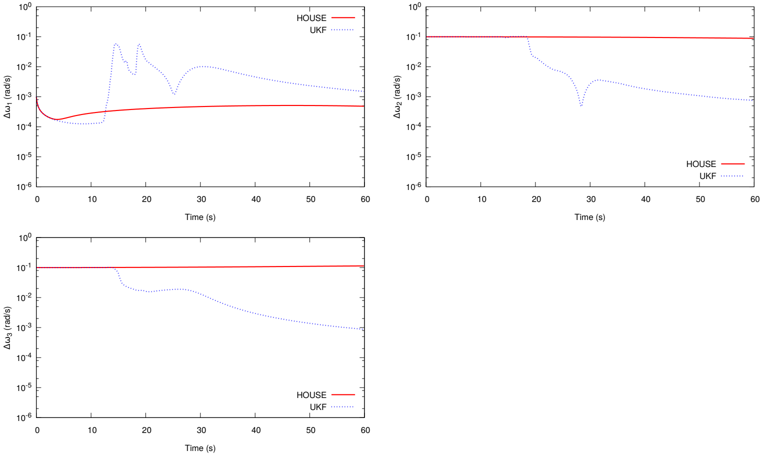
Rigid Body Example: Standard Deviation
Unstable Spinner; Pearson Type IV Noise



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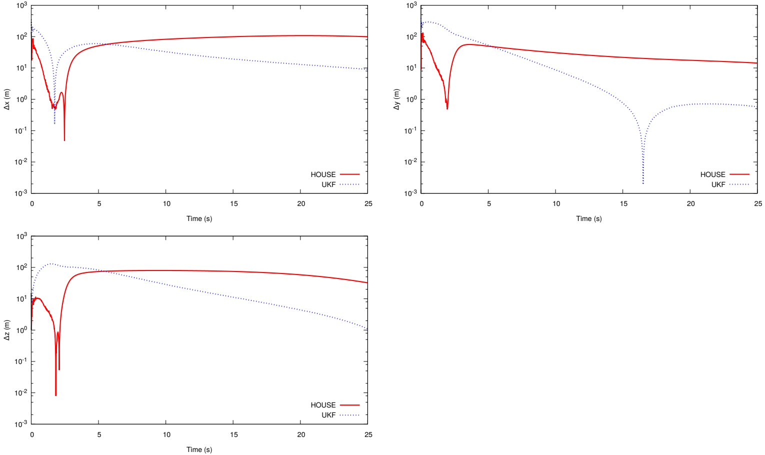
Projectile Example: Estimation Error
Gaussian Noise



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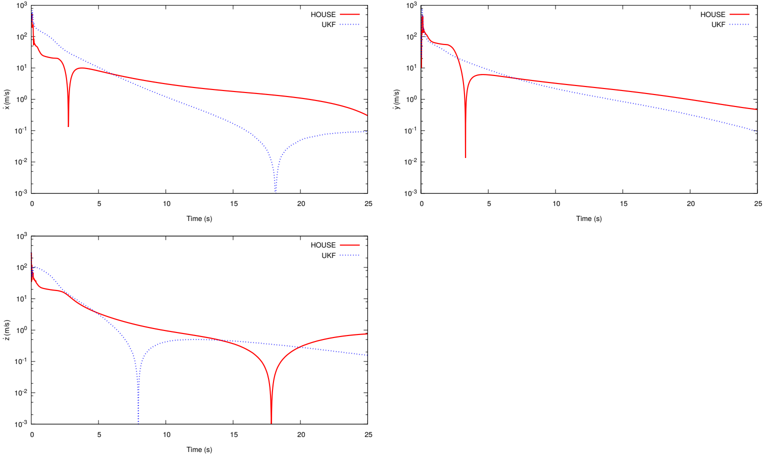
Projectile Example: Estimation Error
Gaussian Noise



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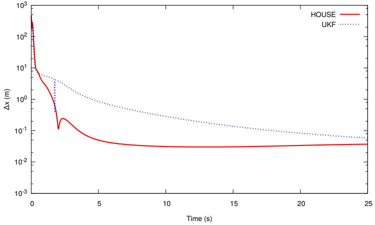
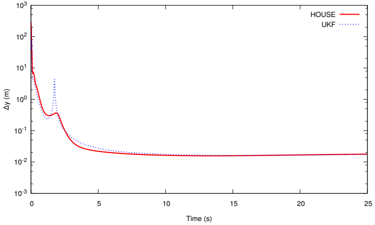
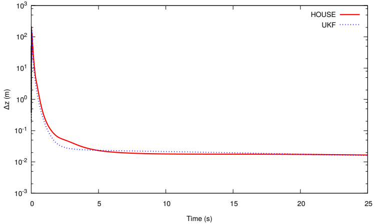
Projectile Example: Standard Deviation *Gaussian Noise*







