Math and Dynamics Review

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The study of astrodynamics, or orbital mechanics, is essentially the study of classical mechanics (sometimes known as Newtonian mechanics). While we now know that these are only approximations, with a more accurate model available via Einstein's general relativity, for the majority of cases we rely on the laws first postulated by Isaac Newton and later expanded by Leonhard Euler. As with any study, the first step is to make sure that we have the appropriate tools and language to describe the phenomena under consideration. Since the early 20th century, thanks to the efforts of Josiah Willard Gibbs, the standard tools for studying classical mechanics are vector algebra and vector calculus, which we will review here. Remember that these handouts are not complete on their own. They are intended to accompany the recorded lectures, and to help in your note-taking and studying.

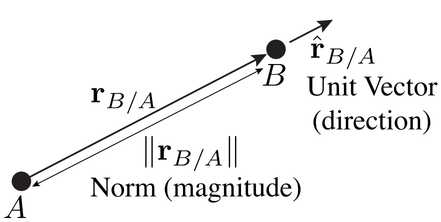
Vector Space Properties

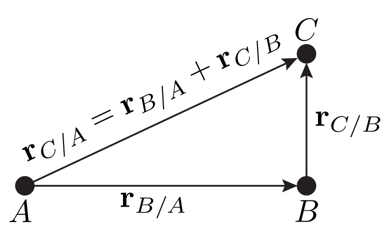
A vector space (V) is a collection of vectors ($\mathbf{a}, \mathbf{b}, \mathbf{c} \ldots \in V$) over a field of scalars ($x, y, z, \ldots \in \mathcal{F}$), with two operators: Vector Addition and Scalar Multiplication with the following properties:

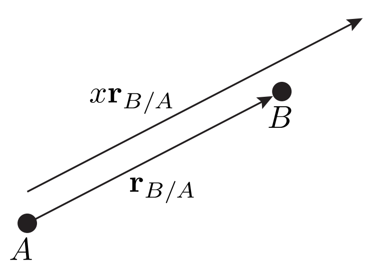
1. Commutativity of vector addition: $\forall \mathbf{a}, \mathbf{b} \in V : \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. Associativity of vector addition: $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in V : (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
3. Identity element of vector addition: $\exists \mathbf{0} \in V$ s.t. $\mathbf{a} + \mathbf{0} = \mathbf{a} \ \forall \mathbf{a} \in V$
4. Inverse elements of vector addition: $\forall \mathbf{a} \in V \ \exists {-\mathbf{a}} \in V$ s.t. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. Compatibility of scalar multiplication: $\forall x, y \in \mathcal{F}, \ \mathbf{a} \in V : x(y\mathbf{a}) = (xy)\mathbf{a}$
6. Distributivity of scalar multiplication over vector addition: $\forall x \in \mathcal{F}, \ \mathbf{a}, \mathbf{b} \in V : x(\mathbf{a} + \mathbf{b}) = x\mathbf{a} + x\mathbf{b}$
7. Distributivity of scalar multiplication over scalar addition: $\forall x, y \in \mathcal{F}, \ \mathbf{a} \in V : (x + y)\mathbf{a} = x\mathbf{a} + y\mathbf{a}$
8. Identity element of scalar multiplication: $\exists 1 \in \mathcal{F}$ s.t. $1\mathbf{a} = \mathbf{a} \ \forall \mathbf{a} \in V$

Euclidean (Geometric) Vectors

A Euclidean vector has a magnitude and a direction. A position vector $\mathbf{r}_{B/A}$ has a magnitude of the distance between points A and B and a direction pointing from A to B.







$$-\mathbf{r}_{B/A} = \mathbf{r}_{A/B}$$

$$\hat{\mathbf{r}}_{B/A} = \frac{\mathbf{r}_{B/A}}{\|\mathbf{r}_{B/A}\|}$$

$$\mathbf{r}_{C/B} + \mathbf{r}_{B/A} = \mathbf{r}_{C/A}$$

$$\Rightarrow \mathbf{r}_{C/A} \underbrace{-\mathbf{r}_{B/A}}_{\equiv +\mathbf{r}_{A/B}} = \mathbf{r}_{C/B}$$

$$x\mathbf{r}_{B/A} = x\|\mathbf{r}_{B/A}\|\hat{\mathbf{r}}_{B/A}$$

The **basis** of a vector space is:

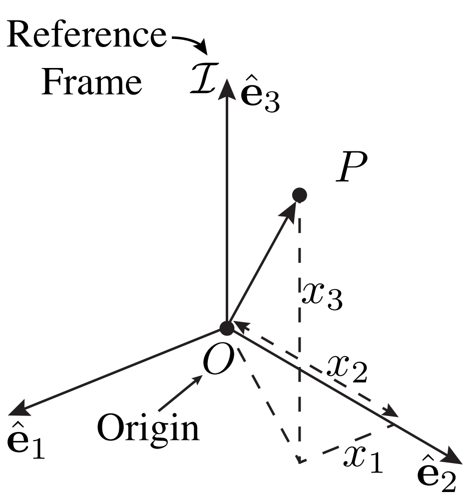
1. A linearly independent set of vectors spanning the vector space
2. A subset of vectors in the space such that all vectors in the space may be written as a weighted sum of the subset
3. **Not** unique

Define set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for $\mathbf{v}_i \in V$ for vector space V.

- S is **linearly independent** if $\sum_i a_i \mathbf{v}_i = 0 \Leftrightarrow a_i \equiv 0 \; \forall i, a_i \in \mathcal{F}$
- S **spans** V if $\exists\, a_i \in \mathcal{F}$ such that $\mathbf{b} = \sum_i a_i \mathbf{v}_i \; \forall \mathbf{b} \in V$

A reference frame is a basis for a 3D Euclidean vector space.

Reference Frames (Bases) and Vector Components



$$\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$$

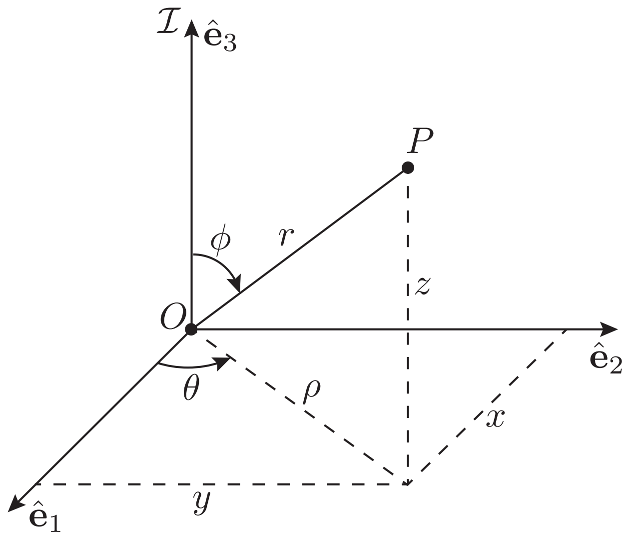
$$\mathbf{r}_{P/O} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3$$

x_i are **Cartesian** coordinates

$$\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathcal{I}}$$

Coordinate Systems

A single reference frame can have an infinite number of coordinate systems



Polar/Cylindrical Coordinates
θ - Azimuthal Angle

$$\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}} = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{bmatrix}_{\mathcal{I}}$$

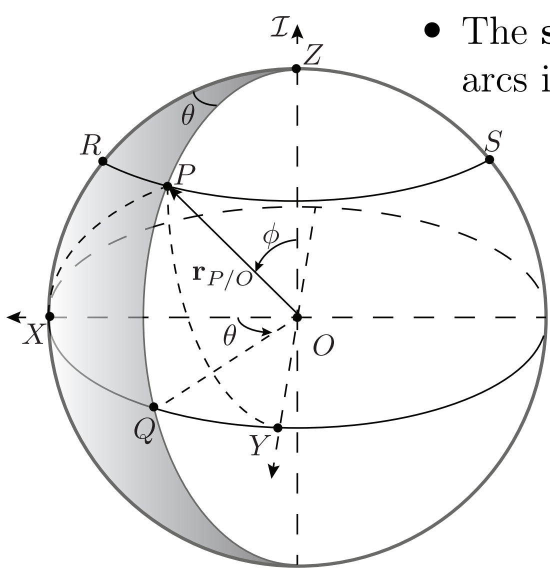
Spherical Coordinates
ϕ - Polar (Zenith) Angle

$$\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}} = r\begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}_{\mathcal{I}}$$

NB: θ and ϕ definitions are frequently reversed. Spherical coordinates are sometimes defined with an elevation angle (the complement to the zenith)

Spherical Trigonometry

- A plane passing through a sphere's center intersects the sphere in a **great circle**, which has **poles** perpendicular to the plane.



Adapted from Green (1985)

- The **spherical angle** between intersecting great circle arcs is the angle between their planes:

$$\text{Spherical Angle } XZQ \equiv \angle XOQ \equiv \theta$$
$$\equiv \text{Great Circle Arc } \widehat{XQ}$$

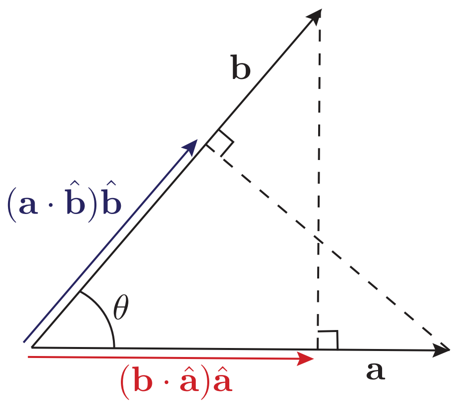
- A plane that does not pass through the sphere's center intersects the sphere in a **small** circle.

$$\left[\hat{\mathbf{r}}_{P/O}\right]_{\mathcal{I}} = \begin{bmatrix} \cos\widehat{XP} \\ \sin\widehat{YP} \\ \cos\widehat{ZP} \end{bmatrix}_{\mathcal{I}}$$

Vector Products

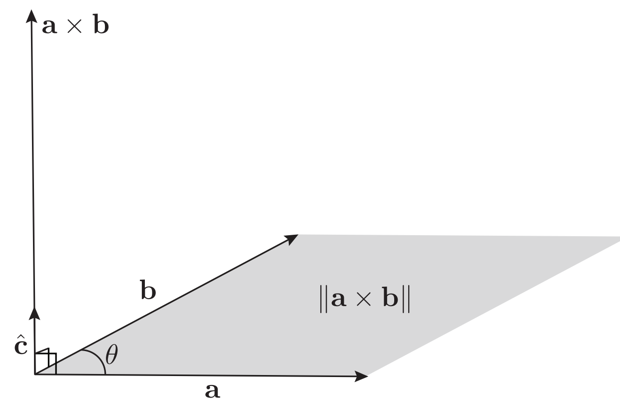
(Scalar) Dot Product

- $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $x\mathbf{a} \cdot y\mathbf{b} = xy(\mathbf{a} \cdot \mathbf{b})$



(Vector) Cross Product

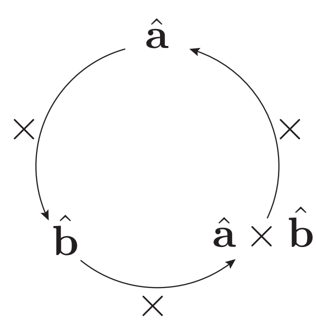
- $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta \hat{\mathbf{c}}$
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $y\mathbf{a} \times \mathbf{b} = y\,(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times y\mathbf{b}$



$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|\|\mathbf{a}\| \cos(0) = \|\mathbf{a}\|^2$$

If vector $\mathbf{a}$ is perpendicular to vector $\mathbf{b}$ ($\mathbf{a} \perp \mathbf{b}$):

1. $\mathbf{a} \cdot \mathbf{b} = 0$
2. $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{a}} \times \hat{\mathbf{b}}$ is a reference frame



This is a representation of the right-hand rule. Carrying out the cross-products in the counter-clockwise direction produces positive values. Following the circle clockwise produces negative values (i.e., $\hat{\mathbf{b}} \times \hat{\mathbf{a}} = -\hat{\mathbf{a}} \times \hat{\mathbf{b}}$).

Vector Triple Products

- Scalar Triple Product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$

- Vector Triple Product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$

Vector Products Can Be Written as Matrix Multiplications

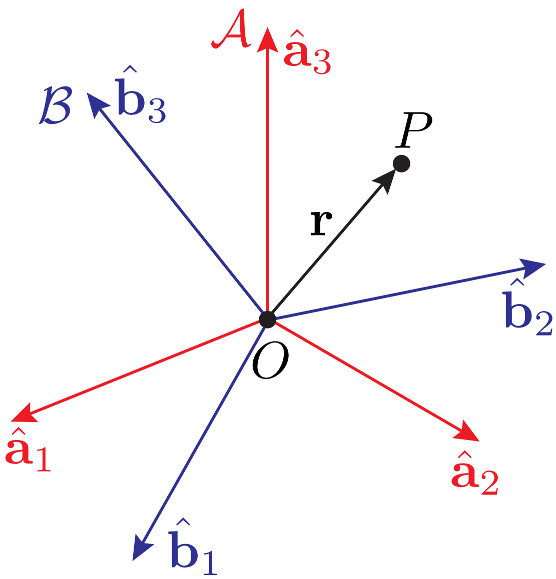
$$\mathcal{I} = (O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \qquad \mathbf{a} = \sum_i a_i \mathbf{e}_i \Rightarrow a_i = \mathbf{a} \cdot \mathbf{e}_i \qquad \mathbf{b} = \sum_i b_i \mathbf{e}_i \Rightarrow b_i = \mathbf{b} \cdot \mathbf{e}_i$$

$$[\mathbf{a}]_{\mathcal{I}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{I}} \qquad [\mathbf{b}]_{\mathcal{I}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}}$$

$$\boxed{\begin{aligned} [\mathbf{a} \cdot \mathbf{b}]_{\mathcal{I}} &= [\mathbf{a}]_{\mathcal{I}}^T [\mathbf{b}]_{\mathcal{I}} \\ [\mathbf{a} \times \mathbf{b}]_{\mathcal{I}} &= [\mathbf{a}\times]_{\mathcal{I}} [\mathbf{b}]_{\mathcal{I}} \\ [\mathbf{b} \times \mathbf{a}]_{\mathcal{I}} &= [\mathbf{b}\times]_{\mathcal{I}} [\mathbf{a}]_{\mathcal{I}} = -[\mathbf{a}\times]_{\mathcal{I}} [\mathbf{b}]_{\mathcal{I}} \end{aligned} \qquad \text{where} \qquad [\mathbf{a}\times]_{\mathcal{I}} \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_{\mathcal{I}}}$$

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2 \Longrightarrow [\mathbf{a} \cdot \mathbf{a}]_{\mathcal{I}} = [\mathbf{a}]_{\mathcal{I}}^T [\mathbf{a}]_{\mathcal{I}} = a_1^2 + a_2^2 + a_3^2 \quad \text{so} \quad \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Multiple Reference Frames and Direction Cosine Matrices



$$\boxed{[\mathbf{r}]_{\mathcal{B}} = {}^{\mathcal{B}}C^{\mathcal{A}}\,[\mathbf{r}]_{\mathcal{A}}}$$

Direction Cosine Matrix (DCM)

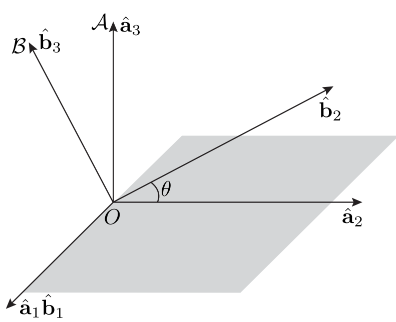
DCMs are Orthogonal Matrices:

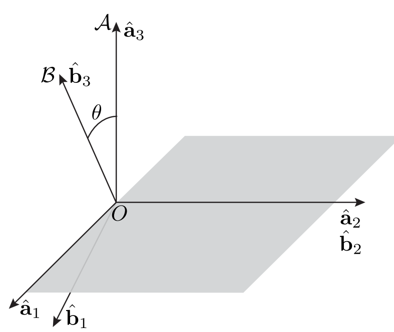
$$ {}^{\mathcal{A}}C^{\mathcal{B}} = \left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{-1} = \left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{T} $$

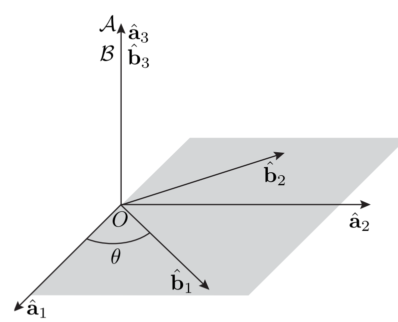
DCMs are Composed by Multiplication:

$$ {}^{\mathcal{I}}C^{\mathcal{F}_1}\,{}^{\mathcal{F}_1}C^{\mathcal{F}_2}\,{}^{\mathcal{F}_2}C^{\mathcal{F}_3}\ldots{}^{\mathcal{F}_{\mathcal{N}-1}}C^{\mathcal{F}_{\mathcal{N}}} = {}^{\mathcal{I}}C^{\mathcal{F}_{\mathcal{N}}} $$

Simple Direction Cosine Matrices







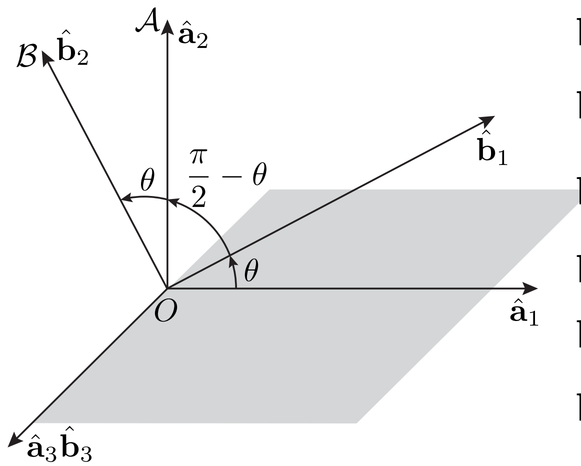
$$ {}^{\mathcal{B}}C^{\mathcal{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\triangleq\, C_1(\theta)} \qquad \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{\triangleq\, C_2(\theta)} \qquad \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\triangleq\, C_3(\theta)} $$

Any DCM can be decomposed into three rotations about non-repeating frame axes.

More on DCMs

Each entry of a DCM is the cosine of the angle between each pair of unit vectors of the two frames the DCM maps between:

$$\left.\begin{aligned}\mathcal{A} &= (O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)\\ \mathcal{B} &= (O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)\end{aligned}\right\} \left[{}^{\mathcal{A}}C^{\mathcal{B}}\right]_{ij} = \hat{\mathbf{a}}_i \cdot \hat{\mathbf{b}}_j \Longrightarrow {}^{\mathcal{B}}C^{\mathcal{A}} = \left({}^{\mathcal{A}}C^{\mathcal{B}}\right)^T \Longrightarrow \left[{}^{\mathcal{B}}C^{\mathcal{A}}\right]_{ij} = \hat{\mathbf{b}}_i \cdot \hat{\mathbf{a}}_j$$



$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_1 = \cos\theta$$

$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_2 = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_1 = \cos\left(\theta + \frac{\pi}{2} - \theta + \theta\right) = -\sin\theta$$

$$\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_2 = \cos\theta$$

$$\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_3 = 1$$

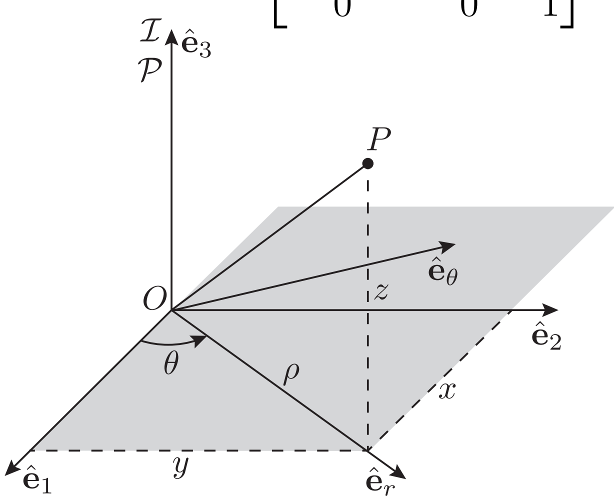
$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_3 = \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_3 = \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_1 = \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_2 = \cos\left(\frac{\pi}{2}\right) = 0$$

Polar/Cylindrical Reference Frames

$$\mathcal{P} = (O, \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$$

$${}^{\mathcal{P}}C^{\mathcal{I}} \equiv C_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\left[\mathbf{r}_{P/O}\right]_{\mathcal{P}}}_{\begin{bmatrix}\rho\\0\\z\end{bmatrix}_{\mathcal{P}}} = {}^{\mathcal{P}}C^{\mathcal{I}} \underbrace{\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}}}_{\begin{bmatrix}x\\y\\z\end{bmatrix}_{\mathcal{I}}}$$



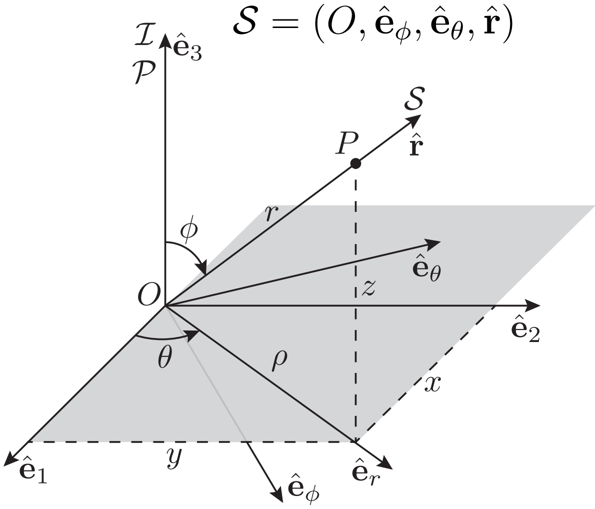
$$\begin{bmatrix}\rho\\0\\z\end{bmatrix}_{\mathcal{P}} = {}^{\mathcal{P}}C^{\mathcal{I}} \begin{bmatrix}x\\y\\z\end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} x\cos(\theta) + y\sin(\theta) \\ -x\sin(\theta) + y\cos(\theta) \\ z \end{bmatrix}_{\mathcal{P}}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}_{\mathcal{I}} = \underbrace{{}^{\mathcal{I}}C^{\mathcal{P}}}_{\left({}^{\mathcal{P}}C^{\mathcal{I}}\right)^T} \begin{bmatrix}\rho\\0\\z\end{bmatrix}_{\mathcal{P}} = \begin{bmatrix}\rho\cos(\theta)\\ \rho\sin(\theta)\\ z\end{bmatrix}_{\mathcal{I}}$$

Useful for tracking an object moving in-plane whose position is most easily described in polar coordinates. $\hat{\mathbf{e}}_r$ will always point at the object.

Spherical Reference Frames

$$ {}^{\mathcal{S}}C^{\mathcal{I}} = C_2(\phi)C_3(\theta) = \begin{bmatrix} \cos(\phi)\cos(\theta) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \\ \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta) & \cos(\phi) \end{bmatrix} $$



$$ \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{I}}}_{[\mathbf{r}_{P/O}]_{\mathcal{I}}} = \underbrace{{}^{\mathcal{I}}C^{\mathcal{S}}}_{\left({}^{\mathcal{S}}C^{\mathcal{I}}\right)^T} \underbrace{\begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}_{\mathcal{S}}}_{[\mathbf{r}_{P/O}]_{\mathcal{S}}} = r \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}_{\mathcal{I}} $$

Useful for tracking an object moving in 3D whose position is most easily described in spherical coordinates. $\hat{\mathbf{r}}$ will always point at the object.

Vector Derivatives

- A vector $\mathbf{r}_{P/O} = a_1\mathbf{a}_1 + a_2\mathbf{a}_2 + a_3\mathbf{a}_3$ is differentiable in time at a time t_1 with respect to frame $\mathcal{A} = (O, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ if $a_1(t), a_2(t), a_3(t)$ are differentiable at $t = t_1$. Then:

$$ \left.\frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\mathbf{r}_{P/O}\right|_{t=t_1} = \left.\frac{\mathrm{d}a_1}{\mathrm{d}t}\right|_{t=t_1}\mathbf{a}_1 + \left.\frac{\mathrm{d}a_2}{\mathrm{d}t}\right|_{t=t_1}\mathbf{a}_2 + \left.\frac{\mathrm{d}a_3}{\mathrm{d}t}\right|_{t=t_1}\mathbf{a}_3 $$

- The unit vectors defining a frame **always** have zero time derivatives with respect to that frame (but not necessarily to other frames)

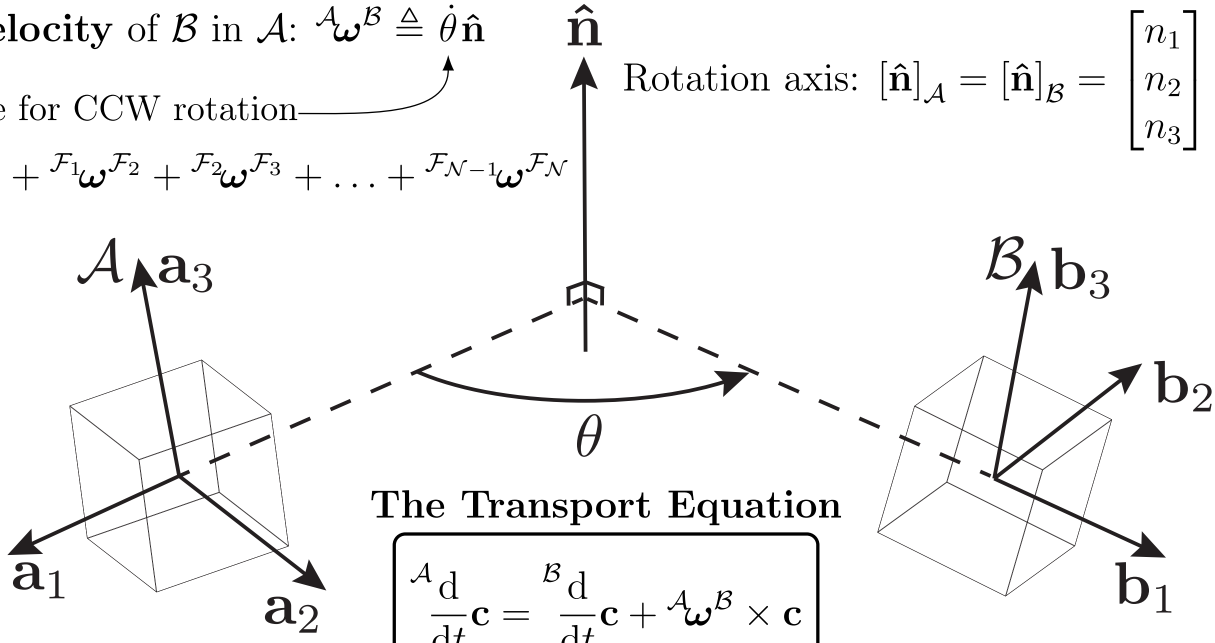
Vector Differentiation Across Reference Frames

Angular Velocity of $\mathcal{B}$ in $\mathcal{A}$: ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \triangleq \dot{\theta}\,\hat{\mathbf{n}}$

Positive for CCW rotation

Rotation axis: $[\hat{\mathbf{n}}]_{\mathcal{A}} = [\hat{\mathbf{n}}]_{\mathcal{B}} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{F}_{\mathcal{N}}} = {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{F}_1} + {}^{\mathcal{F}_1}\boldsymbol{\omega}^{\mathcal{F}_2} + {}^{\mathcal{F}_2}\boldsymbol{\omega}^{\mathcal{F}_3} + \ldots + {}^{\mathcal{F}_{\mathcal{N}-1}}\boldsymbol{\omega}^{\mathcal{F}_{\mathcal{N}}}$$



The Transport Equation

$$\boxed{\frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\mathbf{c} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{c} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{c}}$$

NB: Counter-clockwise is defined by looking *down* along the axis of rotation.

Newton's Laws of Motion

1. *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare*
   Every body preserves in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon
2. *Mutationem motus proportionalem esse vi motrici impressae; et fieri secundum lineam rectam qua vis illa imprimitur*
   The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed
3. *Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi*
   To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts

Newton's Second Law

Inertial Frame Derivative   Inertially Fixed Point   Mass (Assumed Constant)

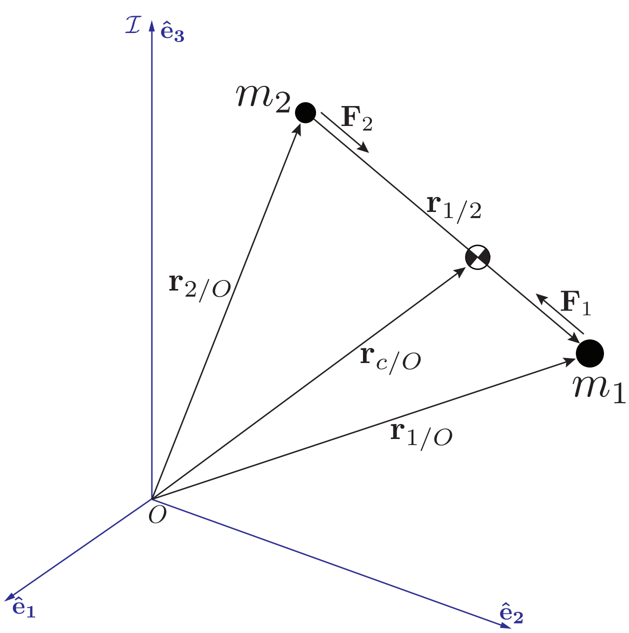
$$\mathbf{F}_P = {}^{\mathcal{I}}\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{p}_{P/O}\right) = {}^{\mathcal{I}}\frac{\mathrm{d}}{\mathrm{d}t}\left(m_P\,{}^{\mathcal{I}}\mathbf{v}_{P/O}\right) = m_P\,{}^{\mathcal{I}}\mathbf{a}_{P/O}$$

Resultant Force on P Linear Momentum   Inertial Velocity and Acceleration

$$\mathbf{M}_{P/O} = {}^{\mathcal{I}}\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_{P/O}\right) = {}^{\mathcal{I}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_{P/O} \times {}^{\mathcal{I}}\mathbf{p}_{P/O}\right) = \mathbf{r}_{P/O} \times \mathbf{F}_P$$

Net Moment (Torque) about O Angular Momentum of P about O

Newton's Law of Gravity



Gravitational Constant

$$\mathbf{F}_1 = -\mathbf{F}_2 = -\frac{G\,m_1 m_2}{\|\mathbf{r}_{1/2}\|^3}\mathbf{r}_{1/2}$$

Work and Energy

- A force ($\mathbf{F}_P$) does work (W) on a particle P when it displaces the particle along a trajectory (γ_P): $W_P^{\mathbf{F}_P}(\mathbf{r}_{P/O};\gamma_P) \triangleq \int_{\gamma_P} \mathbf{F}_P \cdot {}^{\mathcal{I}}\mathrm{d}\mathbf{r}_{P/O}$
  Path Integral over trajectory
- The **Kinetic Energy** of particle P is defined as: $T_{P/O} \triangleq \frac{1}{2}m_p({}^{\mathcal{I}}\mathbf{v}_{P/O} \cdot {}^{\mathcal{I}}\mathbf{v}_{P/O})$
- The change in kinetic energy from time t_1 to time t_2 is equal to the total work done on the particle during that time
- The work done by **Conservative Forces** depends only on the endpoints of the trajectory. $\oint \mathbf{F}_P \cdot {}^{\mathcal{I}}\mathrm{d}\mathbf{r}_{P/O} = 0$ means that $\mathbf{F}_P$ is conservative.
  Closed Path Integral
- Conservative Forces can always be written as the gradient of a scalar **Potential** (U): $\mathbf{F}_P^{(\mathrm{cons})} = -\boldsymbol{\nabla} U_{P/O}^{(\mathbf{F}_P)}$ so
  $U_{P/O}^{(\mathbf{F}_P)}(t_2) = U_{P/O}^{(\mathbf{F}_P)}(t_1) - W_P^{(\mathbf{F}_P)}(t_1, t_2)$

Total Work and Energy

- Total Energy: $E_{P/O}(t) \triangleq T_{P/O}(t) + U_{P/O}(t)$
- Total Work: $W_P^{\mathrm{tot}}(\mathbf{r}_{P/O};\gamma_P) = \underbrace{W_P^{\mathrm{c}}(t_1,t_2)}_{\substack{\text{Work due to conservative forces} \\ = \text{negative change in potential energy}}} + \underbrace{W_P^{\mathrm{nc}}(\mathbf{r}_{P/O};\gamma_P)}_{\substack{\text{Work due to non-conservative forces} \\ = \text{change in total energy}}}$
- Conservation of Energy: no non-conservative forces $\equiv$ constant total energy

$$E_{P/O}(t_2) = E_{P/O}(t_1) + W_P^{(\mathrm{nc})}(\mathbf{r}_{P/O};\gamma_P)$$

Numerical Integration for Initial Value Problems

In general, numerical integrators are trying to solve the IVP:

$$\text{For } \dot{\mathbf{x}} = f(\mathbf{x}, t); \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n; \quad f : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n$$
$$\text{Find } \mathbf{x}(t); \quad t \in [t_0, \pm\infty)$$

$$\text{Forward Euler Method: } \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t f(\mathbf{x}_k, t_k)$$

$$\text{Runge-Kutta: } \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \sum_{i=1}^{s} b_i g_i \begin{cases} g_1 & = f(\mathbf{x}_k, t_k) \\ g_2 & = f(\mathbf{x}_k + \Delta t(a_{21} g_1), t_k + c_2 \Delta t) \\ g_3 & = f(\mathbf{x}_k + \Delta t(a_{31} g_1 + a_{32} g_2), t_k + c_3 \Delta t) \\ & \dots \\ g_i & = f\left(\mathbf{x}_k + \Delta t \sum_{j=1}^{s} a_{ij} g_j, t_k + c_i \Delta t\right) \end{cases}$$

The values of the coefficients a_{ij}, b_i, c_i are determined by the order of the Runge-Kutta method.

Numerical Integrators in MATLAB

MATLAB Provides many different functions for solving systems of ordinary differential equations:

- `ODE45` General-purpose, medium-order Runge-Kutta method. Good place to start for most problems.
- `ODE23` General-purpose, lower-order Runge-Kutta method. Good for getting faster results with less precision.
- `ODE113` Variable-order method, useful when high precision (low numerical error) is desired, and when the function f is expensive to compute.

For lots more detail, see:
www.mathworks.com/help/matlab/ordinary-differential-equations.html

Numerical Integration for 2nd order differential equations

- Our equations of motion typically have the form $\ddot{\mathbf{x}} = f(\mathbf{x}, \dot{\mathbf{x}}, t)$

- We can always turn these into first-order equations by defining a new state:

$$\mathbf{z} \triangleq \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} \quad \Longrightarrow \quad \dot{\mathbf{z}} = \begin{bmatrix} \mathbf{z}_2 \\ f(\mathbf{z}_1, \mathbf{z}_2, t) \end{bmatrix}$$

Reference Frames, Coordinate Systems, and Time

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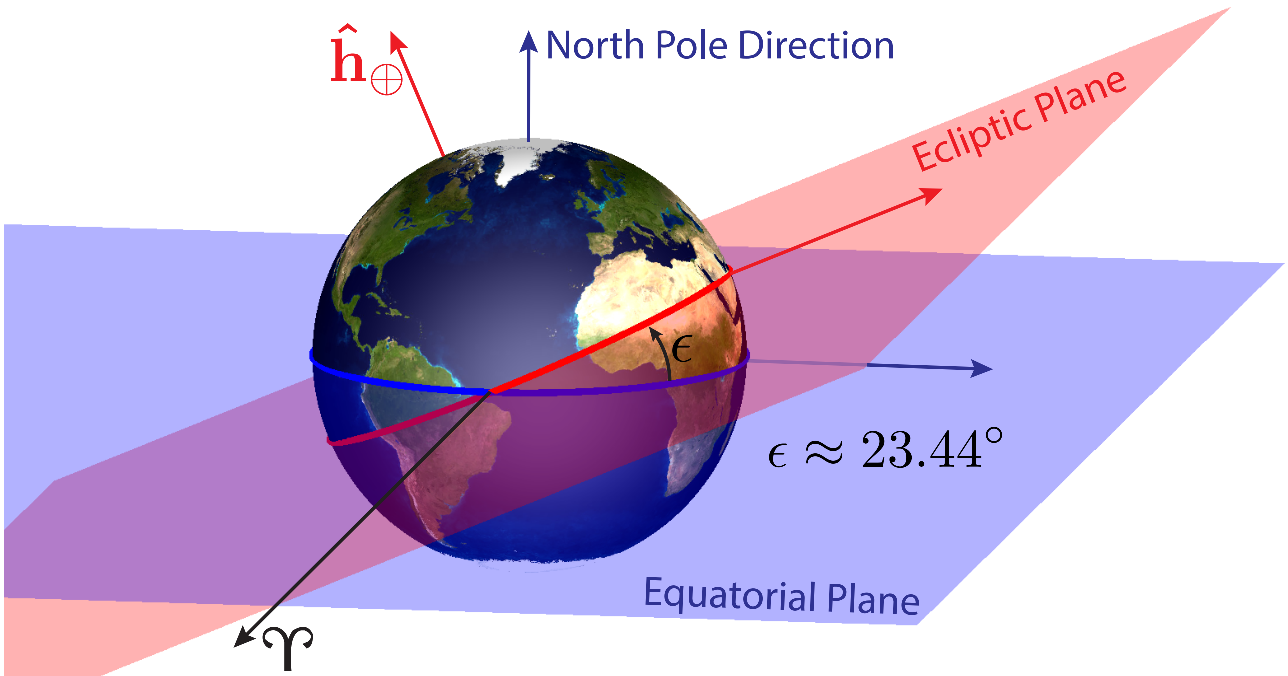
MAE 4060/5065, Fall 2021

Reference Frames, Coordinate Systems, and Time

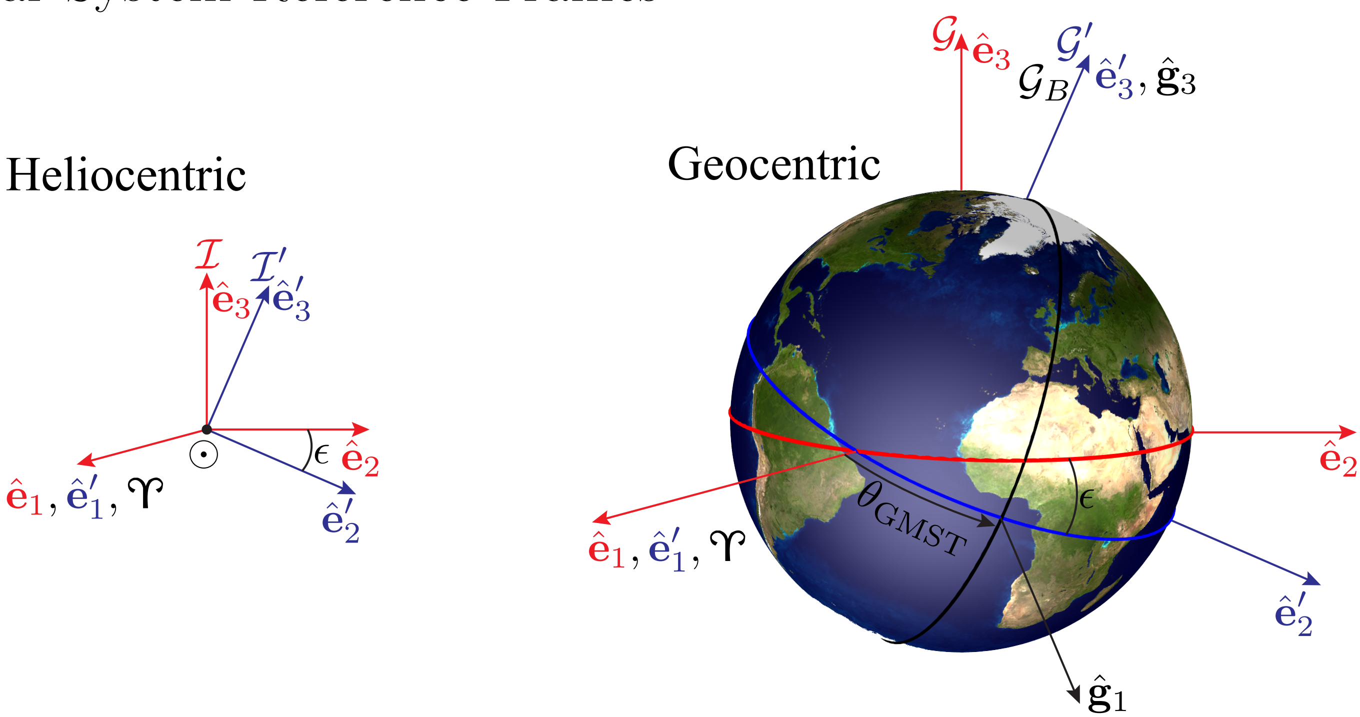
One of the very first steps in solving a dynamics problem involving objects in space (after you have created your system model and listed out all assumptions being made), is to select the reference frames and coordinate systems you'll be working with. For astrodynamics problems, there will usually be one (or a handful) of systems that make the most sense, based on the central body of your orbit, or what, specifically, you are trying to calculate. Here, we review the standard reference frames and coordinate systems used to describe orbits about the Earth and throughout the solar system, as well as standards for measuring time.

Solar System Reference Planes



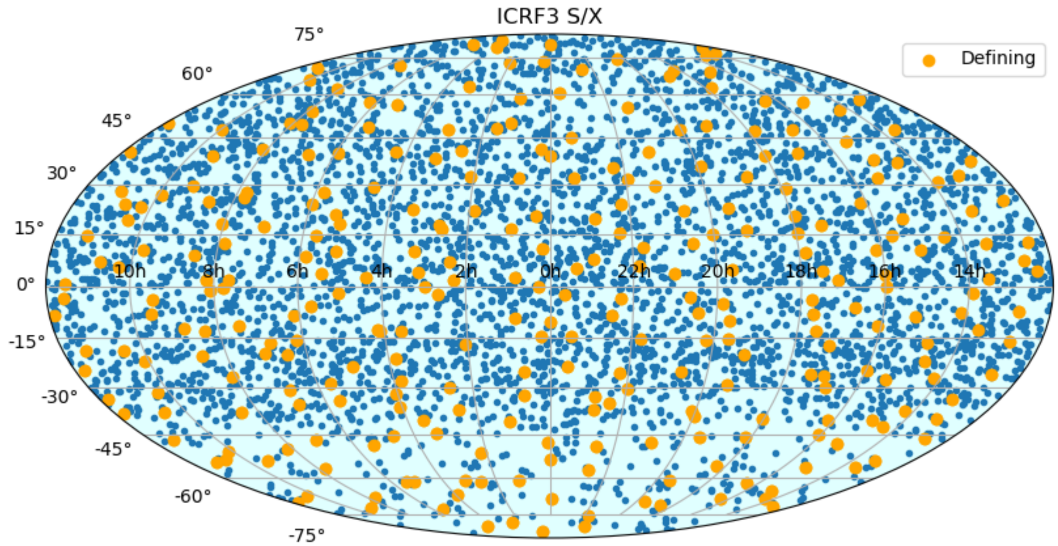
ϵ is known as the *obliquity of the ecliptic*. The ecliptic was historically defined as the mean plane of Earth's orbit, while the equatorial plane was defined as the plane of the Earth's equator. In reality, neither of these planes is inertially fixed, but modern (inertially fixed) definitions attempt to match the historical meanings as closely as possible.

Solar System Reference Frames



$\mathcal{I}, \mathcal{G}$ Ecliptic, Inertial; $\mathcal{I}', \mathcal{G}'$ Equatorial, Inertial; $\mathcal{G}_B$ Equatorial, Non-Inertial

International Celestial Reference System (ICRS)



ICRF3: 4,356 Extragalactic Sources, 303 defining.
http://hpiers.obspm.fr/icrs-pc/newwww/icrf/index.php

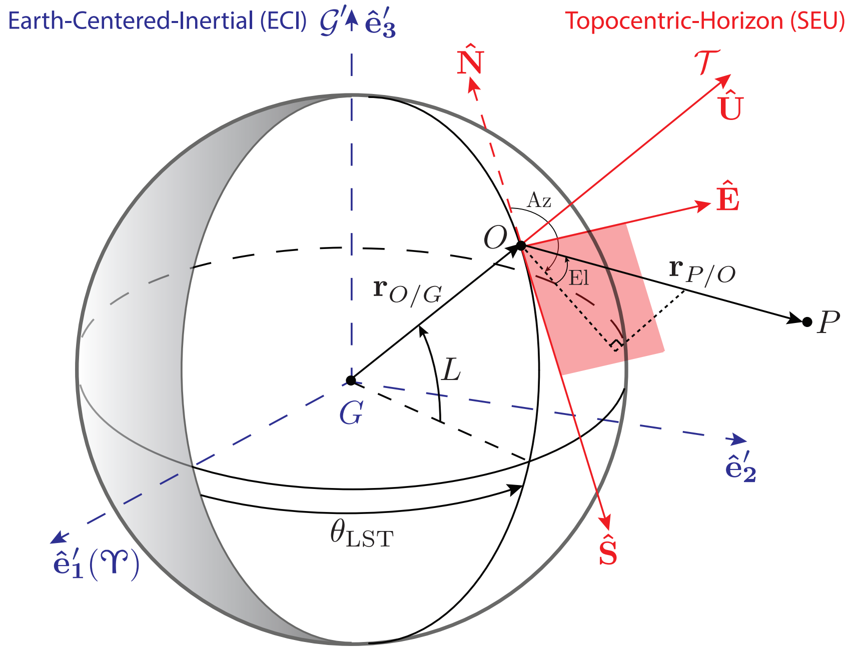
ICRS is the standard by which the reference frame is defined. ICRF1-3 are realizations fo the standard based on updated measurements. ICRS attempts to approximate equatorial coordiantes, with a coordinate origin at the solar system barycenter, a pole direction approximating the north pole direction, and an equinox direction approximating ♈.

Spherical Coordinate Systems

Name	Origin	Reference Plane	Prime Direction	Azimuth Angle	Elevation Angle
Geographic	Geocentric	Equator	Prime Meridian	Longitude (λ)	Latitude (φ or L)
Horizontal (Topocentric)	Observer Location	Horizon	North	Azimuth (Az)	Altitude/Elevation (Alt/El)
Equatorial	Geocentric or Heliocentric	Celestial Equator	Vernal Equinox	Right Ascension (α)	Declination (δ)
Ecliptic	Geocentric or Heliocentric	Ecliptic	Vernal Equinox	Ecliptic Longitude (λ)	Ecliptic Latitude (β)
Galactic	Heliocentric	Galactic Plane	Galactic Center	Galactic Longitude (l)	Galactic Latitude (b)

NB: These are defined via an elevation (rather than zenith) angle.

Topocentric-Horizon Coordinate System



The majority of observations still occur from the surface of the Earth, and so it is important to define reference frames that assist in the transformation of these observations to inertial frame components. The topocentric-horizon reference frame and coordinate system is defined with the origin at the observer's location, and with the frame rotating with the Earth. The coordinates are spherical: azimuth (Az) measured in the $\hat{\mathbf{E}} - \hat{\mathbf{S}}$ plane from north ($\hat{\mathbf{N}} = -\hat{\mathbf{S}}$) and *elevation* (El) measured *up* from this plane to the position vector.

Central Body Shape

It is important to remember that the Earth and other central bodies are not spherical. In fact, the smaller the body, the more non-spherical it is likely to be. The Earth and other planetary bodies are best described as 'lumpy, oblate spheroids'. This description allows for multiple levels of approximation. The first is to simply treat the body as a sphere. For the Earth, the polar and equatorial radius differ by approximately 0.3%, so this is a fair approximation, but inadequate for precision work. The next level of approximation is to fit a mean oblate spheroid (the surface of revolution produced by an ellipse), called a **reference geoid** (or datum surface). The final level is to decompose the true shape/mass distribution of the body into an orthogonal basis set known as spherical harmonics, which we will consider when we discuss orbital perturbations. While many different reference geoids exist for the Earth, we will focus on one of the most widely used ones (and the one used by the GPS constellation: the World Geoditic System 1984 (WGS84) geoid.

The Reference Geoid

World Geodetic System (WGS) 84

$\omega_e = 7.292115 \times 10^{-5}$ rad/s

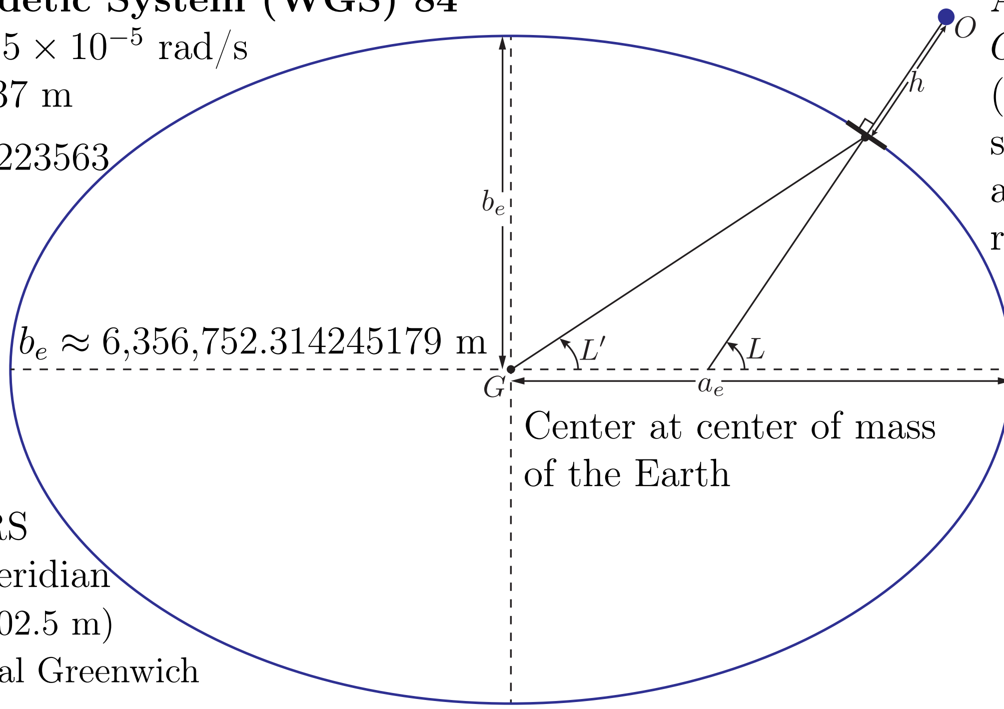
$a_e = 6{,}378{,}137$ m

$\dfrac{1}{f} = 298.257223563$

$f \triangleq \dfrac{a-b}{a}$

$b_e \approx 6{,}356{,}752.314245179$ m

A surface point O at Lon/Lat (λ, L) is at some height h above the reference geoid



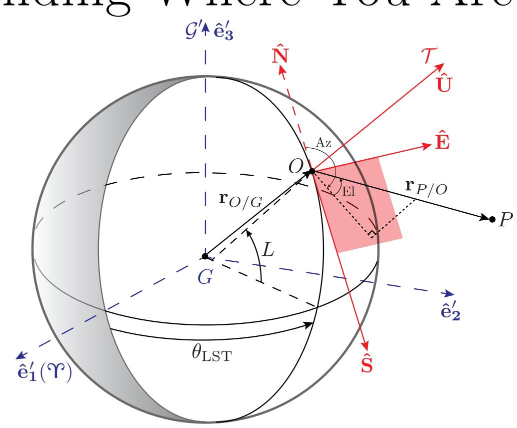
Center at center of mass of the Earth

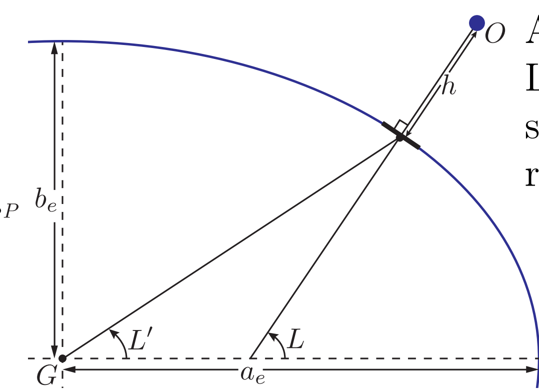
$\lambda = 0$ at IERS Reference Meridian ~5.3 arcsec(102.5 m) East of original Greenwich

See also: https://www.iers.org/IERS/EN/Home/home_node.html

L is the **geoditic** latitude and L' is the **geocentric** latitude.