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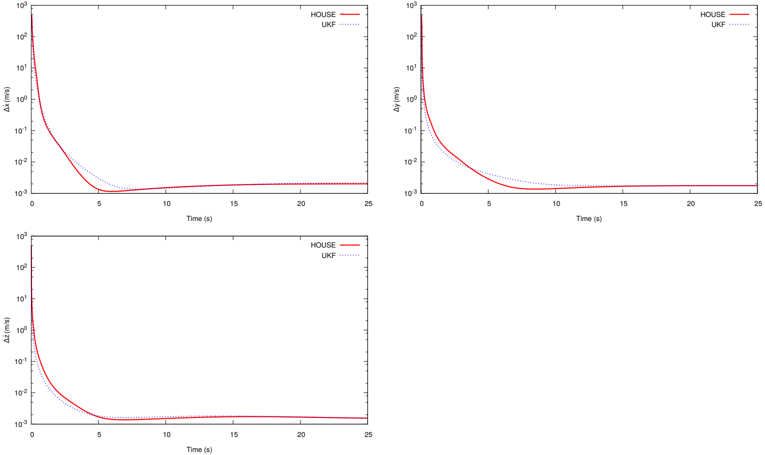
Projectile Example: Standard Deviation
Gaussian Noise



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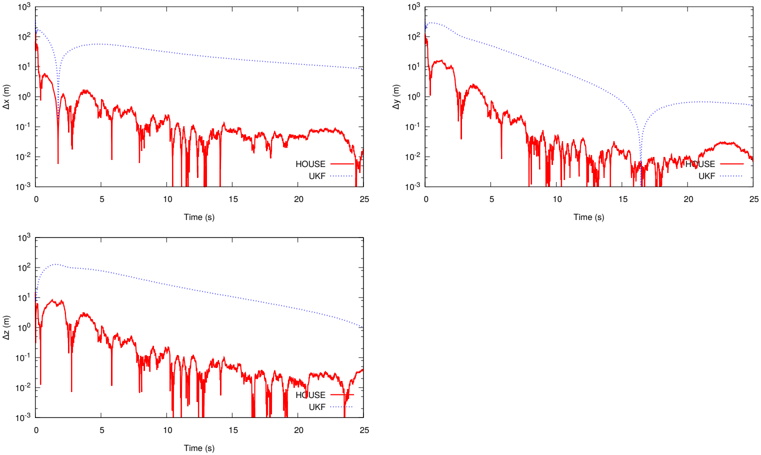
Projectile Example: Estimation Error
Pearson Type IV Noise



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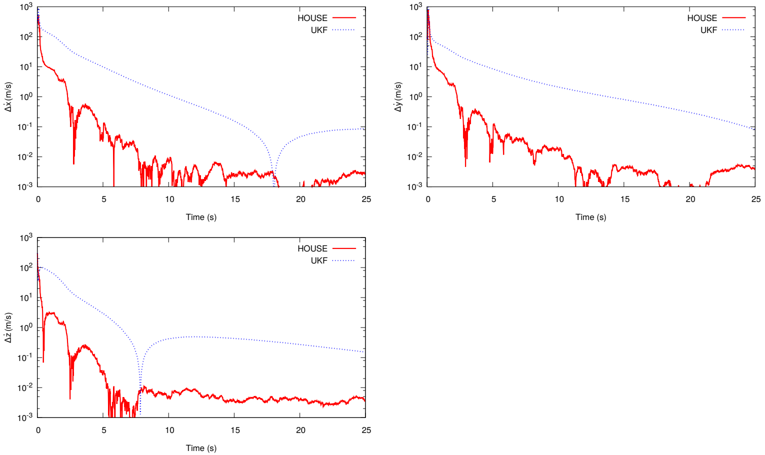
Projectile Example: Estimation Error