Finding Where You Are





A surface point O at Lon/Lat (λ, L) is at some height h above the reference geoid

Geoid described by a_e and e_e where: $e_e^2 = 2f - f^2$

$$\left[\mathbf{r}_{O/G}\right]_{\mathcal{G}'} = \begin{bmatrix} x\cos\theta_{\mathrm{LST}} \\ x\sin\theta_{\mathrm{LST}} \\ z \end{bmatrix}_{\mathcal{G}'}$$

$$x = \left(\frac{a_e}{\sqrt{1 - e_e^2 \sin^2 L}} + h\right)\cos L$$

$$z = \left(\frac{a_e(1 - e_e^2)}{\sqrt{1 - e_e^2 \sin^2 L}} + h\right)\sin L$$

Space and Time

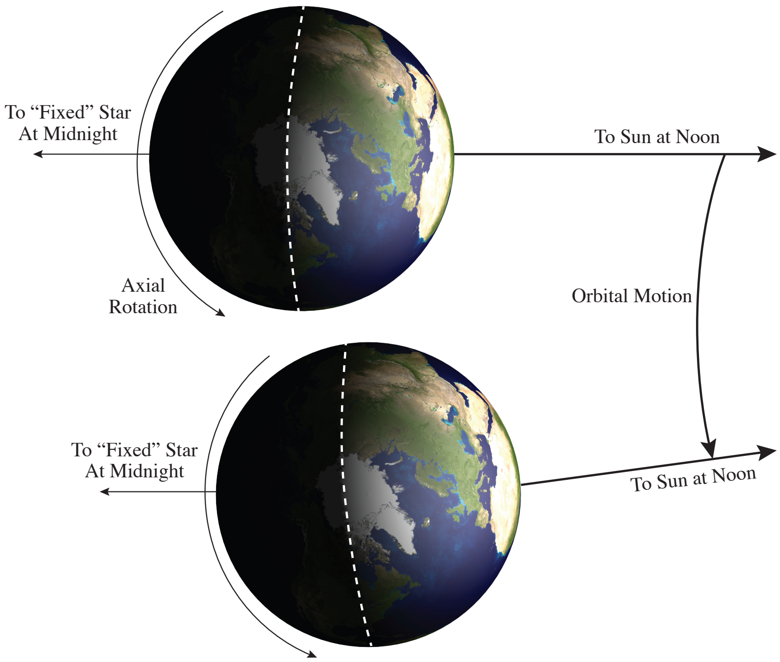
The expressions in the previous slide use the **geoditic** latitude to find the components of the position vector of the surface point with respect to the **center** of the Earth. As we want the $\hat{\mathbf{U}}$ to represent the vertical from this surface point, we similarly use L (and **not** L') when finding the transformation between the $\mathcal{G}'$ and $\mathcal{T}$ frames. The transformation from the Earth-Centered inertial frame to the Topocentric-Horizon frame is effectively the same as our usual spherical frame definition: a θ_{LST} rotation about the $\hat{\mathbf{e}}_3'$ direction, followed by a $\pi/2 - L$ rotation about the $\hat{\mathbf{E}}$ direction. Remember that the final frame is defined as $\mathcal{T} = (O, \hat{\mathbf{S}}, \hat{\mathbf{E}}, \hat{\mathbf{U}})$, so that $\hat{\mathbf{E}}$ is the second unit direction. In cases where you wish to use a purely spherical Earth model, L and L' are identical, and all the same expression still hold (but are greatly simplified, as e_e would be treated as zero in this case).

Note that we still don't know how to calculate θ_{LST} - the angle between the inertial ♈direction and the meridian of the observer. As this is a time-varying quantity, we first need to understand how we measure time before we return to this quantity.

SI Seconds

- The second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the Cesium-133 atom.
  –NIST (`http://physics.nist.gov/cuu/Units/second.html`)
- This definition refers to a cesium atom at rest at a temperature of 0 K.
  –BIPM
  (`http://www.bipm.org/en/publications/si-brochure/second.html`)

Solar vs Sidereal Time

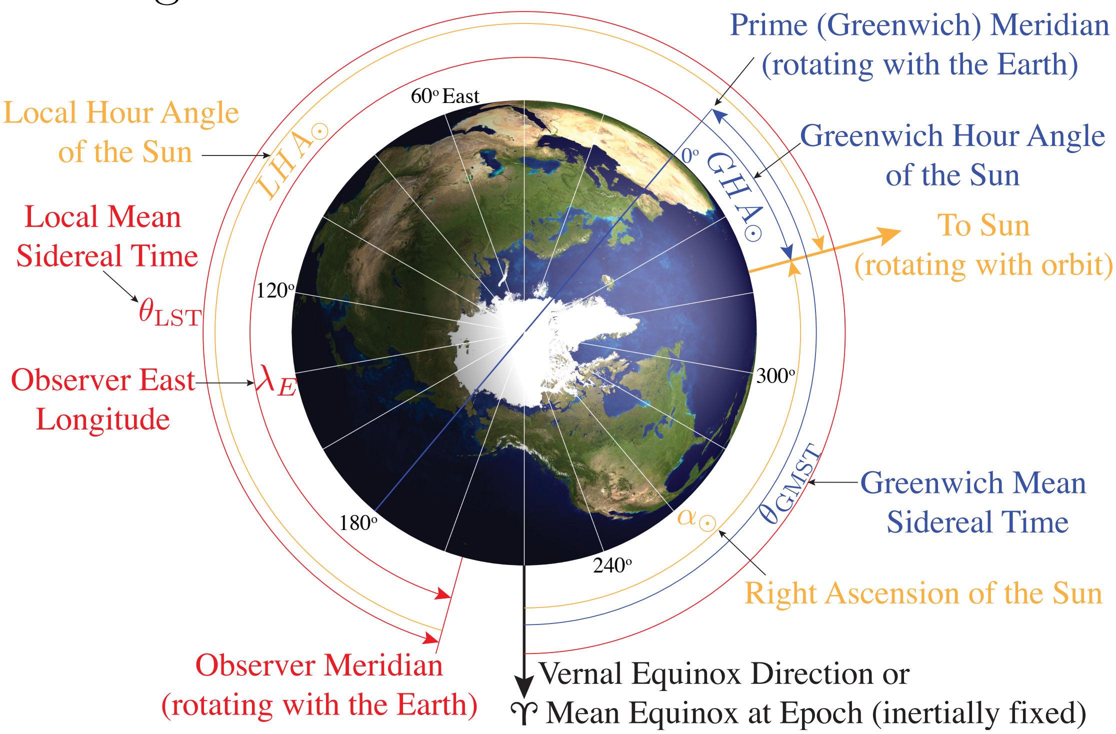


- Mean Solar Day (d): 24 SI hours = 86400 SI seconds
- Solar (Tropical) Year: 365.242190402 d
- Mean Sidereal Day: 23h56m4.09054s
- Sidereal Year: 365.256363004 d

Solar vs Sidereal Days

A solar day is length of time it takes for the sun to return to the same position in the sky. It varies by observer location and in time, but has a mean value set to 24 hours exactly. The sidereal day is the length of time that it takes for distant (fixed) stars to return to the same position in the sky. As the Earth has moved by approximately 1 degree in its orbit during the course of a day, the sidereal day is approximately 4 minutes ($1^\circ = {}^{24^{\mathrm{h}} \times 60^{\mathrm{min}}}/_{360^\circ}$). The sidereal day defines the Earth's 'true' rotation rate, such that the Earth rotates fully on its axis once per sidereal day, but the solar day is more useful in our day to day lives, and the one that is used in all civil applications.

Hour Angles and Sidereal Time



Hour Angle

is defined as the time from when an object was directly overhead. Negative hour angles imply that the object is approaching.

Time Measurements

- Local Hour Angle and Greenwich Hour Angle: $LHA = GHA + \lambda_E$
- Local Solar Time (local midnight is 0 hours): $LHA_\odot + 12^{\rm h}$
- Greenwich Solar Time: $\theta_{\rm GMST} - \alpha_\odot + 12^{\rm h}$ where $\theta_{\rm GMST}$ is the location of the Prime (Greenwich) Meridian with respect to the vernal equinox
- There are two different 'suns': The **apparent** sun (where the sun actually is) and the **fictitious mean sun** (a sun moving uniformly along the celestial equator). We can define apparent and mean solar times.
- The **mean** solar time at Greenwich is defined as **Universal Time**:

$$\text{UT0} = GHA_\odot + 12^{\rm h} = LHA_\odot + 12^{\rm h} - \lambda_E$$

- The motion of Earth's pole affects all these measurements. Correcting for this gives you UT1 ($|\text{UT1} - \text{UT0}| \approx 30$ ms)

Finding When You Are

$$\theta_{\text{LST}} = \theta_{g0} + \omega_e(t - t_0) + \lambda_E$$

Reference Value at Epoch

OR

MEAN SIDEREAL TIME, 2019

Greenwich mean sidereal time at 0^{h} UT

	h		h		h		h
Jan. 0	6·6250	Apr. 0	12·5389	July 0	18·5185	Oct. 0	0·5638
Feb. 0	8·6620	May 0	14·5102	Aug. 0	20·5555	Nov. 0	2·6008
Mar. 0	10·5019	June 0	16·5472	Sept. 0	22·5925	Dec. 0	4·5721

Greenwich mean sidereal time (GMST) on day d of month at hour t UT

$$= \text{GMST at } 0^{\text{h}} \text{ UT on day } 0 + 0^{\text{h}}\!.065\,71\,d + 1^{\text{h}}\!.002\,74\,t$$

$$\text{Local mean sidereal time} = \text{GMST} \begin{smallmatrix} + \text{ east} \\ - \text{ west} \end{smallmatrix} \text{ longitude}$$

https://aa.usno.navy.mil/publications/reports/ap19_for_web.pdf

$\omega_e = 7.292115 \times 10^{-5}$ rad/s (WGS84)

$\theta_{g0} = 100.4606184° + 36,000.77005361 T_{\text{UT1}} + 0.00038793 T_{\text{UT1}}^2 - 2.6 \times 10^{-8} T_{\text{UT1}}^3$
$T_{\text{UT1}} =$ number of Julian centuries from J2000.0

Julian Date

- Defined as days since January 1, 4713 BCE, 12^{h} UT
- 1 Julian year is *exactly* 365.25 days, 1 Julian century is 100 Julian years
- Define Modified Julian Date: MJD $\triangleq$ JD - 2,400,000.5

$$\begin{aligned}\text{JD} =& 367\text{Y} - \text{int}\left(\frac{7\left(\text{Y} + \text{int}\left(\frac{\text{M}+9}{12}\right)\right)}{4}\right) + \text{int}\left(\frac{275\text{M}}{9}\right) + \text{D} + 1721013.5 + \frac{\text{UT}}{24} \\ & - \frac{1}{2}\text{sgn}\left(100\text{Y} + \text{M} - 190002.5\right) + \frac{1}{2}\end{aligned}$$

$$\text{int}(x) = \begin{cases} \lfloor x \rfloor & x \geq 0 \\ \lceil x \rceil & x < 0 \end{cases} \qquad \text{sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

In MATLAB: see `juliandate` and `datetime`

More Time Systems

- **Coordinated Universal Time** (UTC) is an approximation to UT1 defined such that $|\text{UT1} - \text{UTC}| < 0.9$ seconds
- UTC is based on **International Atomic Time** (TAI), a weighted average of >400 atomic clocks in over 50 national laboratories worldwide
- Leap seconds are added to TAI to get UTC. In 2021, TAI is 37 seconds ahead of UTC, with the last leap second added on 12/31/2016 23:59:60 UTC
- GPS time is UTC as of 1/16/1980. As of 2021, GPS - UTC = 18 seconds. TAI - GPS will equal 19 seconds forever.

The Two-Body Problem

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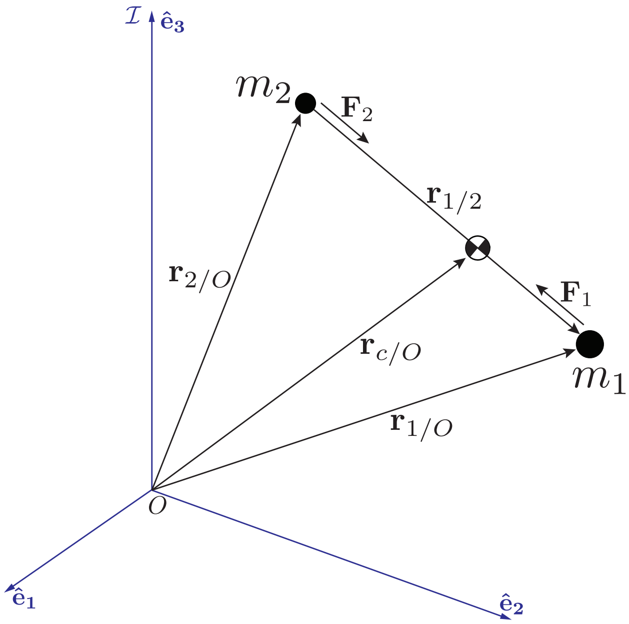
MAE 4060/5065, Fall 2021

The Two-Body Problem

The two-body problem (two point masses interacting via gravity, with no other forces present) is the fundamental building block of celestial mechanics. In fact, the two-body problem is the only orbital mechanics problem with an exact solution, allowing you to express the positions of both bodies in the past, present, and future via a single analytical expression. Although in practice you are unlikely to ever deal with an exact two-body system, many complex systems (including the solar system) behave like collections of two-body orbits that gradually change over time, making two-body concepts broadly applicable to a variety of other cases.

Newton's Law of Gravity and the Two-Body Problem



Gravitational Constant

$$\mathbf{F}_1 = -\mathbf{F}_2 = -\frac{G\, m_1 m_2}{\|\mathbf{r}_{1/2}\|^3}\mathbf{r}_{1/2}$$

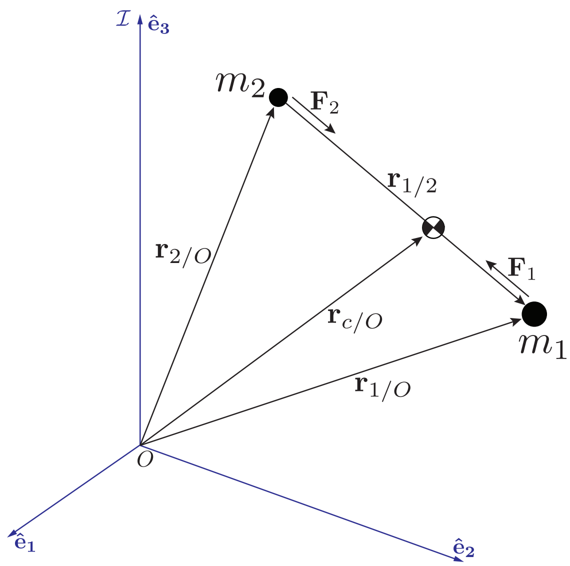
Orbital Radius: $\mathbf{r} \equiv \mathbf{r}_{1/2}$ (or $\mathbf{r}_{2/1}$)
Gravitational Parameter: $\mu \triangleq G(m_1 + m_2)$

$$\boxed{\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = 0}$$

The Two-Body Problem is a Central Force Problem, so:

1. $\mathbf{M}_{P/O} = \mathbf{r}_{P/O} \times \mathbf{F}_P = 0$ where $\mathbf{r}_{P/O} \neq 0$
2. The resultant force acts in the direction of $\mathbf{r}_{P/O}$
3. Angular momentum is conserved

Specific Angular Momentum



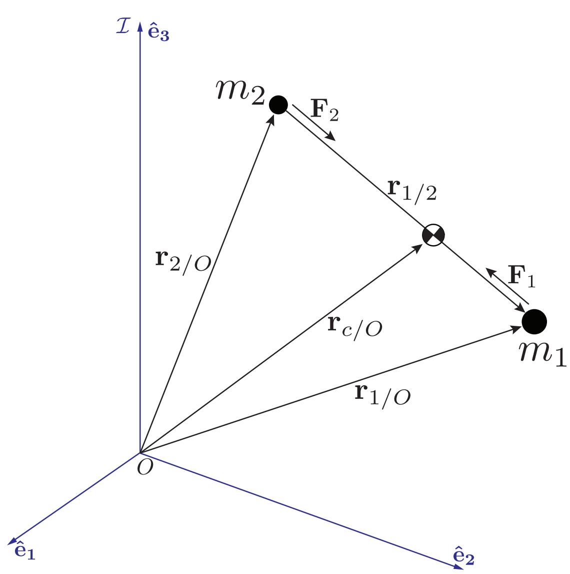
Specific Angular Momentum: $\mathbf{h} \triangleq \mathbf{r} \times \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r}$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = 0 \Rightarrow \frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} \times \mathbf{h} = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{\|\mathbf{r}\|}\mathbf{r}\right)$$

$$\Longrightarrow \quad \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{\|\mathbf{r}\|} + \mathbf{e}\right)$$

Constant of Integration
The **eccentricity vector**
Also: the **Laplace–Runge–Lenz (LRL) vector**

The Two-Body Problem Solution



$$\underbrace{\mathbf{r} \cdot \left(\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} \times \mathbf{h}\right)}_{\left(\mathbf{r} \times \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r}\right) \cdot \mathbf{h} \equiv \|\mathbf{h}\|^2} = \underbrace{\mathbf{r} \cdot \mu\left(\frac{\mathbf{r}}{\|\mathbf{r}\|} + \mathbf{e}\right)}_{\mu\left(\|\mathbf{r}\| + \|\mathbf{r}\|\|\mathbf{e}\|\cos\nu\right)}$$

Angle Between $\mathbf{r}$ and $\mathbf{e}$
True Anomaly

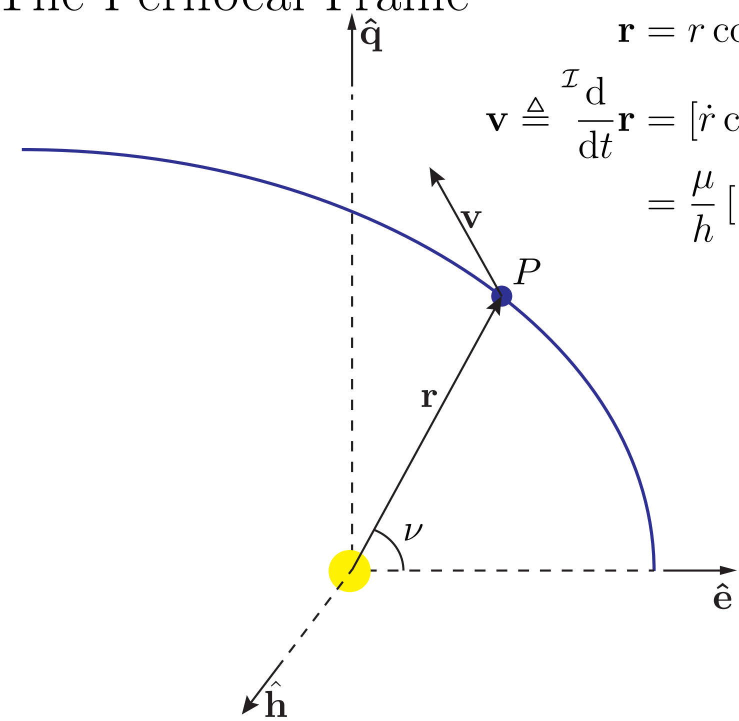
$$\boxed{r \triangleq \|\mathbf{r}\| = \frac{h^2/\mu}{1 + e\cos(\nu)}}$$

$$h \triangleq \|\mathbf{h}\| \qquad e \triangleq \|\mathbf{e}\| \qquad \mathbf{r} \cdot \mathbf{e} = re\cos\nu$$

In general, in the two-body problem:

1. The orbit position and velocity vectors are both orthogonal to the specific angular momentum vector of the orbit.
2. The orbit position, velocity, and eccentricity vectors all lie in an invariant plane in inertial space. We call this plane the **perifocal plane** and use it to define a useful reference frame.
3. A single, simultaneous measurement of both the orbit position and velocity vectors (assuming μ is known) fully defines the orbit.

The Perifocal Frame



$$\mathbf{r} = r\cos(\nu)\hat{\mathbf{e}} + r\sin(\nu)\hat{\mathbf{q}}$$

$$\begin{aligned}\mathbf{v} \triangleq \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} &= [\dot{r}\cos(\nu) - r\dot{\nu}\sin(\nu)]\,\hat{\mathbf{e}} + [\dot{r}\sin(\nu) + r\dot{\nu}\cos(\nu)]\,\hat{\mathbf{q}} \\ &= \frac{\mu}{h}\left[-\sin(\nu)\hat{\mathbf{e}} + (e + \cos(\nu))\hat{\mathbf{q}}\right]\end{aligned}$$

$$r = \|\mathbf{r}\| = \frac{h^2/\mu}{1 + e\cos(\nu)}$$

$$\left.\begin{aligned} e &= \|\mathbf{e}\| = \left\|\frac{\mathbf{v}\times\mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}\right\| \\ h &= \|\mathbf{h}\| = \|\mathbf{r}\times\mathbf{v}\| \end{aligned}\right\}\text{Constants}$$

$$\boxed{h = r^2\dot{\nu}}$$

$\hat{\mathbf{e}}, \hat{\mathbf{q}}, \mathbf{r}$, and $\mathbf{v}$ all lie within the perifocal plane.

Turning Points

- In general, the angle between $\mathbf{r}$ and $\mathbf{v}$ is arbitrary, but there exist special cases, called **turning points**, where: $\mathbf{r}_t \perp \mathbf{v}_t$

$$\mathbf{e} = \frac{\mathbf{v} \times \overbrace{\mathbf{h}}^{\mathbf{r} \times \mathbf{v}}}{\mu} - \frac{\mathbf{r}}{r} = \frac{1}{\mu}\left(\|\mathbf{v}\|^2 - \frac{\mu}{r}\right)\mathbf{r} - \underbrace{\left(\frac{\mathbf{r} \cdot \mathbf{v}}{\mu}\right)}_{=0 \text{ for } \mathbf{r} \perp \mathbf{v}}\mathbf{v}$$

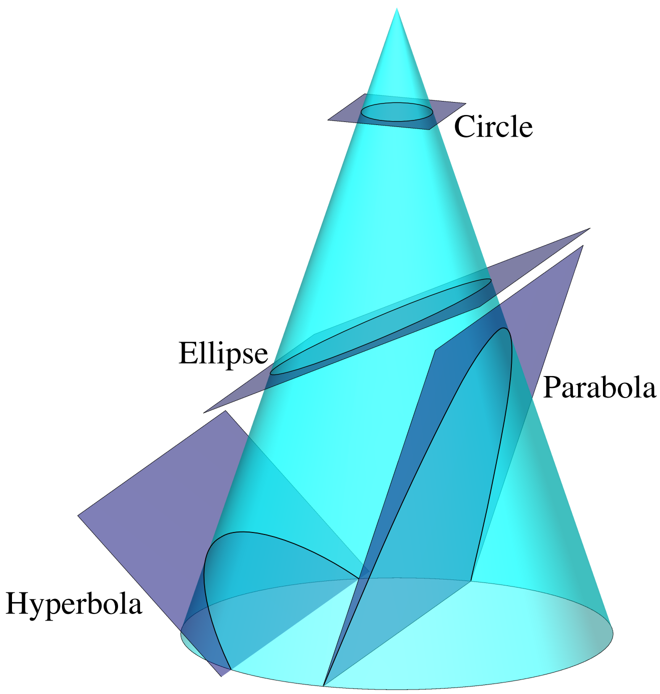
- So at turning points: $\mathbf{r}_t \parallel \mathbf{e}$
- If $\mathbf{r}_t \parallel \mathbf{e}$, $\nu_t = 0, \pi$. So $r_t = \dfrac{h^2/\mu}{1 + e\cos(\nu_t)} = \dfrac{h^2/\mu}{1 \pm e}$.
  Therefore, turning points occur at the minimum and maximum orbital radius.

Turning Point Nomenclature

- A turning point is also called an **apsis** (plural: **apsides**)
- The closest approach between bodies is known as **periapsis** (also pericenter or periapse)
- The furthest distance between bodies is known as **apoapsis** (also apocenter or apoapse)
- When orbiting specific bodies, we frequently replace apsis with a body-specific suffix:

Body	Periapsis	Apoapsis
Earth	Perigee	Apogee
Sun	Perihelion	Aphelion
Jupiter	Perijove	Apojove
Star	Periastron	Apoastron

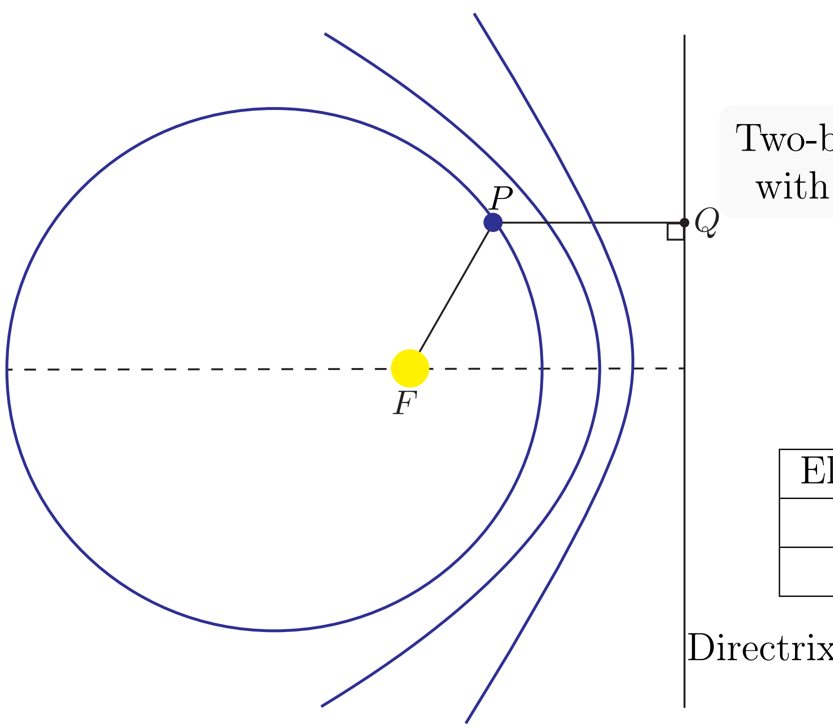
Kepler's First Law



Kepler's First Law

The orbit of a planet is an ellipse (conic section) with the Sun at a focus

Conic Sections

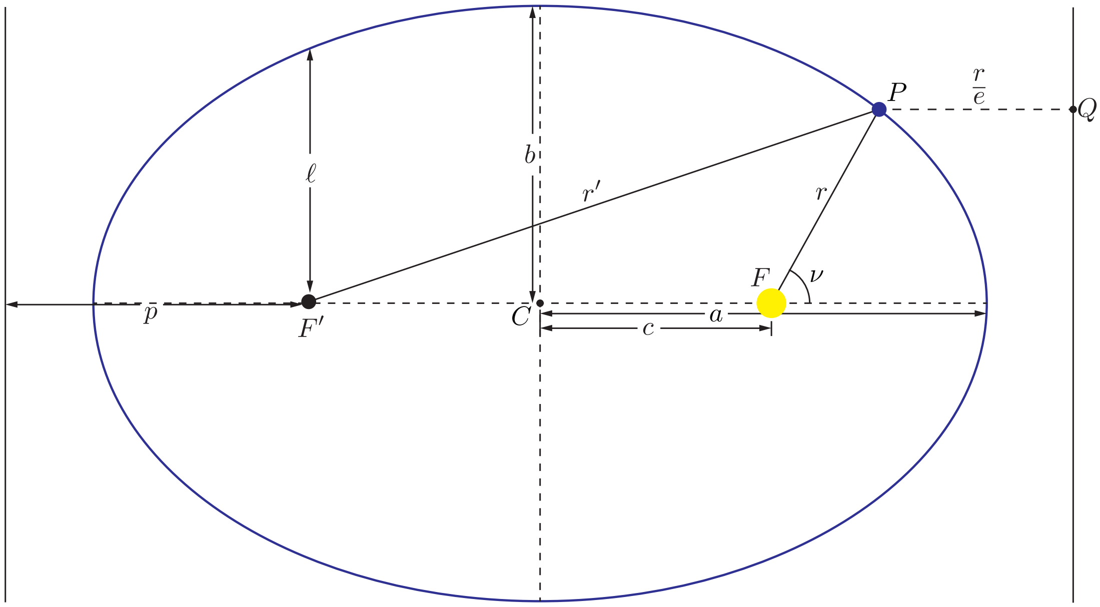


Two-body orbits are conic sections with the central body at a focus

$$\overline{FP} = e\overline{PQ}$$

Ellipse (Circle)	$0 < e < 1$
Parabola	$e = 1$
Hyperbola	$e > 1$

Elliptical Orbits

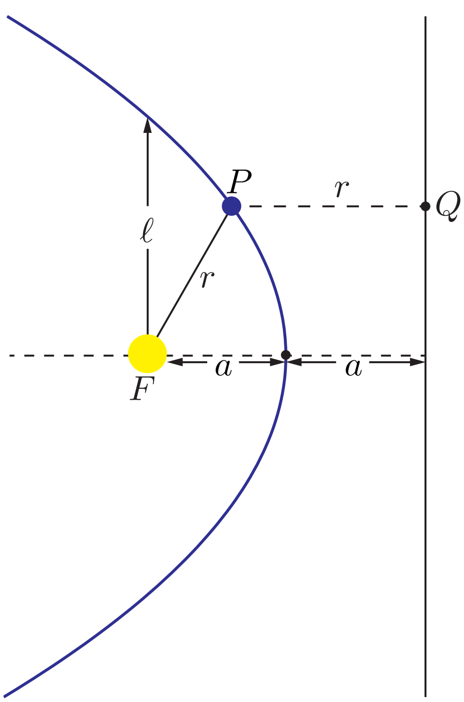


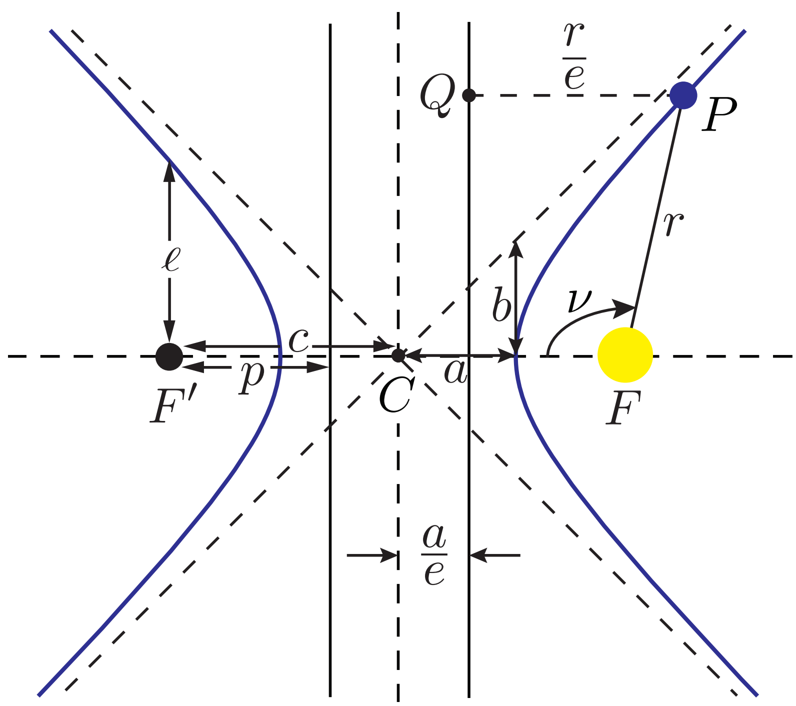
$$r' + r = 2a$$

$$r = \frac{a(1-e^2)}{1+e\cos(\nu)} = \frac{h^2/\mu}{1+e\cos(\nu)} = \frac{\ell}{1+e\cos(\nu)}$$

*This last equation applies for all conic sections.

Parabolic and Hyperbolic Orbits





Conic Section Parameters

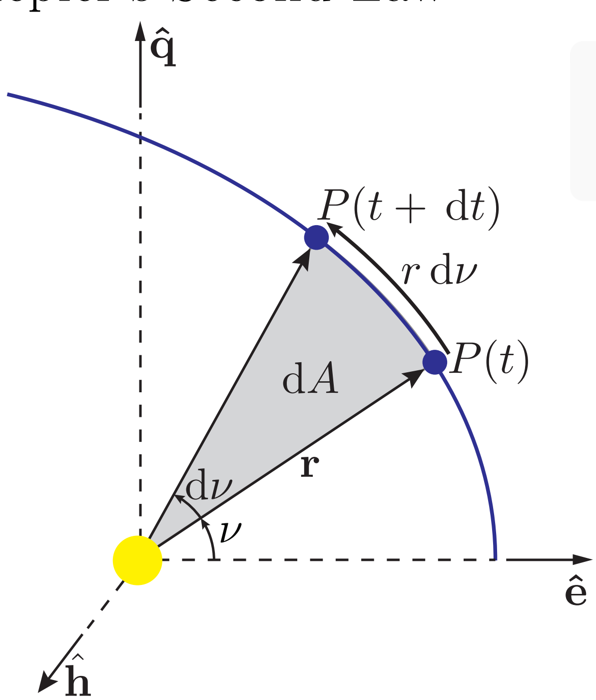
semi-parameter: $\ell = r(\nu = \pi/2) =$ height above focus
linear eccentricity: $c = ae =$ distance from center to focus
focal parameter: $p = \ell/e =$ distance from focus to vertex

NB: p and ℓ frequently have reversed definitions, depending on the text.

	Definition	e	c	ℓ	p
circle	$x^2 + y^2 = a^2$	0	0	a	∞
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{a^2 - b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2 - b^2}}$
parabola	$y^2 = 4ax$	1	∞	$2a$	$2a^*$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\sqrt{1 + \frac{b^2}{a^2}}$	$\sqrt{a^2 + b^2}$	$\frac{b^2}{\lvert a \rvert}$	$\frac{b^2}{\sqrt{a^2 + b^2}}$

$^*a$ is the focus to vertex distance for a parabola

Kepler's Second Law



A line segment joining a planet and the Sun sweeps out equal areas in equal time

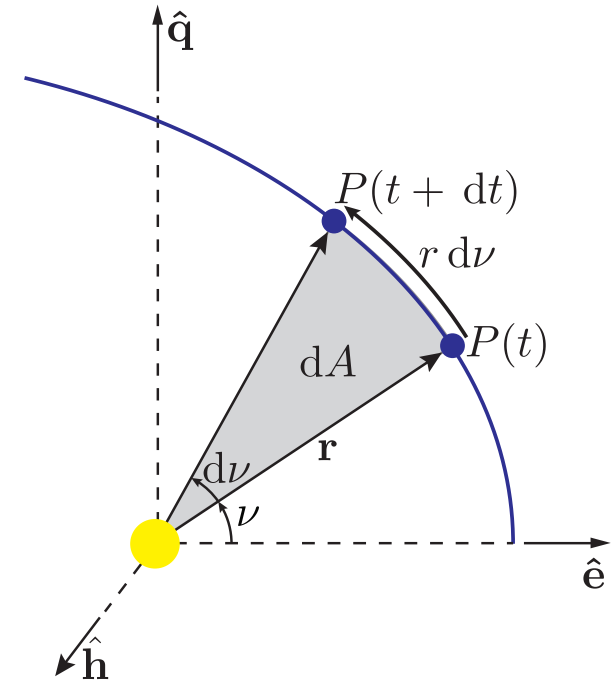
Recall: $r\dot{\nu} = \frac{h}{r}$

$$\mathrm{d}A = \frac{1}{2} r (r\,\mathrm{d}\nu) = \frac{1}{2} \underbrace{r^2 \frac{\mathrm{d}\nu}{\mathrm{d}t}}_{h} \mathrm{d}t$$

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{h}{2}}$$

Corollary: a body is traveling the fastest at periapsis and slowest at apoapsis

Kepler's Third Law



The square of the orbital period (T_P) is proportional to the cube of the semi-major axis

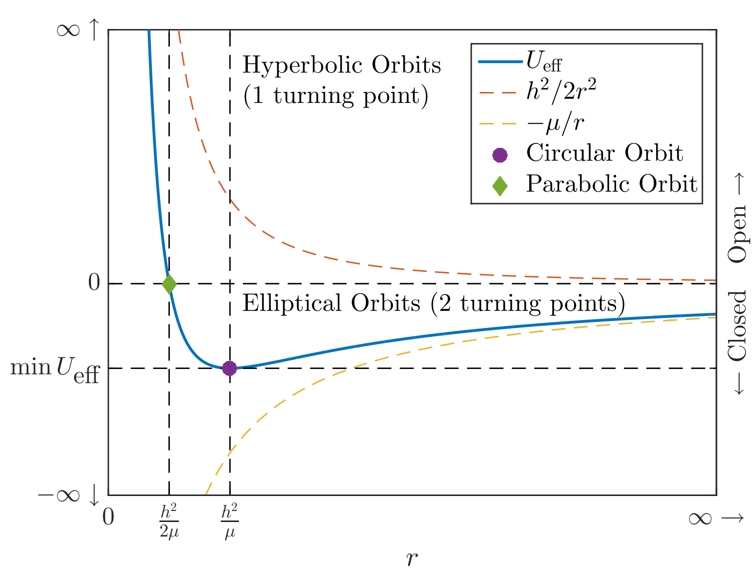
$$\int_0^{T_P} \frac{\mathrm{d}A}{\mathrm{d}t}\,\mathrm{d}t = \int_0^{T_P} \frac{h}{2}\,\mathrm{d}t \quad \Longrightarrow \quad A = \frac{h}{2}T_P$$

For an ellipse: $A = \pi ab = \pi a^{\frac{3}{2}}\sqrt{\ell} = \dfrac{h}{2}T_P$

$$\boxed{T_P = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}}$$

Specific Energy and Effective Potential

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = \text{constant}$$



$$\boxed{\mathcal{E} = -\frac{\mu}{2a}}$$

The Vis-Viva Equation

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\mathcal{E} = \frac{\dot{r}^2}{2} + \underbrace{U(r) + \frac{h^2}{2r^2}}_{\triangleq\, U_{\mathrm{eff}}}$$

Special Orbits

- On a circular orbit, $r = a$ everywhere, so: $v_c = \sqrt{\dfrac{\mu}{r}}$

- On a parabolic orbit, $a = \infty$, so $v_p = \sqrt{\dfrac{2\mu}{r}} = \sqrt{2}v_c$

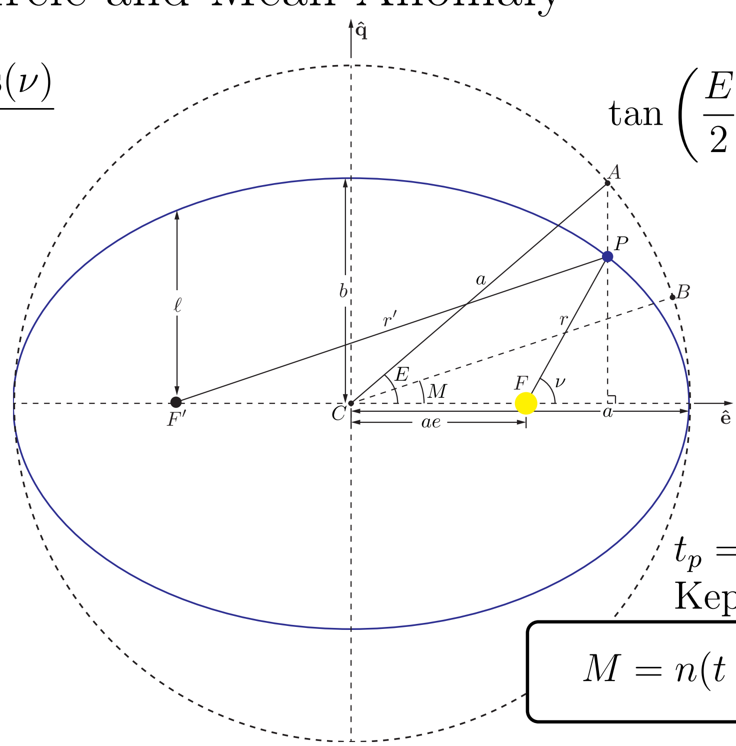
- A parabolic orbit is where an orbit switches from closed to open, and so v_p is the **Escape Velocity**

The Auxiliary Circle and Mean Anomaly

Eccentric Anomaly:

$$\cos(E) = \frac{ae + r\cos(\nu)}{a}$$

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\nu}{2}\right)$$



Mean Motion:

$$n \triangleq \frac{2\pi}{T_P} = \sqrt{\frac{\mu}{a^3}}$$

t_p = time of periapsis

Kepler's Time Equation:

$$\boxed{M = n(t - t_p) = E - e\sin(E)}$$

Solving Kepler's Time Equation

Kepler's time equation is still a transcendental one, and so cannot be analytically inverted to solve for eccentric anomaly (and therefore true anomaly) as a function of mean anomaly (time). The benefit of this equation is that it is more easily numerically solvable than attempting to find true anomaly directly from time. Much of the history of astrodynamics has been devoted to coming up with new and better approaches for inverting Kepler's time equation. While literally dozens of distinct methods exist, here we will focus on just one: Newton-Raphson iteration. This approach has the benefit of being easy to implement in almost any computer language, is relatively computationally efficient, and, with the proper choice of initial conditions, is typically guaranteed to converge to any desired precision within a finite number of iterations.

Newton-Raphson Iteration

- Given: $x : f(x) = 0, x \in \mathbb{R}; \quad f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$
- Iterate: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- Until converged (answer stops changing to your desired precision)

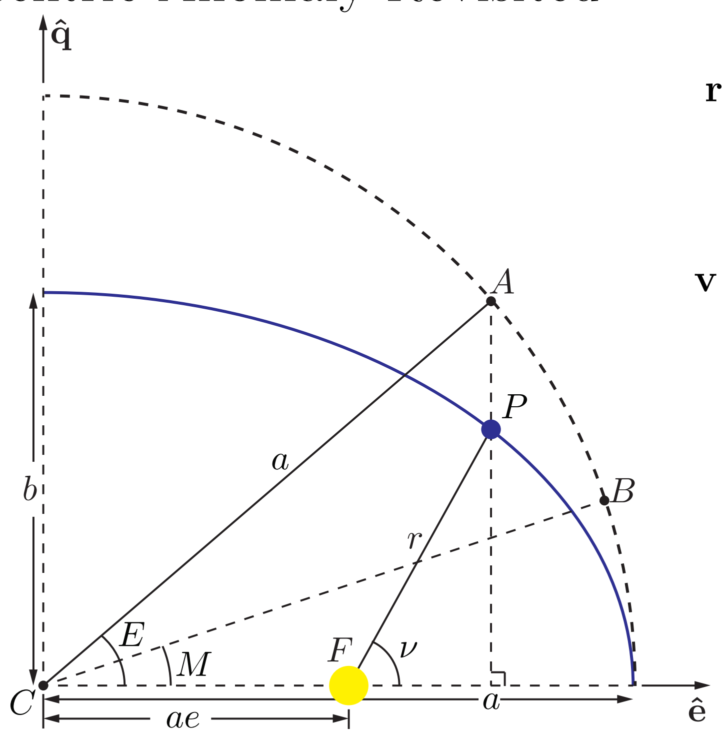
Newton-Raphson Iteration for Kepler's Time Equation

$$M - (E - e\sin(E)) = 0$$

$$E_{n+1} = E_n - \frac{M - E_n + e\sin(E_n)}{e\cos(E_n) - 1}$$

$$E_0 = \begin{cases} \dfrac{M}{1-e} & \dfrac{M}{1-e} < \sqrt{\dfrac{6(1-e)}{e}} \\[2ex] \left(\dfrac{6M}{e}\right)^{\frac{1}{3}} & \text{else} \end{cases}$$

Eccentric Anomaly Revisited



$$\begin{aligned}\mathbf{r} &= r\cos(\nu)\hat{\mathbf{e}} + r\sin(\nu)\hat{\mathbf{q}} \\ &= a\left(\cos(E) - e\right)\hat{\mathbf{e}} + b\sin(E)\hat{\mathbf{q}}\end{aligned}$$

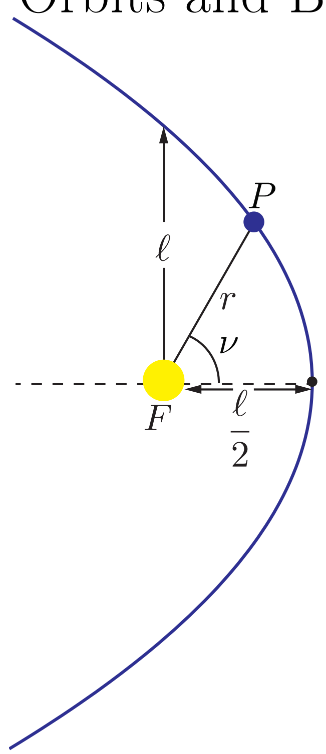
$$\begin{aligned}\mathbf{v} &= -a\dot{E}\sin(E)\hat{\mathbf{e}} + b\dot{E}\cos(E)\hat{\mathbf{q}} \\ &= \frac{an}{r}\left(-a\sin(E)\hat{\mathbf{e}} + b\cos(E)\hat{\mathbf{q}}\right)\end{aligned}$$

$$\boxed{\dot{E} = \frac{n}{1 - e\cos(E)}}$$

Parabolic and Hyperbolic Time Equations

We can define a parabolic and hyperbolic anomaly (B and H) that are effectively equivalent to the eccentric anomaly in terms of their function—proxies for relating true anomaly to time. In the case of parabolic orbits, we can actually solve the time equation analytically, as it is a third order polynomial (and thus has a known, exact solution). On the other hand, hyperbolic anomaly is defined to behave in the exact same way as eccentric anomaly, only with hyperbolic instead of trigonometric functions in the time equation. This is why hyperbolic anomaly is defined in terms of an area rather than an angle. The exact same Newton-Raphson procedure can be applied for solving the hyperbolic time equation.

Parabolic Orbits and Barker's Equation



$$B \triangleq \tan\left(\frac{\nu}{2}\right)$$

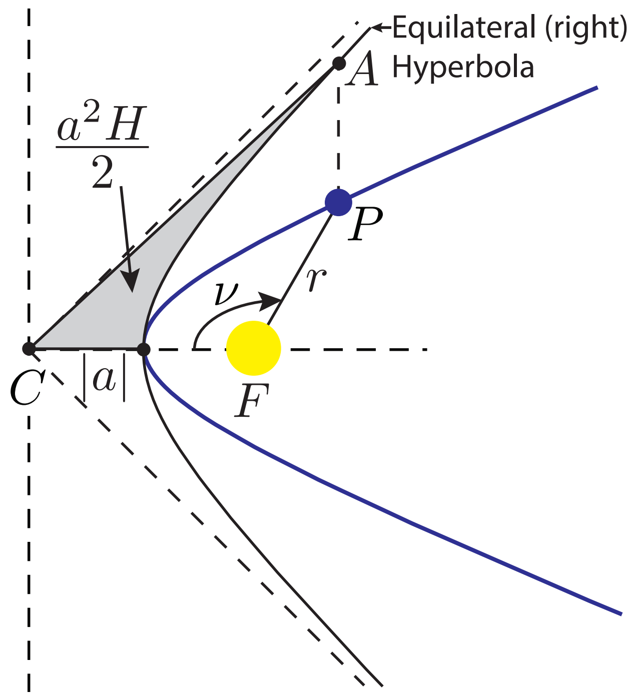
$$r = \frac{\ell}{2}\left(1 + B^2\right)$$

$$\nu = \sin^{-1}\left(\frac{\ell B}{r}\right)$$

$$n_p \triangleq 2\sqrt{\frac{\mu}{\ell^3}}$$

$$\boxed{n_p(t - t_p) = B + \frac{B^3}{3}}$$

Hyperbolic Orbits



$$\sinh(H) = -\frac{r\sin(\nu)}{a\sqrt{e^2-1}}$$

$$\cosh(H) = \frac{ae + r\cos(\nu)}{a}$$

$$r = a\left(1 - e\cosh(H)\right)$$

$$\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right)$$

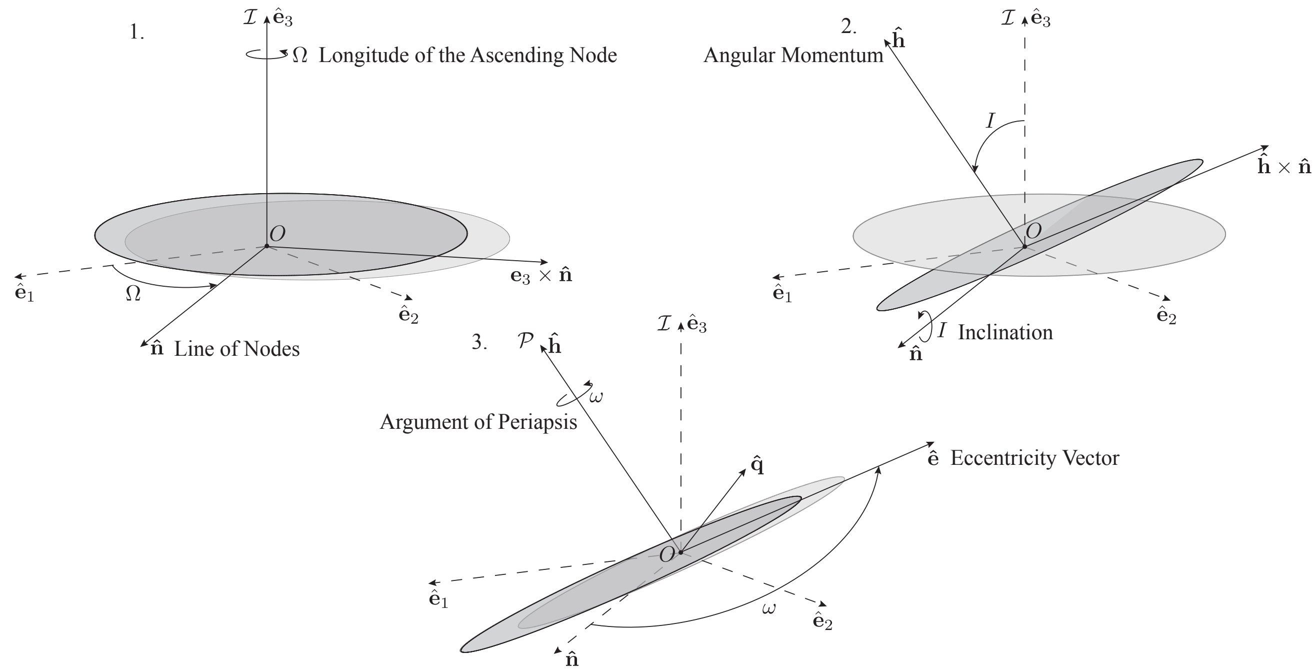
$$n_h \triangleq \sqrt{-\frac{\mu}{a^3}}$$

$$\boxed{n_h\left(t - t_p\right) = e\sinh(H) - H}$$

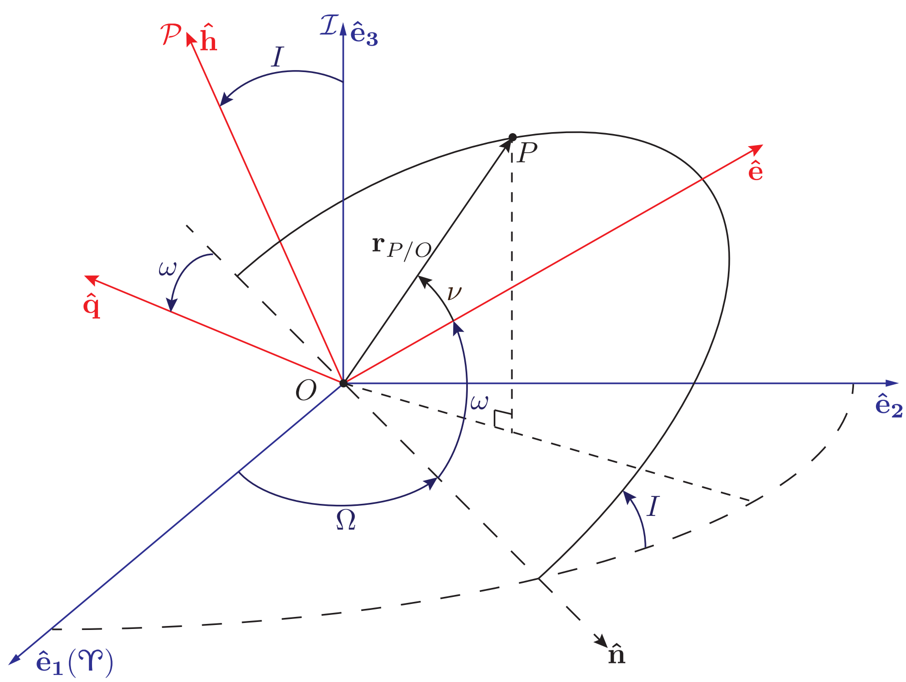
Orbits in 3D

Recall that the orbital radius and velocity vectors ($\mathbf{r}$ and $\mathbf{v}$) fully define an orbit, and contain six scalar values. In our geometric description of an orbit, however, we have only used three scalar values - the semi-major axis (a), eccentricity (e), and some measure of time or anomaly (t, ν, or E, B, H). This is because our geometric description has been entirely in the perifocal frame. To make it generally applicable, we must also add the orientation of a specific perifocal frame within an arbitrary inertial frame (note that the perifocal frame *is* inertial, as it is based on an invariant plane in space, but for an arbitrary orbit, the perifocal frame typically won't correspond to a frame in which we define standard measurements). As we will learn later in the course, the orientation of one reference frame with respect to another can be encoded via three angles (called Euler angles), representing rotations about a particular axis of a series of reference frames. By convention, the relationship between the perifocal and inertial reference frames is given by a 3-1-3 rotation set, detailed below.