Pearson Type IV Noise



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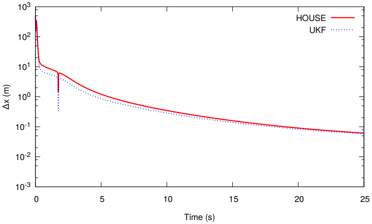
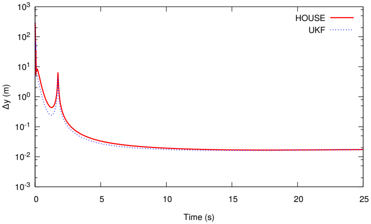
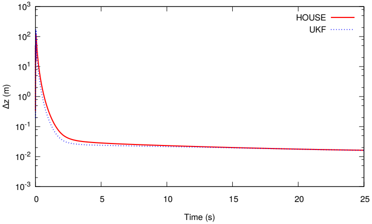
Projectile Example: Standard Deviation
Pearson Type IV Noise







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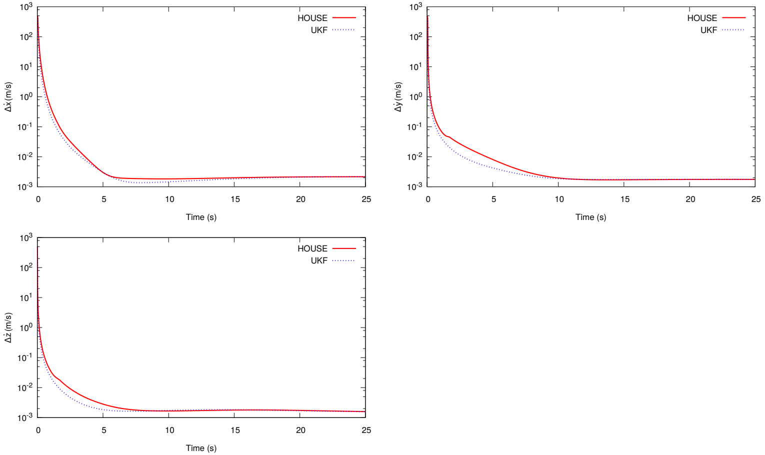
Projectile Example: Standard Deviation
Pearson Type IV Noise



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Equivalence of Quadrature Rules

$$\int_{\Omega} \phi(\mathbf{x}) p(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_{\tilde{\Omega}} \phi\left((\sqrt{\mathbf{P}_{xx}})\tilde{\mathbf{x}} + \bar{\mathbf{x}} \right) \tilde{p}(\tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}} = \int_{\tilde{\Omega}} \tilde{\phi}(\tilde{\mathbf{x}}) \tilde{p}(\tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}}$$

$$= w_{2n+1} \tilde{\phi}(\mathbf{0}) + \sum_{j=1}^{n} \left(w_j \tilde{\phi}(\alpha_j \mathbf{e}^{(j)}) + w_{n+j} \tilde{\phi}(-\beta_j \mathbf{e}^{(j)}) \right)$$

$$= w_{2n+1} \phi(\bar{\mathbf{x}}) + \sum_{j=1}^{n} \left(w_j \phi(\bar{\mathbf{x}} + \alpha_j \mathbf{c}^{(j)}) + w_{n+j} \phi(\bar{\mathbf{x}} - \beta_j \mathbf{c}^{(j)}) \right)$$

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Equivalence of Quadrature Rules

$$\int_{\tilde{\Omega}} \tilde{\phi}(\tilde{\mathbf{x}})\tilde{p}(\tilde{\mathbf{x}})\mathrm{d}\tilde{\mathbf{x}} = \int_{\tilde{\Omega}} \phi\left((\sqrt{\mathbf{P}_{xx}})\tilde{\mathbf{x}} + \bar{\mathbf{x}}\right)\tilde{p}(\tilde{\mathbf{x}})\mathrm{d}\tilde{\mathbf{x}} = \int_{\Omega} \phi(\mathbf{x})p(\mathbf{x})\mathrm{d}\mathbf{x}$$

$$= w_{2n+1}\phi(\bar{\mathbf{x}}) + \sum_{j=1}^{n}\left(w_j\phi(\bar{\mathbf{x}} + \alpha_j\mathbf{c}^{(j)}) + w_{n+j}\phi(\bar{\mathbf{x}} - \beta_j\mathbf{c}^{(j)})\right)$$

$$= w_{2n+1}\tilde{\phi}(\mathbf{0}) + \sum_{j=1}^{n}\left(w_j\tilde{\phi}(\alpha_j\mathbf{e}^{(j)}) + w_{n+j}\tilde{\phi}(-\beta_j\mathbf{e}^{(j)})\right)$$

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