Inertial → Perifocal: 3-1-3 (Ω, I, ω) Body Rotation



Orbits in 3D



Orbits in 3D (Math Version)

$$
{}^{\mathcal{P}}C^{\mathcal{I}} = \begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(I) & \sin(I) \\ 0 & -\sin(I) & \cos(I) \end{bmatrix} \begin{bmatrix} \cos(\Omega) & \sin(\Omega) & 0 \\ -\sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} =
$$

$$
\begin{bmatrix} -\sin(\Omega)\sin(\omega)\cos(I) + \cos(\Omega)\cos(\omega) & \sin(\Omega)\cos(\omega) + \sin(\omega)\cos(I)\cos(\Omega) & \sin(I)\sin(\omega) \\ -\sin(\Omega)\cos(I)\cos(\omega) - \sin(\omega)\cos(\Omega) & -\sin(\Omega)\sin(\omega) + \cos(I)\cos(\Omega)\cos(\omega) & \sin(I)\cos(\omega) \\ \sin(I)\sin(\Omega) & -\sin(I)\cos(\Omega) & \cos(I) \end{bmatrix}
$$

$$
\boxed{\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}} = {}^{\mathcal{I}}C^{\mathcal{P}} \begin{bmatrix} r\cos\nu \\ r\sin\nu \\ 0 \end{bmatrix}_{\mathcal{P}} = r \begin{bmatrix} \cos(\Omega)\cos(\nu+\omega) - \sin(\Omega)\sin(\nu+\omega)\cos(I) \\ \sin(\Omega)\cos(\nu+\omega) + \sin(\nu+\omega)\cos(I)\cos(\Omega) \\ \sin(I)\sin(\nu+\omega) \end{bmatrix}}
$$

Special Cases

- $I = 0$, Longitude of Periapsis:

$$\pi \equiv \varpi \triangleq \omega + \Omega$$

- $e = 0$, Argument of Lattitude:

$$u \equiv \theta \triangleq \nu + \omega$$

- $e = I = 0$, True Longitude:

$$l \triangleq \varpi + \nu = \Omega + \omega + \nu$$

NB: All of these can be defined in the general case as well, but only the argument of latitude is always a true angle (since ν and ω always lie in the perifocal plane) whereas π and l are compound angles measured across two planes in cases where $I \neq 0$.

Numerical Encoding and Canonical Values

It is important to remember that computers (for the most part) don't actually encode numbers to infinite precision. The majority of the time, when dealing with non-integer values on a computer, you are utilizing floating point values, which encode numbers to a fixed precision. In most situations, this will not lead to any issues at all, but can lead to highly surprising results when operating on pairs of values of very different magnitudes (e.g., adding very small numbers to very large numbers). At the same time, certain values in astrodynamics are notoriously difficult to measure to great precision. These two considerations lead us to briefly consider how exactly computers store numerical values, and to introduce the concept of canonical units.

IEEE 754: Standard for Floating-Point Arithmetic

- A floating point number is represented by two values:
  1. s: The significand (mantissa, coefficient)—fixed length (p) digit string in base b
  2. e: The exponent—a signed integer

$$f \approx \frac{s}{b^{p-1}} b^e$$

- The IEEE 754 double precision (binary64, default in MATLAB) data type has: $b = 2$, $p = 52$, and 11 exponent bits ($e \in [-1022, 1023]$).
- See MATLAB `eps` command.

Canonical Units

- In astrodynamics, we constantly deal with values of hugely differing scales: $G = 6.67430(\pm 0.00015) \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$,
  ${}^{m_\odot}/\left(m_\oplus + m_☾\right) = 328900.56(\pm 0.02)$,
  $Gm_\odot = 1.32712440018 \times 10^{20}(\pm 8 \times 10^9)\,\mathrm{m^3\,s^{-2}}$.
  See: https://ssd.jpl.nasa.gov/?constants for lots more.
- Define canonical distance, time and mass units (DU, TU, MU) such that $\mu = G(m_1 + m_2) = 1\ \mathrm{DU^3\ TU^{-2}}$ and $G = 1\ \mathrm{DU^3\ TU^{-2}\ MU^{-1}}$
- If 1 MU $= m_1 + m_2$ then:

$$\mathrm{TU} = \sqrt{\frac{\mathrm{DU}^3}{\mathrm{MU}}}$$

- For planetary orbits, we typically take DU to be the planetary radius. For Earth satellites, typically DU $= R_\oplus$
- For heliocentric orbits, typically use astronomical units:
  1 DU $=$ 1 AU $=$ 149597870700 m

Orbital Perturbations

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MAE 4060/5065, Fall 2021

Orbital Perturbations

A two-body orbit can be thought of as a static structure in space, but in practice real orbits evolve in time due to gravitational and non-gravitational effects not captured in the two-body model. In many cases, we can think of these additional effects as perturbations—forces that are small compared with the primary gravitational attraction between the two bodies, that lead to very gradual changes in the Keplerian orbital elements. We introduce the concept of osculating orbital elements—orbital elements representing the best fit of a two-body orbit to the true orbit at a given point in time, which then evolve in response to perturbations.

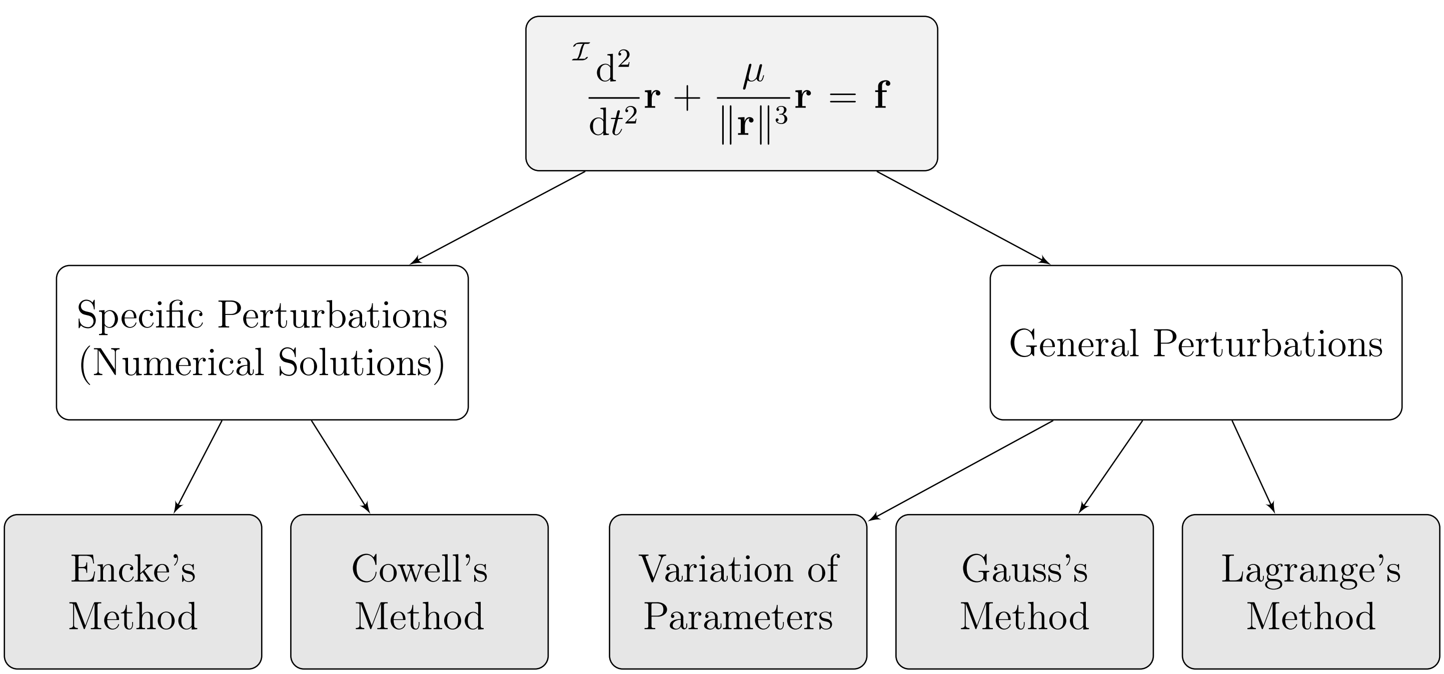
Osculating Orbital Elements

- Recall: $\mathbf{r}(t), \mathbf{v}(t) \quad \iff \quad \underbrace{a, e, I, \Omega, \omega, t_p}_{\triangleq \mathbf{c}}, t$
- For a two-body orbit, $\mathbf{c}$ is constant and we can always define the orbit as: $\mathbf{r}(\mathbf{c}, t), \mathbf{v}(\mathbf{c}, t)$
- If $\mathbf{c}$ varies in time then its elements are called **osculating** orbital elements
- To describe a time-varying orbit, we modify our two-body differential equation:

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \underbrace{\mathbf{f}}_{\text{Perturbing Specific Force}}$$

- Our goal is to find $\dot{\mathbf{c}} = f(\mathbf{c}, \mathbf{f})$

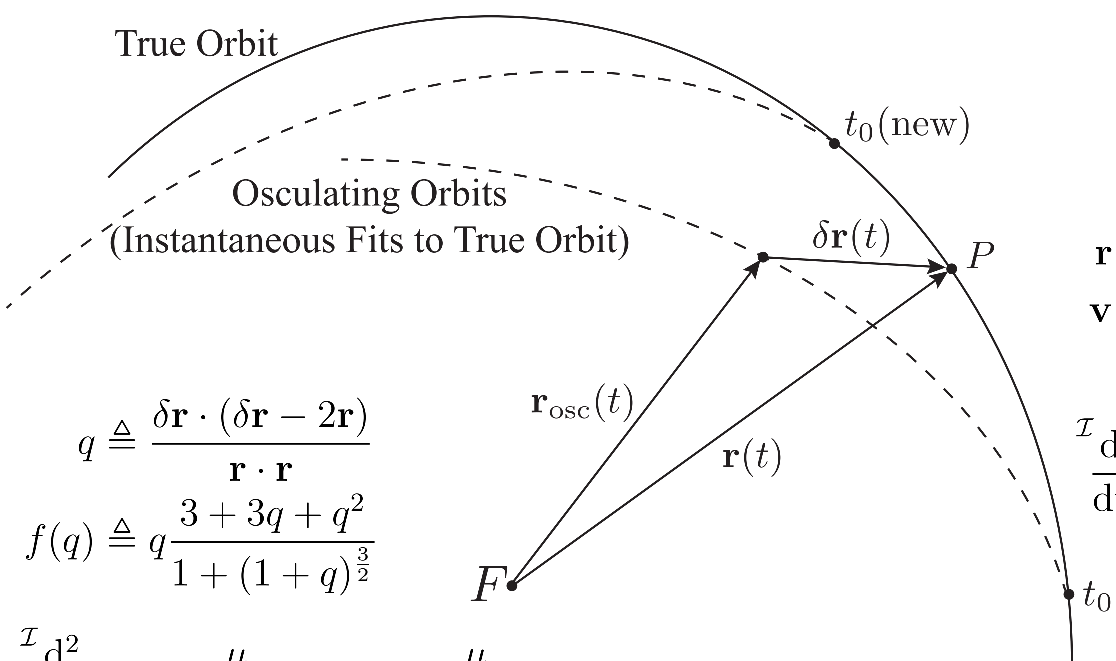
Perturbations Roadmap



Perturbation Techniques

Encke's method utilizes numerical integration of the deviation from a reference orbit due to any perturbations. Cowell's method specifically models the perturbations due to N mutually non-interacting bodies. Variation of parameters and Gauss's method apply for all forces, whereas Lagrange's method is usually used for conservative forces as the perturbation is expressed as a scalar potential.

Encke's Method



$$\mathbf{r}(t_0) \equiv \mathbf{r}_{\text{osc}}(t_0)$$
$$\mathbf{v}(t_0) \equiv \mathbf{v}_{\text{osc}}(t_0)$$

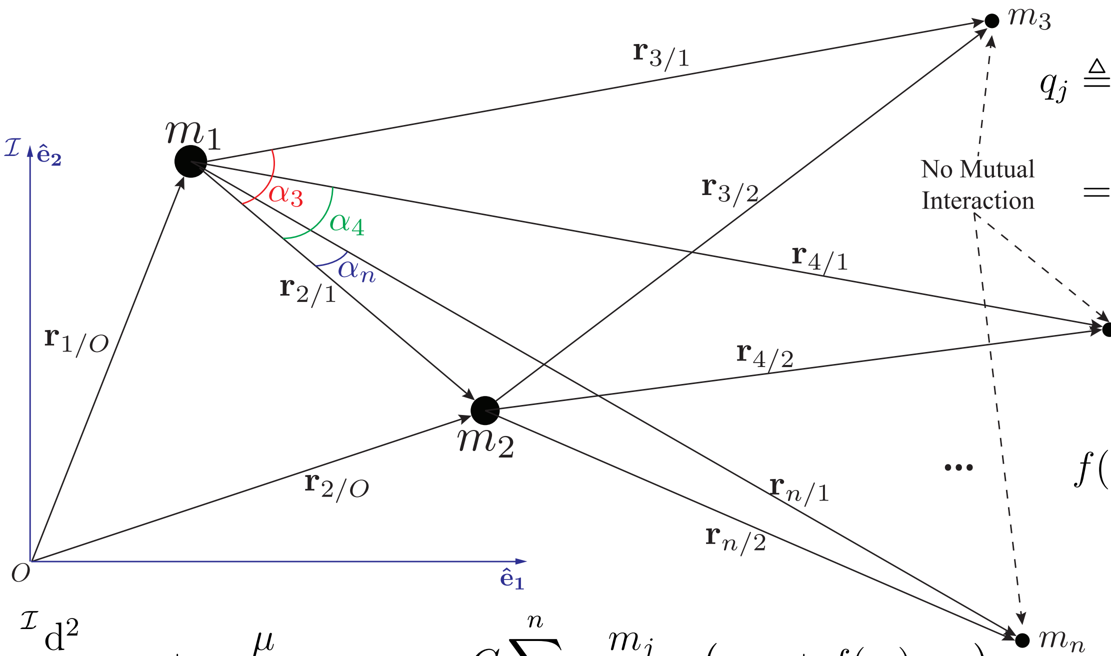
$$\mathbf{r}(t) = \mathbf{r}_{\text{osc}}(t) + \delta\mathbf{r}(t)$$
$$\mathbf{v}(t) = \mathbf{v}_{\text{osc}}(t) + \delta\mathbf{v}(t)$$

$$q \triangleq \frac{\delta\mathbf{r}\cdot(\delta\mathbf{r} - 2\mathbf{r})}{\mathbf{r}\cdot\mathbf{r}}$$
$$f(q) \triangleq q\frac{3 + 3q + q^2}{1 + (1+q)^{\frac{3}{2}}}$$

$$^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{\text{osc}} + \frac{\mu}{\|\mathbf{r}_{\text{osc}}\|^3}\mathbf{r} = 0$$
$$^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = \mathbf{f}$$

$$^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta\mathbf{r} + \frac{\mu}{\|\mathbf{r}_{\text{osc}}\|^3}\delta\mathbf{r} = -\frac{\mu}{\|\mathbf{r}_{\text{osc}}\|^3}f(q)\mathbf{r} + \mathbf{f}$$

Cowell's Method



$$q_j \triangleq \frac{\mathbf{r}_{2/1} \cdot \left(\mathbf{r}_{2/1} - 2\mathbf{r}_{j/1}\right)}{\mathbf{r}_{j/1} \cdot \mathbf{r}_{j/1}} = \frac{r_{2/1}}{r_{j/1}} \left(\frac{r_{2/1}}{r_{j/1}} - 2\cos\alpha_j \right)$$

$$f(q) \triangleq q \left(\frac{3 + 3q + q^2}{1 + (1+q)^{\frac{3}{2}}} \right)$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/1} + \frac{\mu}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = -G \sum_{j=3}^{n} \frac{m_j}{\|\mathbf{r}_{2/j}\|^3} \left(\mathbf{r}_{2/1} + f(q_j)\mathbf{r}_{j/1}\right)$$

Variation of Conserved Quantities

- In the unperturbed two-body problem, we have two conserved vector quantities:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} \quad \text{and} \quad \mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

- If we add a specific perturbing force to our two-body equations of motion, how do these change?
- Given: $\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} \equiv \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \underbrace{-\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \mathbf{f}}$ we can look at the variation of $\mathbf{h}$:

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r} \times \mathbf{v}\right) = \cancel{\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} \times \mathbf{v}} + \mathbf{r} \times \overbrace{\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{v}}$$

Variation of Specific Angular Momentum and the Eccentricity Vector

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

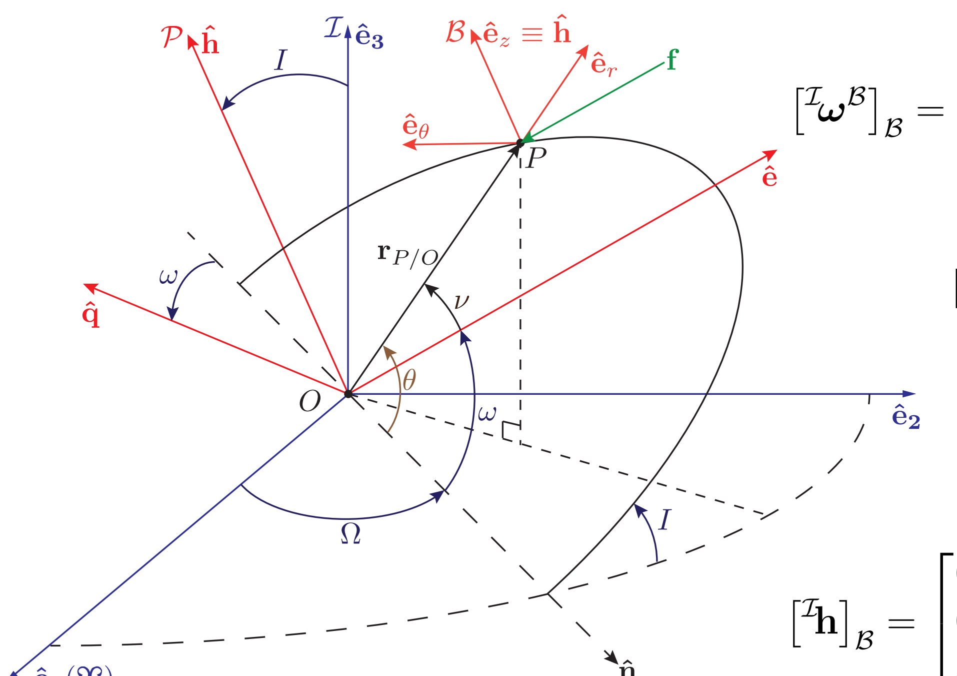
$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{v} = -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \mathbf{f}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} = \mathbf{r} \times \mathbf{f}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{e} = \frac{1}{\mu}\left(\mathbf{f} \times \mathbf{h} + \mathbf{v} \times \mathbf{r} \times \mathbf{f}\right)$$

Variation of Parameters Reference Frames



$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\Omega}\hat{\mathbf{e}}_3 + \dot{I}\hat{\mathbf{n}} + \dot{\theta}\hat{\mathbf{h}}$$

$$\left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} = \begin{bmatrix} \dot{I}\cos(\theta) + \dot{\Omega}\sin(I)\sin(\theta) \\ -\dot{I}\sin(\theta) + \dot{\Omega}\sin(I)\cos(\theta) \\ \dot{\Omega}\cos(I) + \dot{\theta} \end{bmatrix}_{\mathcal{B}}$$

$$[\mathbf{r}]_{\mathcal{B}} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \qquad \left[{}^{\mathcal{I}}\mathbf{v}\right]_{\mathcal{B}} = \begin{bmatrix} v_r \\ v_\theta \\ 0 \end{bmatrix}_{\mathcal{B}}$$

$$[\mathbf{f}]_{\mathcal{B}} = \begin{bmatrix} f_r \\ f_\theta \\ f_h \end{bmatrix}_{\mathcal{B}}$$

$$\left[{}^{\mathcal{I}}\mathbf{h}\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}_{\mathcal{B}} \qquad [\mathbf{e}]_{\mathcal{B}} = \begin{bmatrix} e\cos\nu \\ -e\sin\nu \\ 0 \end{bmatrix}_{\mathcal{B}}$$

Gauss's Perturbation Equations (the setup)

$$\frac{{}^{\mathcal{I}}\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \frac{{}^{\mathcal{B}}\mathrm{d}\mathbf{h}}{\mathrm{d}t} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{h} = \mathbf{r} \times \mathbf{f} \quad \Rightarrow$$

$$\begin{bmatrix} 0 \\ 0 \\ \dot{h} \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} h\left(-\dot{I}\sin(\theta) + \dot{\Omega}\sin(I)\cos(\theta)\right) \\ -h\left(\dot{I}\cos(\theta) + \dot{\Omega}\sin(I)\sin(\theta)\right) \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 \\ -f_h r \\ f_\theta r \end{bmatrix}_{\mathcal{B}}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \frac{{}^{\mathcal{B}}\mathrm{d}\mathbf{e}}{\mathrm{d}t} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{e} = \frac{1}{\mu}\left(\mathbf{f} \times \mathbf{h} + \mathbf{v} \times (\mathbf{r} \times \mathbf{f})\right) \quad \Rightarrow$$

$$\begin{bmatrix} -e\left(\dot{\omega} - \dot{\theta}\right)\sin(\omega - \theta) + \dot{e}\cos(\omega - \theta) \\ e\left(\dot{\omega} - \dot{\theta}\right)\cos(\omega - \theta) + \dot{e}\sin(\omega - \theta) \\ 0 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} -e\left(\dot{\Omega}\cos(I) + \dot{\theta}\right)\sin(\omega - \theta) \\ e\left(\dot{\Omega}\cos(I) + \dot{\theta}\right)\cos(\omega - \theta) \\ e\left(\dot{I}\sin(\omega) - \dot{\Omega}\sin(I)\cos(\omega)\right) \end{bmatrix}_{\mathcal{B}} = \frac{1}{\mu}\left(\begin{bmatrix} f_\theta h \\ -f_r h \\ 0 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} f_\theta r v_\theta \\ -f_\theta r v_r \\ -f_h r v_r \end{bmatrix}_{\mathcal{B}}\right)$$

Gauss's Perturbation Equations (the solution)

$$\boxed{\begin{aligned} \dot{I} &= \frac{f_h r}{h}\cos(\theta) & \dot{e} &= \frac{e f_\theta}{h} r \sin^2(\nu) + \frac{f_r h}{\mu}\sin(\nu) + \frac{2 f_\theta}{\mu} h \cos(\nu) \\ \dot{\Omega} &= \frac{f_h r \sin(\theta)}{h \sin(I)} & \dot{\omega} &= -\frac{f_h r \sin(\theta)}{h \tan(I)} - \frac{f_\theta r}{2h}\sin(2\nu) - \frac{f_r h}{e\mu}\cos(\nu) + \frac{2 f_\theta h}{e\mu}\sin(\nu) \\ \dot{h} &= f_\theta r & \frac{h}{r^2} &= \dot{\Omega}\cos(I) + \dot{\theta} \end{aligned}}$$

$$\dot{a} = \frac{2a^2}{h}\left[e \sin\nu f_r + (1 + e\cos(\nu)) f_\theta\right]$$

Gauss's Perturbation Equations (other versions)

$$\frac{d\Omega}{dt} = \frac{r\sin\theta}{h\sin i} a_{dh}$$
$$\frac{di}{dt} = \frac{r\cos\theta}{h} a_{dh}$$
$$\frac{d\omega}{dt} = \frac{1}{he}[-p\cos f\, a_{dr} + (p+r)\sin f\, a_{d\theta}] - \frac{r\sin\theta\cos i}{h\sin i} a_{dh}$$
$$\frac{da}{dt} = \frac{2a^2}{h}\left(e\sin f\, a_{dr} + \frac{p}{r} a_{d\theta}\right)$$
$$\frac{de}{dt} = \frac{1}{h}\{p\sin f\, a_{dr} + [(p+r)\cos f + re]a_{d\theta}\}$$
$$\frac{dM}{dt} = n + \frac{b}{ahe}[(p\cos f - 2re)a_{dr} - (p+r)\sin f a_{d\theta}]$$

Battin (1999) Eq. 10.41
NB: $f \equiv \nu,\ p \equiv \ell$
$(a_{dr}, a_{d\theta}, a_{dh}) \equiv (f_r, f_\theta, f_h)$

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left\{e\,\mathrm{SIN}(\nu)F_R + \frac{p}{r}F_S\right\}$$
$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left\{\mathrm{SIN}(\nu)F_R + \left(\mathrm{COS}(\nu) + \frac{e+\mathrm{COS}(\nu)}{1+e\,\mathrm{COS}(\nu)}\right)F_S\right\}$$
$$\frac{di}{dt} = \frac{r\,\mathrm{COS}(u)}{na^2\sqrt{1-e^2}}F_W$$
$$\frac{d\Omega}{dt} = \frac{r\,\mathrm{SIN}(u)}{na^2\sqrt{1-e^2}\,\mathrm{SIN}(i)}F_W$$
$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left\{-\mathrm{COS}(\nu)F_R + \mathrm{SIN}(\nu)\left(1+\frac{r}{p}\right)F_S\right\} - \frac{r\,\mathrm{COT}(i)\,\mathrm{SIN}(u)}{h}F_W$$
$$\frac{dM_o}{dt} = \frac{1}{na^2e}\left\{(p\,\mathrm{COS}(\nu) - 2er)F_R - (p+r)\,\mathrm{SIN}(\nu)F_S\right\} - \frac{dn}{dt}(t-t_o)$$

Vallado (2013) Eq. 9-24
NB: $p \equiv \ell$
$(F_R, F_S, F_W) \equiv (f_r, f_\theta, f_h)$

Lagrange Planetary Equations

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial R}{\partial M}$$
$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1}{na^2e}\left(\left(1-e^2\right)\frac{\partial R}{\partial M} - \sqrt{1-e^2}\frac{\partial R}{\partial \omega}\right)$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\left(\cos I\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right)$$
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e} - \frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I}$$
$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial I}$$
$$\frac{\mathrm{d}M}{\mathrm{d}t} = n \underbrace{- \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}}$$

$$\mathbf{F}_P^{(\mathrm{cons})} = -\boldsymbol{\nabla}U \triangleq \boldsymbol{\nabla}R$$

Recall the mean motion:

$$n = \sqrt{\frac{\mu}{a^3}}$$

Remember: mean anomaly is always changing (at rate n). This term gives the variation in this change due to the perturbation.

Lagrange Planetary Equations (other versions)

$$\frac{d\Omega}{dt} = \frac{1}{nab\sin i}\frac{\partial R}{\partial i}$$
$$\frac{di}{dt} = -\frac{1}{nab\sin i}\frac{\partial R}{\partial \Omega} + \frac{\cos i}{nab\sin i}\frac{\partial R}{\partial \omega}$$
$$\frac{d\omega}{dt} = -\frac{\cos i}{nab\sin i}\frac{\partial R}{\partial i} + \frac{b}{na^3 e}\frac{\partial R}{\partial e}$$
$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \lambda}$$
$$\frac{de}{dt} = -\frac{b}{na^3 e}\frac{\partial R}{\partial \omega} + \frac{b^2}{na^4 e}\frac{\partial R}{\partial \lambda}$$
$$\frac{d\lambda}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{b^2}{na^4 e}\frac{\partial R}{\partial e}$$

Battin (1999) Eq. 10.31

$$\lambda \triangleq nt_p$$
$$b = a\sqrt{1-e^2}$$

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M_o}$$
$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M_o} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega}$$
$$\frac{di}{dt} = \frac{1}{na^2\sqrt{1-e^2}\,\mathrm{SIN}(i)}\left\{\mathrm{COS}(i)\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right\}$$
$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{\mathrm{COT}(i)}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}$$
$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\,\mathrm{SIN}(i)}\frac{\partial R}{\partial i}$$
$$\frac{dM_o}{dt} = -\frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} + n$$

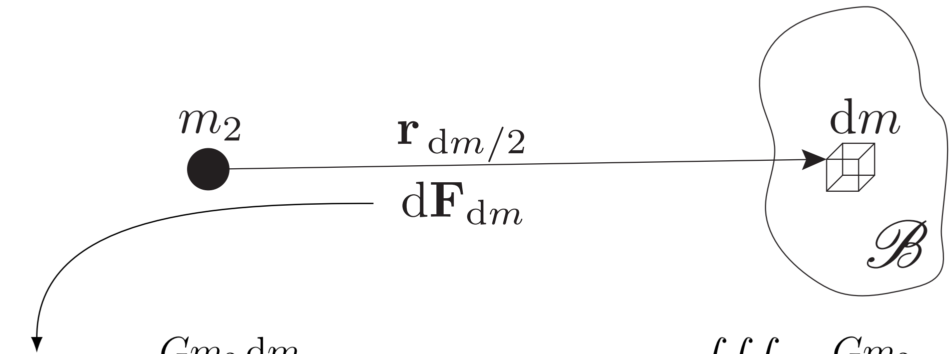
Vallado (2013) Eq. 9-12

$$M = M_0 + n(t - t_p)$$

Orbiting About Extended Bodies

One of the most important perturbations for spacecraft in orbit about the Earth (or another large body) is due to the fact that the central bodies have physical extent (they're not ideal particles) and are non-spherical (remember, the Earth is best described as a lumpy, oblate spheroid). Here we will continue our earlier development of the parametrization of Earth shape, which started by defining the reference geoid. We can introduce the concept of spherical harmonics to decompose the true mass distribution of the Earth into a finite number of measurable terms, and then use this description along with our perturbation equations to find the effects on a satellite orbit.

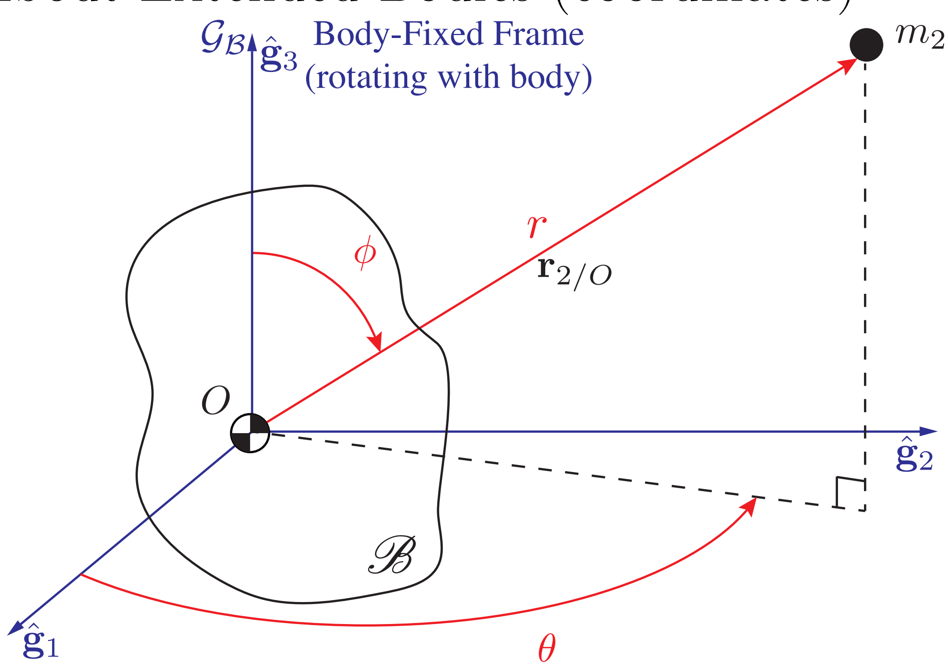
Orbiting About Extended Bodies (forces)



$$\mathrm{d}\mathbf{F}_{\mathrm{d}m} = \frac{Gm_2\,\mathrm{d}m}{\|\mathbf{r}_{\mathrm{d}m/2}\|^3}\mathbf{r}_{\mathrm{d}m/2} \qquad \Rightarrow \qquad \mathbf{F} = \iiint_{\mathscr{B}} \frac{Gm_2}{\|\mathbf{r}_{\mathrm{d}m/2}\|^3}\mathbf{r}_{\mathrm{d}m/2}\,\mathrm{d}m$$

$$\mathbf{F} = m_2 \nabla U \qquad\qquad U = G\iiint_{\mathscr{B}} \frac{\mathrm{d}m}{\|\mathbf{r}_{\,\mathrm{d}m/2}\|}$$

Orbiting About Extended Bodies (coordinates)



NB: Careful! θ, ϕ are often reversed in other texts

Potential of an Azimuthally Symmetric Body

Body Total Mass   Body Equatorial Radius

$$\boxed{U(r,\phi) = \frac{G m_{\mathscr{B}}}{r}\left[1 - \sum_{k=2}^{\infty} J_k \left(\frac{R_{\mathscr{B}}}{r}\right)^k \underbrace{P_k(\cos\phi)}\right]}$$

Non-dimensional coefficients named after Harold Jeffreys.   Legendre Polynomials

$J_1 = 0$ due to symmetry

Legendre Polynomials are Solutions to Legendre's differential equation:

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)\right) + \left(n^2+n\right)P_n(x)$$

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x-1)^{n-k}(x+1)^k = 2^n \sum_{k=0}^{n} x^k \binom{n}{k}\binom{(n+k-1)/2}{n}$$

J Values for Solar System Bodies

($\times 10^{-6}$)	**Earth**	**Mars**	**Moon**	**Venus**	**Mercury**
J_2	1082.6	1955.5	203.23	4.4044	22.5
J_3	-2.5327	31.450	8.4759	-2.1082	4.49
J_4	-1.6196	-15.377	-9.5919	-2.1474	6.5

($\times 10^{-6}$)	**Jupiter**	**Saturn**	**Uranus**	**Neptune**
J_2	14696.572	16290.573	3341.29	3408.43
J_3	-0.042	0.059	—	—
J_4	-586.609	-935.314	-30.44	-33.40

From: Lemoine et al. 1998 (Earth), Lemoine et al. 2001 (Mars), Konopliv et al. 2001 (Moon), Konopliv et al. 1999 (Venus), Iess et al. 2018 (Jupiter), Iess et al. 2019 (Saturn), Smith et al. 2012 (Mercury)

Potential of an Arbitrary Body

Multipole Moments

$$U(r,\theta,\phi) = \frac{Gm_{\mathscr{B}}}{r} + G\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} q_\ell^m \; r^{-(\ell+1)} \; Y_\ell^m(\theta,\phi)$$

$$\text{Spherical Harmonics: } Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\phi)\, e^{im\phi}$$

$$\text{Associated Legendre Polynomials : } P_\ell^m(x) = (-1)^m \; (1-x^2)^{m/2} \; \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(P_\ell(x)\right)$$

Typically use:

$$\boxed{U(r,\theta,\phi) = \frac{Gm_{\mathscr{B}}}{r}\left[1 + \sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^\ell P_\ell^m(\cos\phi)\times\Big(\; C_\ell^m\cos(m\theta) + \; S_\ell^m\sin(m\theta)\Big)\right]}$$

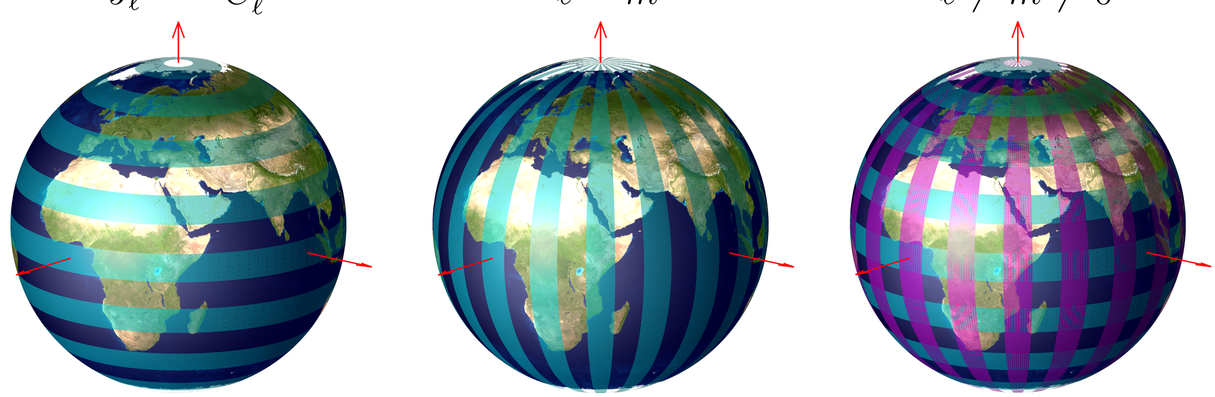
Tabulated Coefficients

Spherical Harmonics

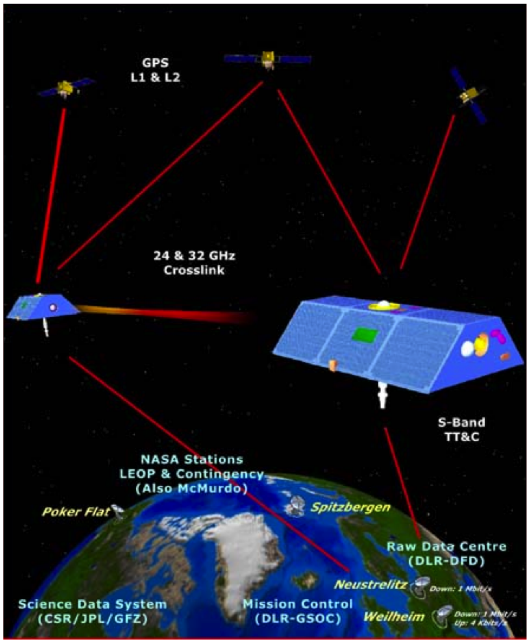
$$U(r,\theta,\phi) = \frac{\mu}{r}\left[1 - \sum_{\ell=2}^{\infty}\left(\frac{R_{\mathscr{B}}}{r}\right)^\ell J_\ell P_\ell(\cos\phi) + \sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^\ell P_\ell^m(\cos\phi)\times\left(C_\ell^m\cos(m\theta) + S_\ell^m\sin(m\theta)\right)\right]$$

ℓ = degree, m = order

Zonal Harmonics	**Sectoral Harmonics**	**Tesseral Harmonics**
Bands of Latitude	Bands of Longitude	Tiles of lat/lon
$J_\ell = -C_\ell^0$	$\ell = m$	$\ell \neq m \neq 0$



State of the Art: GRACE



- Launched in 2002 with original 5 year mission (decommissioned in 2017)
- Followup (GRACE-FO) launched in 2018
- Provides monthly gravity anomaly mapping (degree 60-90)
- Static geopotential maps available from:
  - International Centre for Global Earth Models (ICGEM)
  - National Geospatial-Intelligence Agency (NGA)
- Current Standard is Earth Gravitational Model 2008 (EGM2008)

  http://earth-info.nga.mil/GandG/update/index.php?action=home
- See also: http://icgem.gfz-potsdam.de/tom_longtime

Normalizations – Be Careful!

Description of Files Related to Using the EGM2008 Global Gravitational Model to Compute Geoid Undulations with Respect to WGS 84

(1) EGM2008_to2190_TideFree.gz

This file contains the fully-normalized, unit-less, spherical harmonic coefficients of the Earth's gravitational potential $\{\overline{C}_{nm}, \overline{S}_{nm}\}$ and their associated (calibrated) error standard deviations $\{sigma\overline{C}_{nm}, sigma\overline{S}_{nm}\}$, as implied by the EGM2008 model. The $\{\overline{C}_{nm}, \overline{S}_{nm}\}$ coefficients are consistent with the expression:

$$V(r,\theta,\lambda) = \frac{GM}{r}\left[1 + \sum_{n=2}^{N\max}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}\left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\right)\overline{P}_{nm}(\cos\theta)\right] \quad (1)$$

From: README_WGS84_2.pdf

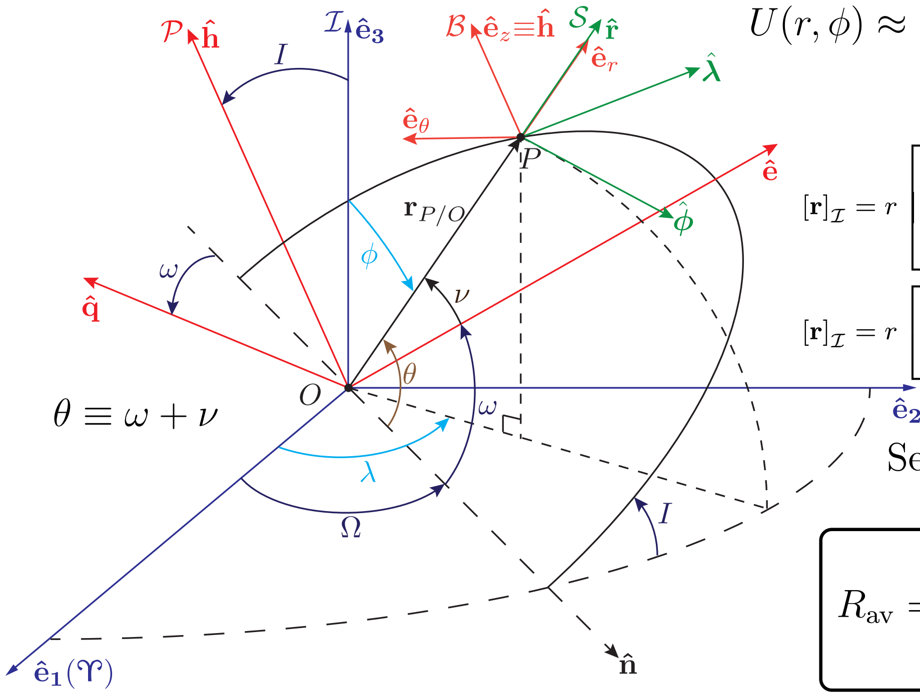
$$C_\ell^m = \left[\frac{(\ell-m)!(2\ell+1)(2-\delta_{0m})}{(l+m)!}\right]^{\frac{1}{2}} \overline{C}_\ell^m$$

For example:

$$\overline{C}_2^0 = -4.841651437908150 \times 10^{-4}$$

$$C_2^0 = \left[\frac{(2-0)!(2\times 2+1)(2-\delta_{20})}{(2+0)!}\right]^{\frac{1}{2}} \overline{C}_2^0 = -0.001082626173852 = -J_2$$

J_2 Perturbation Analysis Setup



$$U(r,\phi) \approx \frac{\mu}{r} \underbrace{- \frac{\mu}{r} J_2 \left(\frac{R_{\mathscr{B}}}{r}\right)^2 \left(\frac{3\cos^2\phi - 1}{2}\right)}_{\text{Perturbing Potential}}$$

$$[\mathbf{r}]_{\mathcal{I}} = r \begin{bmatrix} -\sin(\Omega)\sin(\theta)\cos(I) + \cos(\Omega)\cos(\theta) \\ \sin(\Omega)\cos(\theta) + \sin(\theta)\cos(I)\cos(\Omega) \\ \underbrace{\sin(I)\sin(\theta)} \end{bmatrix}_{\mathcal{I}}$$

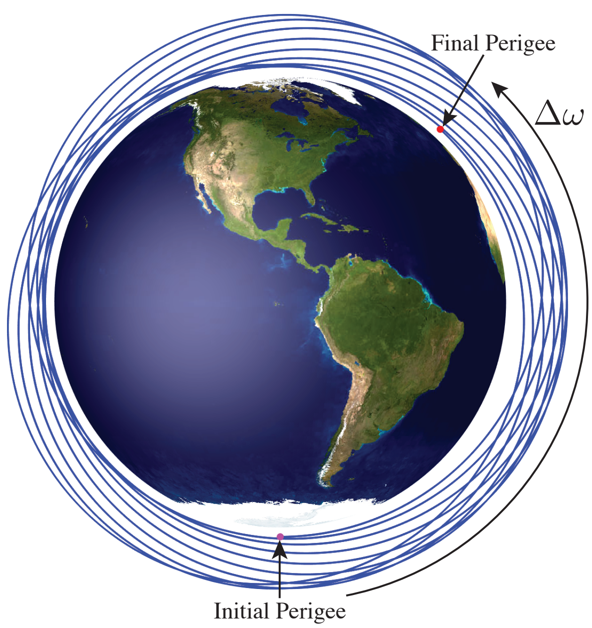
$$[\mathbf{r}]_{\mathcal{I}} = r \begin{bmatrix} \sin(\phi)\cos(\lambda) \\ \sin(\lambda)\sin(\phi) \\ \cos(\phi) \end{bmatrix}_{\mathcal{I}} \qquad \boxed{\Rightarrow \cos\phi = \sin I \sin\theta}$$

Secular (non-periodic) perturbation:

$$\boxed{R_{\text{av}} = -\frac{J_2 \mu R_{\mathscr{B}}^2}{2a^3(1-e^2)^{\frac{3}{2}}} \left(\frac{3}{2}\sin^2(I) - 1\right)}$$

NB: The secular perturbing potential is the result of averaging over one full orbit. This means that encodes effects that span multiple orbits, but ignores any effects that cancel out over the course of a full orbit. As such, it is good for predicting long-term orbit evolution, but not suitable for detailed tracking of a spacecraft.

Apsidal Rotation



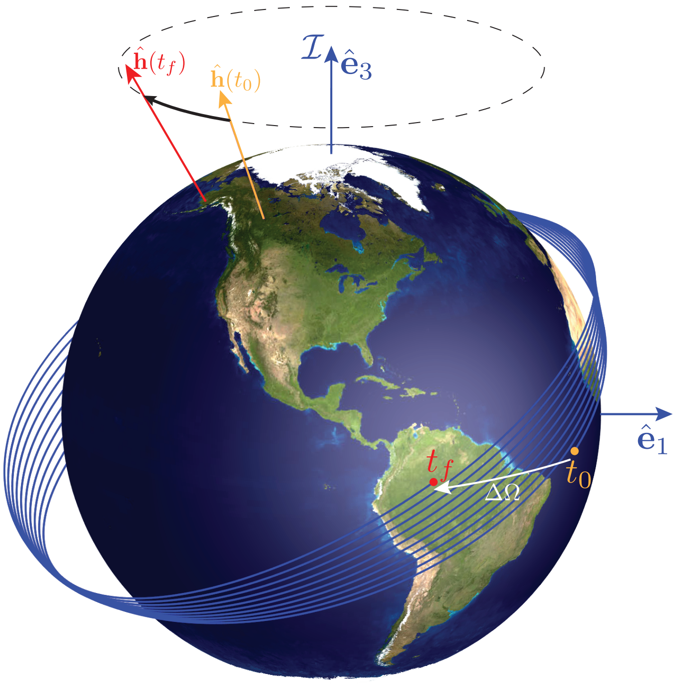
$$\boxed{\begin{aligned} \dot{\omega}_{\text{sec}} &= \frac{3}{2} J_2 n \left(\frac{R_{\mathscr{B}}}{a(1-e^2)}\right)^2 \left(2 - \frac{5}{2}\sin^2(I)\right) \\ \omega(t) &= \omega(t_0) + \dot{\omega}_{\text{sec}}(t - t_0) \end{aligned}}$$

Polar, 1000 km altitude, 0.1 eccentricity orbit over 10 periods

Rotation exaggerated 100x for clarity

Figure based on Vallado (2013)

Nodal Regression



$$\boxed{\begin{aligned}\dot{\Omega}_{\text{sec}} &= -\frac{3nR_{\mathscr{B}}^2 J_2}{2a^2(1-e^2)^2}\cos I \\ \Omega(t) &= \Omega(t_0) + \dot{\Omega}_{\text{sec}}(t-t_0)\end{aligned}}$$

30° inclination, 1000 km altitude, 0.1 eccentricity orbit over 10 periods

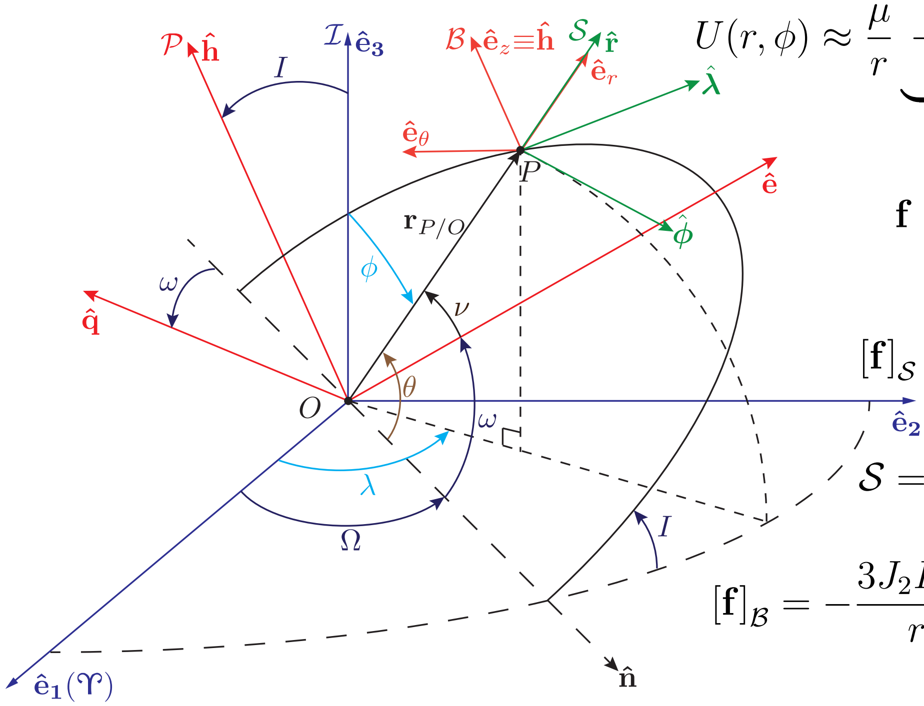
Rotation exaggerated 10x for clarity

Figure based on Vallado (2013)

Geopotential Perturbation Analysis with Gauss's Equations

Remember that Gauss's equations encode the same basic physics as Lagrange's equations, which means that we should be able to get the exact same results (i.e., nodal regression and apsidal rotation) via a force-based analysis. While this is absolutely true, in this case, the analysis is significantly more complex with Gauss's equations than with Lagrange's. Here, we will show how to get equivalent perturbing forces equivalent to the J_2 perturbing potential, but leave the application of Gauss's equations as an exercise for those so inclined. However, there is another important reason why we may wish to derive perturbing forces due to the central body geopotential: we require this form of equations if we wish to numerically integrate the specific path a spacecraft will take along its orbit. Note that the final form of the perturbing force expressions we derive will be in Earth-fixed, rotating (non-inertial) components. To use these within a numerical integration, an additional step is required to transform these force components into whichever inertial frame is being used to encode the spacecraft state.

J_2 Perturbation Analysis Setup (Forces)



$$U(r,\phi) \approx \frac{\mu}{r} \underbrace{- \frac{\mu}{r} J_2 \left(\frac{R_{\mathscr{B}}}{r}\right)^2 \left(\frac{3\cos^2\phi - 1}{2}\right)}_{\text{Perturbing Potential :}}$$

$$\mathbf{f} \triangleq -\boldsymbol{\nabla}\left(\frac{\mu J_2 R_{\mathscr{B}}^2}{r^3}\left(\frac{3\cos^2\phi - 1}{2}\right)\right)$$

$$[\mathbf{f}]_{\mathcal{S}} = \frac{3J_2 R_{\mathscr{B}}^2 \mu}{r^4}\begin{bmatrix} \frac{1}{2}\left(3\cos^2(\phi) - 1\right) \\ \sin(\phi)\cos(\phi) \\ 0 \end{bmatrix}_{\mathcal{S}}$$

$$\mathcal{S} = \left(P, \hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\lambda}}\right) \qquad \mathcal{B} = \left(P, \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{h}}\right)$$

$$[\mathbf{f}]_{\mathcal{B}} = -\frac{3J_2 R_{\mathscr{B}}^2 \mu}{r^4}\begin{bmatrix} \frac{1}{2}\left(1 - 3\sin^2(I)\sin^2(\theta)\right) \\ -\sin^2(I)\sin(\theta)\cos(\theta) \\ \sin(I)\cos(I)\sin(\theta) \end{bmatrix}_{\mathcal{B}}$$

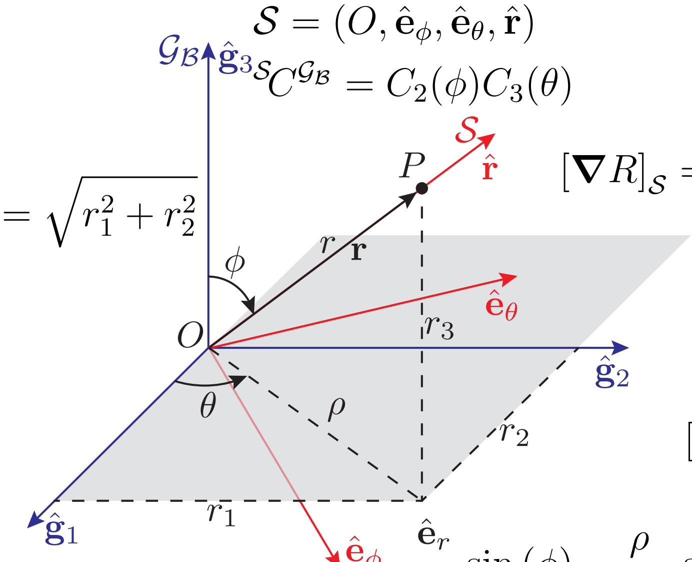
Central Body Shape Perturbing Forces

Perturbing Acceleration $\mathbf{f} = \boldsymbol{\nabla} R = \boldsymbol{\nabla}\left(\frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} P_\ell^m(\cos\phi)\left[C_\ell^m\cos(m\theta) + S_\ell^m\sin(m\theta)\right]\right)$

$$\mathcal{S} = (O, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta, \hat{\mathbf{r}})$$

$${}^{\mathcal{S}}C^{\mathcal{G}_{\mathcal{B}}} = C_2(\phi)C_3(\theta)$$

$$\rho = \sqrt{r_1^2 + r_2^2}$$



$$[\boldsymbol{\nabla} R]_{\mathcal{S}} = \begin{bmatrix} \frac{1}{r}\frac{\partial R}{\partial \phi} \\ \frac{1}{r\sin\phi}\frac{\partial R}{\partial \theta} \\ \frac{\partial R}{\partial r} \end{bmatrix}_{\mathcal{S}} \qquad [\boldsymbol{\nabla} R]_{\mathcal{G}_{\mathcal{B}}} = {}^{\mathcal{G}_{\mathcal{B}}}C^{\mathcal{S}}\,[\boldsymbol{\nabla} R]_{\mathcal{S}}$$

$$[\mathbf{r}]_{\mathcal{G}_{\mathcal{B}}} \triangleq \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}_{\mathcal{G}_{\mathcal{B}}} = {}^{\mathcal{G}_{\mathcal{B}}}C^{\mathcal{S}}\begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} \implies$$

$$\sin(\phi) = \frac{\rho}{r},\ \sin(\theta) = \frac{r_2}{\rho},\ \cos(\phi) = \frac{r_3}{r},\ \cos(\theta) = \frac{r_1}{\rho}$$

Central Body Shape Perturbing Forces (continued)

$$[\mathbf{f}]_{\mathcal{G}_\mathcal{B}} = \begin{bmatrix} \frac{r_1}{r}\left(\frac{\partial R}{\partial r} + \frac{r_3}{r\rho}\frac{\partial R}{\partial \phi}\right) - \frac{r_2}{\rho^2}\frac{\partial R}{\partial \theta} \\ \frac{r_1}{\rho^2}\frac{\partial R}{\partial \theta} + \frac{r_2}{r}\left(\frac{\partial R}{\partial r} + \frac{r_3}{r\rho}\frac{\partial R}{\partial \phi}\right) \\ \frac{r_3}{r}\frac{\partial R}{\partial r} - \frac{\rho}{r^2}\frac{\partial R}{\partial \phi} \end{bmatrix}_{\mathcal{G}_\mathcal{B}}$$

$$\frac{\partial R}{\partial r} = -\frac{\mu}{r^2}\sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell}(\ell+1)\,P_\ell^m(\cos\phi)\left[C_\ell^m\cos(m\theta) + S_\ell^m\sin(m\theta)\right]$$

$$\frac{\partial R}{\partial \theta} = \frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} mP_\ell^m(\cos\phi)\left[-C_\ell^m\sin(m\theta) + S_\ell^m\cos(m\theta)\right]$$

$$\frac{\partial R}{\partial \phi} = \frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell}\left(\ell\cos(\phi)P_\ell^m(\cos\phi) - (\ell+m)P_{\ell-1}^m(\cos\phi)\right)\left[\frac{C_\ell^m\cos(m\theta) + S_\ell^m\sin(m\theta)}{\sin(\phi)}\right]$$

Specialized Orbits

Orbital perturbations due to the Earth's geopotential can be either helpful or harmful, in that they can either enable orbital capabilities that are impossible with standard two-body orbits, or they can cause orbits based on two-body analyses to lose some of their desired features over time. Here we will consider two orbital design problems (one of each type). In the first, we will take advantage of nodal regression to create new orbital capabilities, while in the second, we will see that apsidal rotation must be accounted for explicitly in the orbital design in order to ensure the desired functionality.

Design Problem 1: You wish for your satellite to observe the same points on the surface at the same local time of day

- Non-synchronous orbits will cross different points on the ground at different times of day
- Solution: use nodal regression to match the mean motion of the sun:

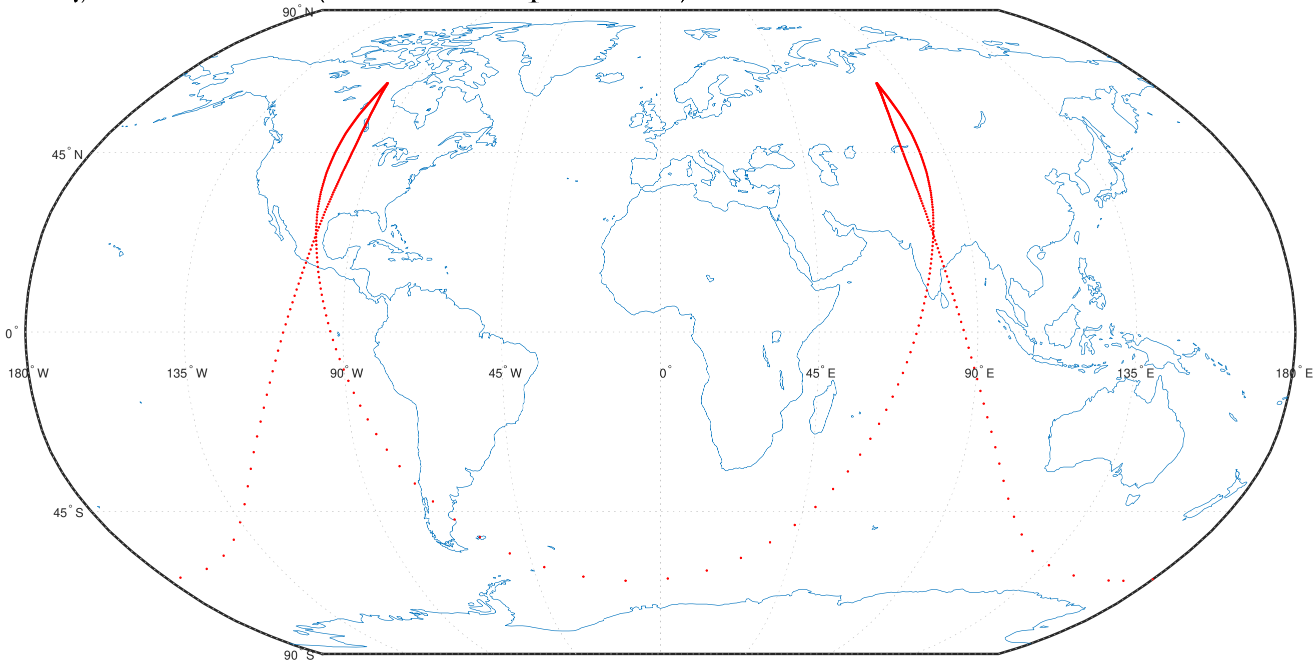
$$\dot{\Omega}_{\text{sec}} = -\frac{3nR_{\mathscr{B}}^2 J_2}{2a^2(1-e^2)^2}\cos I \;\; = \frac{2\pi}{\text{Tropical Year}}$$

- Various combinations of a, e, I will make this work. For **Low-Earth Orbits** (LEO), **Sun-Synchronous** orbits are near **Polar** ($I \sim 90°$)
- Spacecraft will pass over the same points on the surface at the same *mean* solar time (not apparent)
- A synchronous orbit on the **terminator** is called a dawn/dusk orbit—the spacecraft always sees the mean sun (but will still have eclipses)

Design Problem 2: You wish to have extended views of one high-latitude region of the Earth

- Typically solved with a **Geostationary Orbit (GEO)**—orbit period is exactly one sidereal day
- Secondary problem: GEOs only stay over one spot in near-equatorial orbits, and therefore are unsuitable for high/low latitudes
- Solution: create a highly eccentric orbit with a period of exactly 12 sidereal hours with apogee over the region of interest
- Tertiary problem: apsidal rotation will move apogee away from the preferred region over many orbits:
  $\dot{\omega}_{\text{sec}} = \frac{3}{2}J_2 n\left(\frac{R_{\mathscr{B}}}{a(1-e^2)}\right)^2\left(2-\frac{5}{2}\sin^2(I)\right)$
- But what if $\frac{5}{2}\sin^2(I) = 2$? Then $\dot{\omega} = 0$. This corresponds to $I \sim 63.43°$
- Still have a problem of nodal regression, but can deal with that by adjusting the orbital period to counteract the effect.

Molniya Orbits (12 hour period)

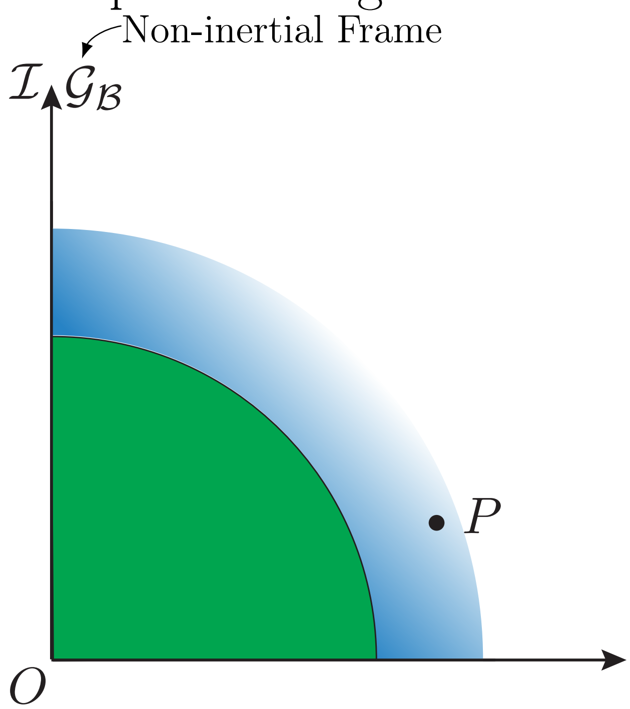


Points are spaced equally in time, so that the spacecraft spends most of its time (at apogee) over North American and Asia.

Atmospheric Drag

For low Earth orbits (and low orbits about any body with a significant atmosphere), atmospheric drag will typically be the most important perturbation. Unfortunately, it is also much harder to model than the geopotential perturbation, as the atmosphere evolves very quickly, and in response to many different forcing functions. Here, we will develop a basic model of the atmosphere and apply our perturbation equations to find its secular (orbit-average) effects. We will also consider the main contributors to atmospheric variation and define a basic exponential atmosphere model useful for numerical propagation of orbits.

Atmospheric Drag



Cross-sectional Area Relative Velocity

$$\mathbf{F}_{\text{drag}} = -\frac{1}{2}\, C_D\, A\, \rho\, v_{\text{rel}}^2\, \hat{\mathbf{v}}_{\text{rel}}$$

Drag Coefficient Atmospheric Density

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{I}}\mathbf{v}_{P/O} - \underbrace{{}^{\mathcal{I}}\mathbf{v}_{\text{atm}/O}}_{\cancel{{}^{\mathcal{G}_\mathcal{B}}\mathbf{v}_{\text{atm}/O}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{G}_\mathcal{B}} \times \mathbf{r}_{P/O}}$$

$$\boxed{\mathbf{v}_{\text{rel}} \approx {}^{\mathcal{I}}\mathbf{v}_{P/O} - {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{G}_\mathcal{B}} \times \mathbf{r}_{P/O}}$$

$$\textbf{Ballistic Coefficient} \triangleq \frac{m}{C_D A}$$

Secular Perturbations Due to Atmospheric Drag

Planet Rotation Rate

$$Q \triangleq \left(1 - \frac{\omega_r (1-e)^{3/2}}{n\sqrt{1+e}} \cos(I)\right)$$

$$\Delta a_{\text{rev}} \approx -2\pi \frac{Q^2 A C_D}{m} a^2 \rho_p \left(I_0 + 2eI_1 + \frac{3e^2}{4}(I_0 + I_2) + \frac{e^3}{4}(3I_1 + I_3)\right) \exp\left(\frac{-ae}{H}\right)$$

Spacecraft Mass Density at Periapsis Atmospheric Scale Height

$$\Delta e_{\text{rev}} \approx -2\pi \frac{Q^2 A C_D}{m} a \rho_p \left(I_1 + \frac{e}{2}(I_0 + I_2) - \frac{e^2}{8}(5I_1 - I_3) + \frac{e^3}{16}(5I_0 + 4I_2 - I_4)\right) \exp\left(\frac{-ae}{H}\right)$$

Here, $I_{0\ldots4}$ are modified Bessel functions of the first kind with argument $z = \dfrac{ae}{H}$:

$$I_s(z) = \frac{1}{\pi}\int_0^\pi \exp\left(z\cos\theta\right)\cos(s\theta)\, \mathrm{d}\theta$$

Secular Perturbations Due to Atmospheric Drag (2)

$$\Delta I_{\text{rev}} \approx -\pi \frac{QAC_D}{2nm} \omega_r a \rho_p \sin(I) \left(I_0 - 2eI_1 + (I_2 - 2eI_1)\cos(2\omega) \right) \exp\left(\frac{-ae}{H} \right)$$

$$\Delta \Omega_{\text{rev}} \approx -\pi \frac{QAC_D}{2nm} \omega_r a \rho_p \left(I_2 - 2eI_1 \right) \sin(2\omega) \exp\left(\frac{-ae}{H} \right)$$

$$\Delta \omega_{\text{rev}} \approx -\Delta \Omega_{\text{rev}} \cos(I)$$

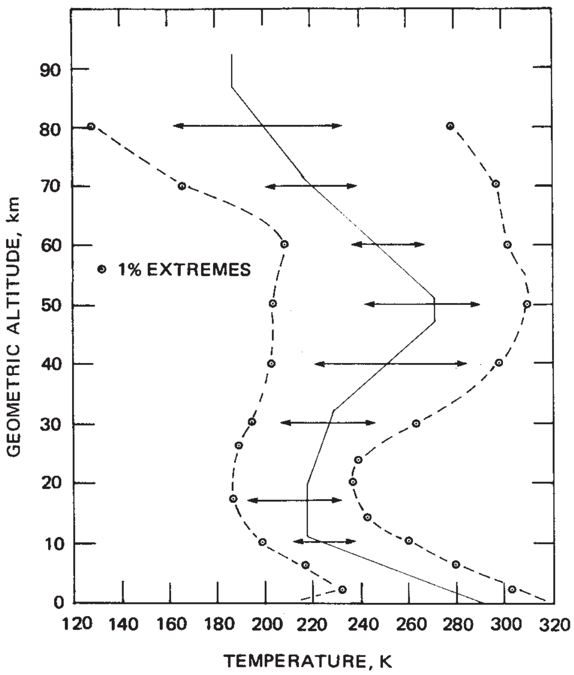
Again, here I_i are the Modified Bessel Functions of the First Kind with argument $\frac{ae}{H}$

The primary effects of drag are to circularize and shrink the orbit.

Atmosphere Variation

- The density of the Earth's upper atmosphere constantly fluctuates. Two major effects are:
  - **Incident Solar Flux** - heating from extreme ultraviolet radiation (EUV) has a near instantaneous effect
  - **Geomagnetic Interactions** - collisions with charged energetic particles cause a delayed heating effect
- Atmospheric fluctuations vary both spatially and temporally, with both random and cylcic behavior. Important cycles include:
  - **Diurnal Variations** - the atmosphere bulges in a direction lagging the direction of the sun (around 2:00 PM local time)
  - **Solar Rotation Cycle** - the same solar active regions come into view approximately every 27 days
  - **Solar Magnetic Activity Cycle** - the sun cycles in activity over a period of 11 years, as measured by the number of observed sun spots

Atmosphere Temperature Variation

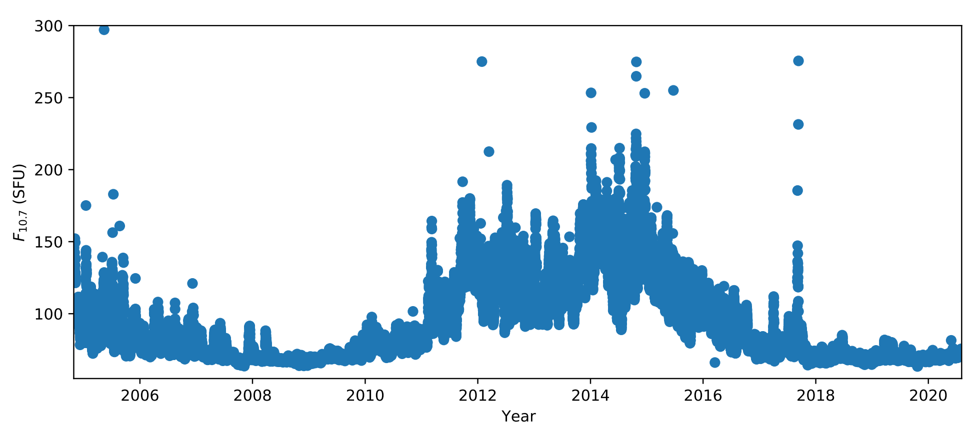


- Range of systematic variability of temperature around U.S. Standard Atmosphere, 1976.
- Arrows indicate min/max monthly measured temperatures.
- Dots are estimates of 1% min/max global temperatures.

From: U.S. Standard Atmosphere, 1976.
https://ntrs.nasa.gov/search.jsp?R=19770009539

Measuring Solar Activity

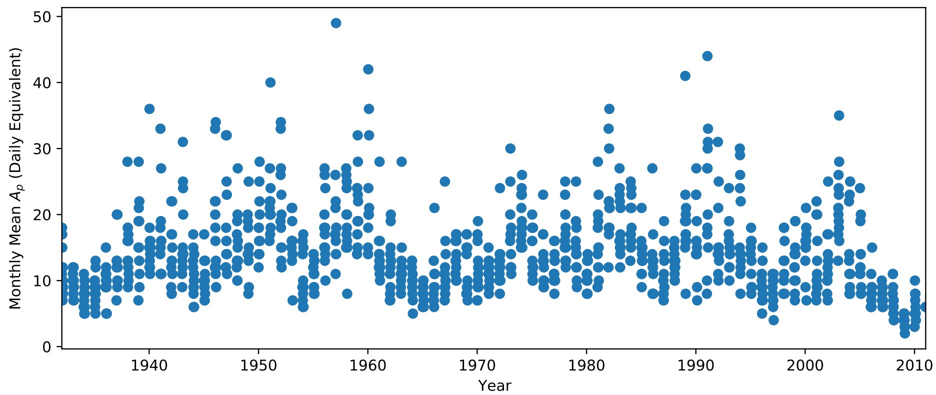
- The atmosphere absorbs all UV radiation, so we cannot directly measure EUV flux from the ground but both EUV and radiation with a wavelength of 10.7 cm (2800 MHz) originate in the same layers of the sun.
- $F_{10.7}$ is used as a proxy for EUV, and has been measured since 1940
- Define one **Solar Flux Unit** (SFU) as 1×10^{-22} watt m^{-2} Hz^{-2}



Data from: https://www.spaceweather.gc.ca/solarflux/sx-en.php

Measuring Geomagnetic Activity

- The Earth's magnetic field also varies temporally and spatially and is typically fit with a spherical harmonic model (same as the geopotential)
- Define a geomagnetic planetary index K_p to measure worldwide geomagnetic activity. Can also use the daily planetary amplitude A_p. Both are in units of gamma $= 10^{-9}$ Tesla $= 10^{-9}$ kg s m^{-1}.



Data from: https://www.ngdc.noaa.gov/stp/GEOMAG/kp_ap.html

The Exponential Atmosphere

Assuming a static atmosphere where density decays exponentially with altitude:

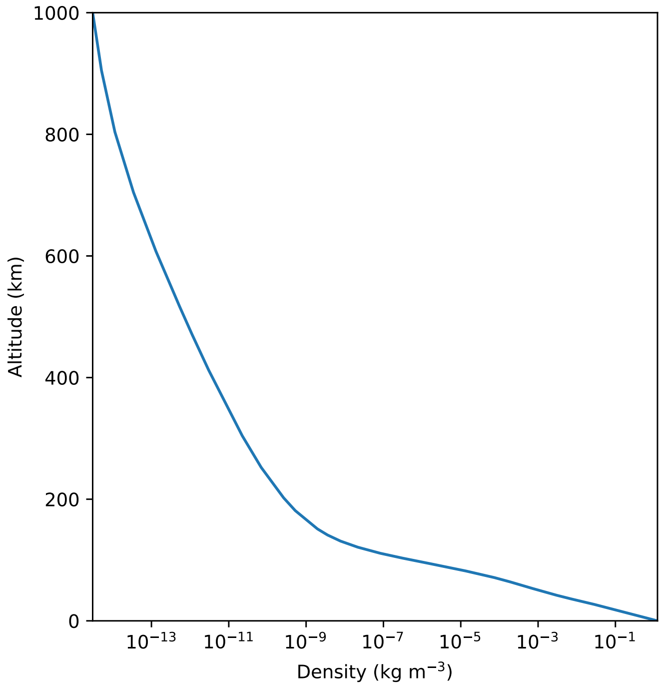
$$\rho = \rho_0 \exp\left(-\frac{h - h_0}{H}\right)$$

- ρ_0, h_0 are reference density and reference altitude (tabulated)
- h is the altitude above the ellipsoid
- H is the **Scale Height** - the fractional change in density with height. H is the increase in altitude required for ρ to drop to $^1/_e$ of its initial value:

$$H = \frac{R^\star T}{M g_0}$$

- $R^\star$ is the ideal gas constant: 8.31446261815324 J K^{-1} mol^{-1}
- T and M are the temperature and mean molecular weight of the atmosphere
- g_0 is the gravitational acceleration at the surface (9.80665 m s^{-2} on Earth)

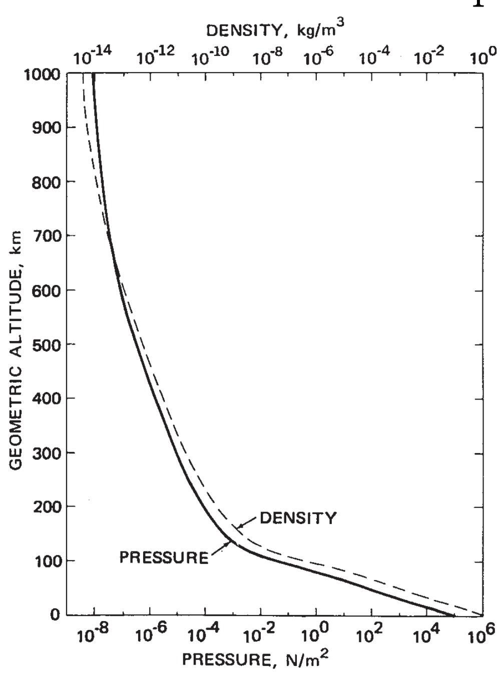
The Exponential Atmosphere (2)

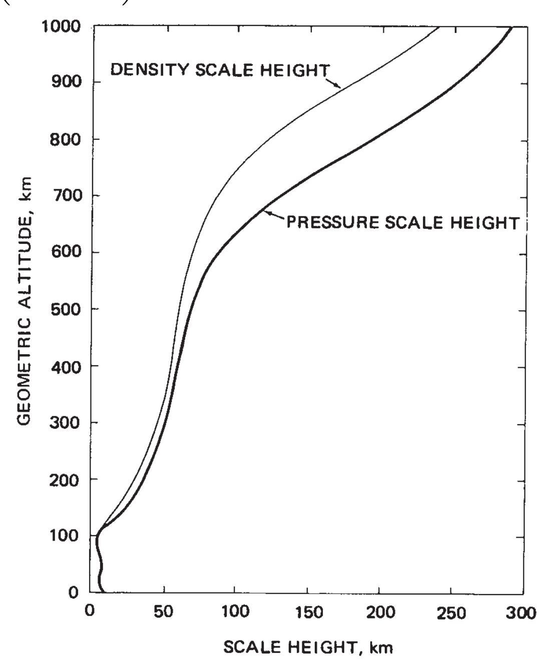


h (km)	h_0 (km)	ρ_0 (kg m^{-3})	H (km)
0-25	0	1.225	7.249
25-30	25	3.899e-2	6.349
30-40	30	1.774e-2	6.682
40-50	40	3.972e-3	7.554
50-60	50	1.057e-3	8.382
60-70	60	3.206e-4	7.714
70-80	70	8.770e-5	6.549
80-90	80	1.905e-5	5.799
90-100	90	3.396e-6	5.382
100-110	100	5.297e-7	5.877
110-120	110	9.661e-8	7.263
120-130	120	2.438e-8	9.473
130-140	130	8.484e-9	12.636
140-150	140	3.845e-9	16.149
150-180	150	2.070e-9	22.523
180-200	180	5.464e-10	29.740
200-250	200	2.789e-10	37.105
250-300	250	7.248e-11	45.546
300-350	300	2.418e-11	53.628
350-400	350	9.518e-12	53.298
400-450	400	3.725e-12	58.515
450-500	450	1.585e-12	60.828
500-600	500	6.967e-13	63.822
600-700	600	1.454e-13	71.835
700-800	700	3.614e-14	88.667
800-900	800	1.170e-14	124.64
900-1000	900	5.245e-15	181.05
1000-	1000	3.019e-15	268.00

Data from Wertz (1978)

The U.S. Standard Atmosphere (1976)



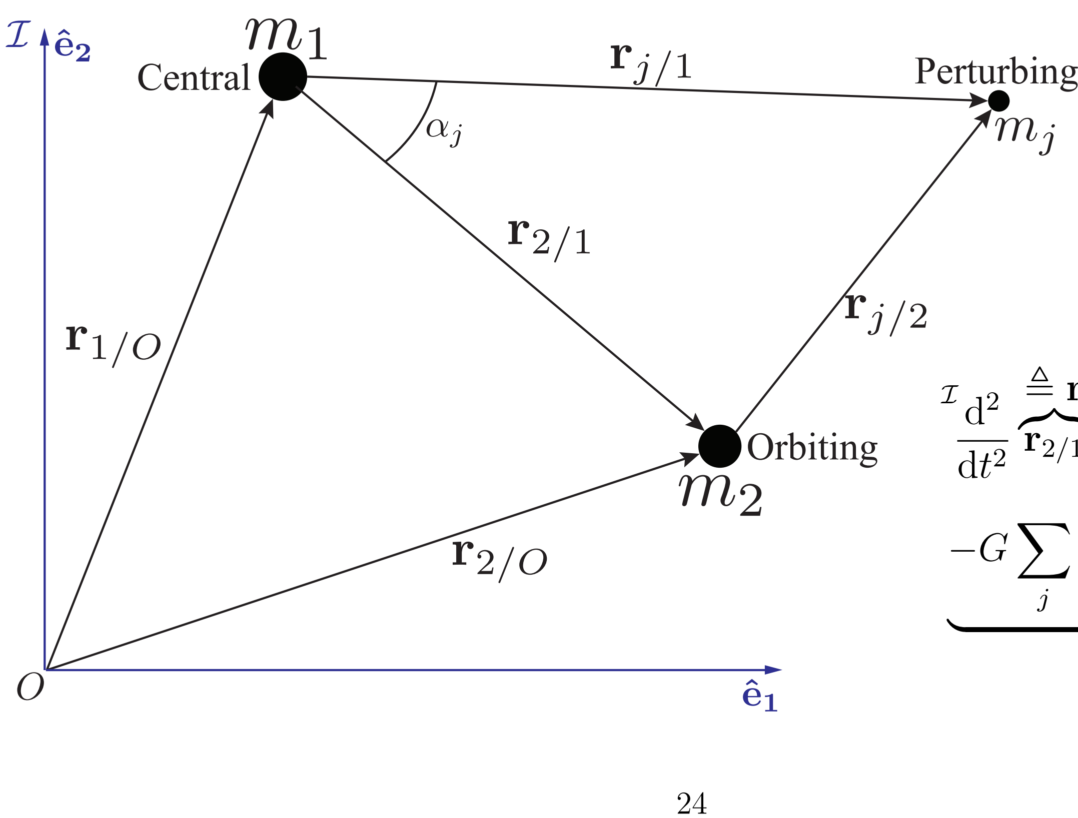


From: https://ntrs.nasa.gov/search.jsp?R=19770009539

Third-Body Perturbations

Another important source of orbital perturbations is the effect of other masses interacting with the two-body system. These become increasingly significant with distance from the central body (for example, during interplanetary flight), but will also effect the orbits of spacecraft about the Earth. The Sun and moon are the two most important third-body perturbers for Earth-orbiting spacecraft, and their effects are collectively known as **lunisolar** perturbations. We can define a critical radius (called the Laplace radius) below which oblateness (geopotential) effects dominate, and beyond which third-body perturbations dominate. For the Earth, the Laplace radius is approximately at 8.41 $R_\oplus$. We have already seen third body perturbers in Cowell's method, but there, we ignored any mutual interaction between the perturbing bodies. Here, we will provide the full formalism for N-body perturbers, and will again look at the secular effects of a perturbing third body.

3^{rd} (N^{th}) Body Perturbations



$$
{}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\overbrace{\mathbf{r}_{2/1}}^{\triangleq \mathbf{r}} + \overbrace{\frac{G(m_1+m_2)}{\|\mathbf{r}_{2/1}\|^3}}^{\triangleq \mu}\mathbf{r}_{2/1} = \underbrace{-G\sum_j m_j\left(\frac{\mathbf{r}_{j/1}}{\|\mathbf{r}_{j/1}\|^3}+\frac{\mathbf{r}_{2/j}}{\|\mathbf{r}_{2/j}\|^3}\right)}_{\triangleq \mathbf{f}}
$$

Secular Perturbations From Third Body in Circular Orbit

$$\boxed{\begin{aligned}\dot{\Omega}_{\text{sec}} &\approx -\frac{3}{16}\frac{\mu_3\left(2+3e^2\right)\left(2-3\sin^2(I_3)\right)}{nr_3^3\sqrt{1-e^2}}\cos I \\ \\ \dot{\omega}_{\text{sec}} &\approx \frac{3}{16}\frac{\mu_3\left(2-3\sin^2(I_3)\right)}{nr_3^3\sqrt{1-e^2}}\left(e^2+4-5\sin^2(I)\right)\end{aligned}}$$

No subscript refers to elements of orbiting body, 3 subscript refers to perturber orbit

- No secular or long-period variations occur in the semi-major axis due to third-body perturbations
- Sinusoidal variations in periapsis and long period variations in e, I, Ω, ω
- Periapsis variations can couple significantly with drag perturbations

Solar Radiation Pressure (SRP)

- Photons are energetic, and can transfer momentum to masses via absorption, re-emission, reflection, and scattering
- This is a perturbation, but also a potential for propulsion (stay tuned)
- SRP generates periodic variations in *all* orbital elements and starts to dominate over atmospheric drag effects above ~ 800 km
- Highly dependent on spacecraft mass and surface area

Tides and Magnetic Field Effects

- **Solid-Earth Tides**: The Earth is deformed by other masses in the solar system (and especially the Moon)
- **Ocean Tides**: Water moves over the surface of the Earth in response to gravitational forces
- Both types of tides change the mass distribution of the central body and therefore the geopotential effects on an orbit
- Interaction with the Earth's magnetic field and charged particles can produce forcing and torquing effects on a spacecraft. This is a perturbation but can also be used for attitude control (stay tuned)

Orbital Maneuvers, Trajectories, and Relative Motion

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MAE 4060/5065, Fall 2021

Orbital Maneuvers, Trajectories, and Relative Motion

At the start of the space age, in-space propulsion was almost entirely chemical, based on systems capable of producing (relatively) large thrusts over (relatively) short intervals of time. Because of this, the standard starting point for the analysis of orbital maneuvers is to adopt an impulsive model—one where changes in velocity can occur instantaneously, without a corresponding change in position. In this model, spacecraft are typically thought of as spending the majority of their time passively following two-body orbits, with control applied for only a tiny fraction of the trajectory. While this model is significantly less applicable to high-efficiency, low-thrust propulsion system, which can thrust over the majority (or entirety) of a trajectory, it remains an essential initial calculation for trajectory planning. It is also the basis for the patched-conic model, which allows us to do preliminary analyses of interplanetary trajectories. Ultimately, however, even chemical systems are not truly impulsive, and the final trajectory must always be evaluated and optimized via numerical integration.

Review of Linear Momentum and Impulse

- The change in linear momentum (proportional to the change in velocity) from t_1 to t_2 is equal to the integral of the total force applied

$$\int_{t_1}^{t_2} \mathbf{F}_P \, \mathrm{d}t = \int_{t_1}^{t_2} \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}({}^{\mathcal{I}}\mathbf{p}_{P/O}) \, \mathrm{d}t = {}^{\mathcal{I}}\mathbf{p}_{P/O}(t_2) - {}^{\mathcal{I}}\mathbf{p}_{P/O}(t_1)$$

$$m_P {}^{\mathcal{I}}\mathbf{v}_{P/O}(t_2) = m_P {}^{\mathcal{I}}\mathbf{v}_{P/O}(t_1) + \underbrace{\int_{t_1}^{t_2} \mathbf{F}_P \, \mathrm{d}t}_{\textbf{Linear Impulse} \triangleq \overline{\mathbf{F}}_P(t_1, t_2)}$$

- If the applied force is nearly constant, then we can approximate: $\overline{\mathbf{F}}_P(t_1, t_2) \approx \mathbf{F}_P \Delta t$ where $\Delta t = t_2 - t_1$ and is typically very small

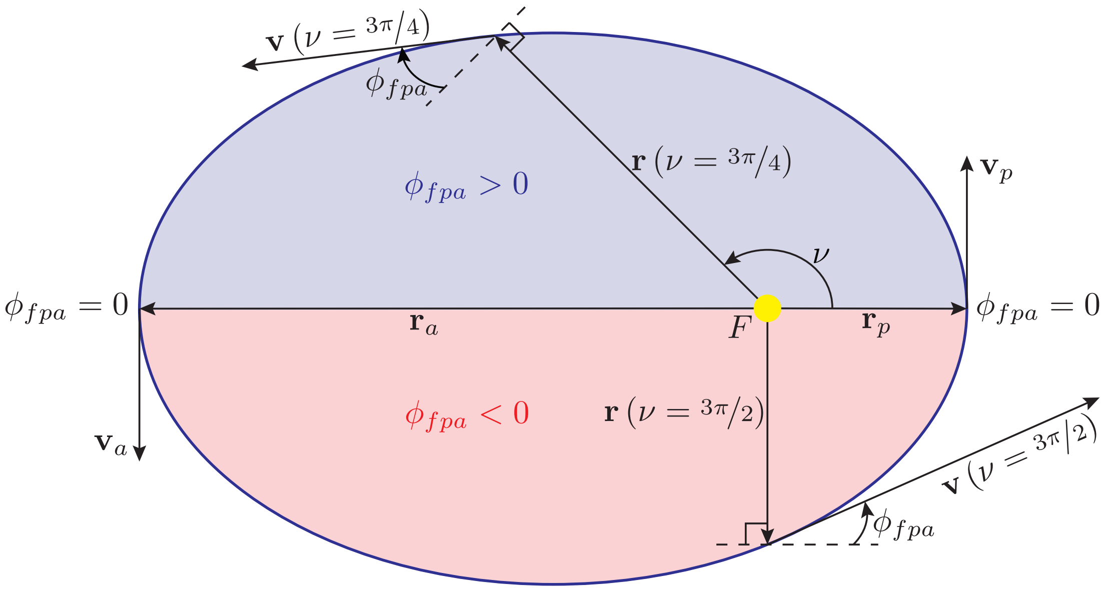
Impulsive Maneuvers and Δv

- If the time over which a force is applied is infinitesimal, then the position of the particle doesn't have time to change. An impulse over infinitesimal time produces a change **only** in velocity
- We call this model an impulsive maneuver (or burn)

$$\text{Original Orbit} \begin{cases} \mathbf{r} \\ \mathbf{v} \end{cases} + \quad \text{burn} \quad = \left. \begin{array}{rl} \mathbf{r} & \text{Original Position} \\ \mathbf{v} + \frac{\overline{\mathbf{F}}_P}{m_P} & \text{New Velocity} \end{array} \right\} \text{New Orbit}$$

- The velocity change due to the maneuver is $\boldsymbol{\Delta}\mathbf{v}$ and its magnitude is called Δv.
- NB: While the velocity change is a vector quantity and can be positive or negative, when talking about maneuvers, we primarily care about total fuel expended, and so all individual Δvs are positive
- NB: Real burns take place over extended periods of time. The only way to accurately model this is via numerical integration

Flight Path Angle



ϕ_{fpa}: The angle between the local horizontal and the velocity vector such that

$$h = \|\mathbf{r} \times \mathbf{v}\| = rv\cos(\phi_{fpa})$$

$$\cos\phi_{fpa} = \frac{r\dot{\nu}}{v} = \frac{1 + e\cos\nu}{\sqrt{1 + 2e\cos\nu + e^2}} \qquad \sin\phi_{fpa} = \frac{\dot{r}}{v} = \frac{e\sin\nu}{\sqrt{1 + 2e\cos\nu + e^2}}$$

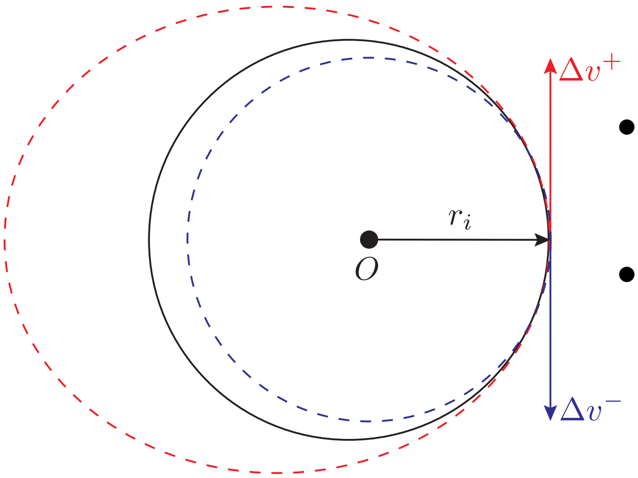
Tangential and Non-Tangential Burns

- **Tangential burns** are those where the added $\mathbf{\Delta v}$ is tangent to the initial and resulting orbits. All other burns are **non-tangential**
- Tangential burns must occur at $\phi_{fpa} = 0$
- Tangential burns are parallel to the orbital velocity vector

NB: Because an impulsive burn does not change the orbital positions, the orbits before and after a burn must intersect. Therefore, at least two impulsive maneuvers are needed to get to an orbit that doesn't intersect the original one.

Tangential Burns

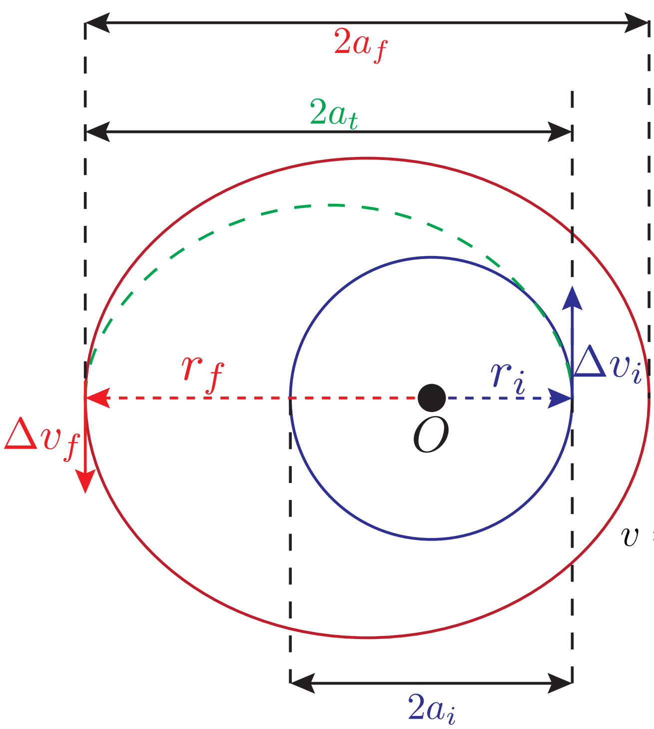
- $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad \Longrightarrow \quad a = \frac{\mu r}{2\mu - rv^2}$
- $e^2 = \mathbf{e}\cdot\mathbf{e} = \left\|\frac{\mathbf{v}\times\mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}\right\|^2 = \frac{\left(rv^2-\mu\right)^2}{\mu^2} + \frac{(\mathbf{r}\cdot\mathbf{v})v^2}{\mu} - \frac{2(\mathbf{r}\cdot\mathbf{v})^2\left(rv^2-\mu\right)}{\mu^2 r}$



- But at $\phi_{fpa} = 0$, $\mathbf{r}\cdot\mathbf{v} = 0$ so:
  $$e = \frac{|rv^2 - \mu|}{\mu}$$
- Increasing velocity at turning points also increases the semi-major axis and eccentricity

NB: As eccentricity cannot go below zero, burning from a circular orbit will always result in an *increase* in eccentricity, regardless of whether the semi-major axis increases or decreases.

Hohmann Transfers



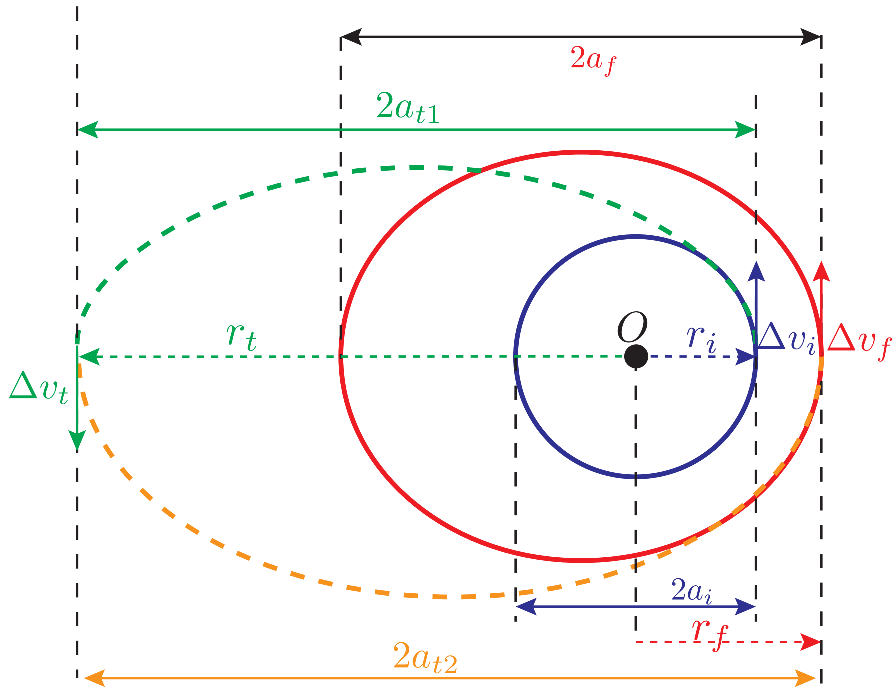
A Hohmann transfer requires two tangential burns.

$$a_{\text{transfer}} \equiv a_t = \frac{r_i + r_f}{2}$$
$$t_{\text{transfer}} = \frac{1}{2}T_P^{\text{transfer}} = \pi\sqrt{\frac{a_t^3}{\mu}}$$

$$\Delta v = |\Delta v_i| + |\Delta v_f|$$

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} \begin{cases} \Delta v_i = \underbrace{\sqrt{\frac{2\mu}{r_i} - \frac{\mu}{a_t}}}_{v_{t_i}} - \underbrace{\sqrt{\frac{2\mu}{r_i} - \frac{\mu}{a_i}}}_{v_i} \\ \Delta v_f = \underbrace{\sqrt{\frac{2\mu}{r_f} - \frac{\mu}{a_f}}}_{v_f} - \underbrace{\sqrt{\frac{2\mu}{r_f} - \frac{\mu}{a_t}}}_{v_{t_f}} \end{cases}$$

Bi-Elliptic Transfers



A bi-elliptic transfer requires three tangential burns and adds a new free parameter: the orbital radius at the second burn (r_t).

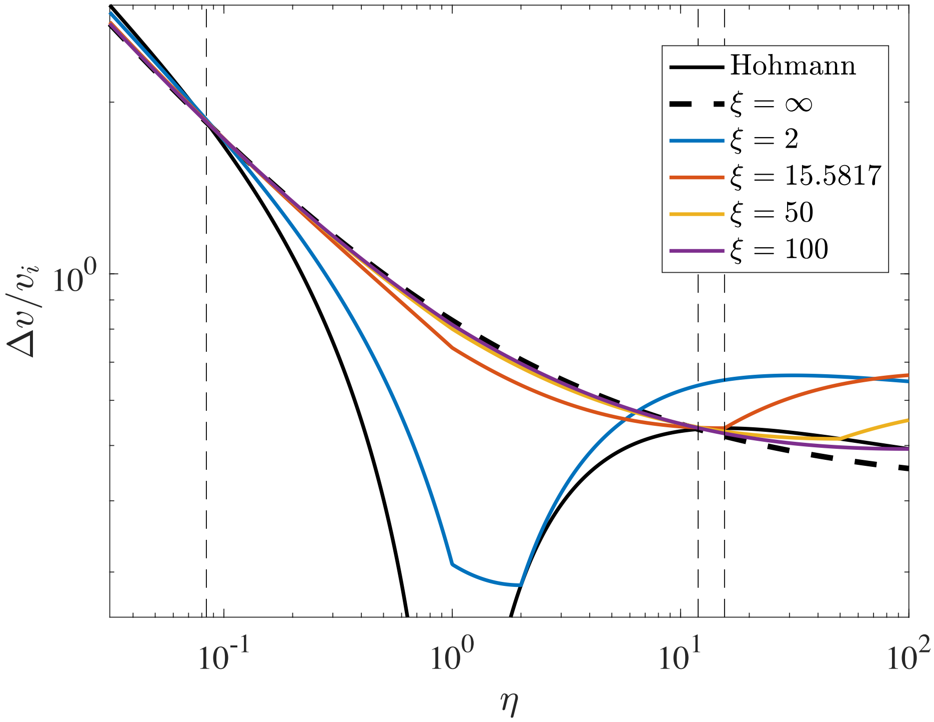
Hohmann vs. Bi-Elliptic

- Define: $\eta \triangleq \dfrac{a_f}{a_i}$ and $\xi \triangleq \dfrac{r_t}{a_i}$
- Hohmann: $\dfrac{|\Delta v_i| + |\Delta v_f|}{v_i} = \left| \sqrt{\dfrac{2\eta}{1+\eta}} + \sqrt{\dfrac{1}{\eta}} - \left(1 + \sqrt{\dfrac{2}{\eta(1+\eta)}}\right) \right|$
- Bi-Elliptic: $\dfrac{|\Delta v_i| + |\Delta v_t| + |\Delta v_f|}{v_i}$

$$= \left| \sqrt{\frac{2\xi}{1+\xi}} - 1 \right| + \left| \sqrt{\frac{2\eta}{\xi(\eta+\xi)}} - \sqrt{\frac{2}{\xi(1+\xi)}} \right| + \left| \sqrt{\frac{1}{\eta}} - \sqrt{\frac{2\xi}{\eta(\eta+\xi)}} \right|$$

- As $r_t \to \infty$: $\displaystyle\lim_{r_t \to \infty} \left(\frac{|\Delta v_i| + |\Delta v_t| + |\Delta v_f|}{v_i} \right) = \sqrt{2} - 1 + \left| \sqrt{\frac{1}{\eta}} - \sqrt{\frac{2}{\eta}} \right|$

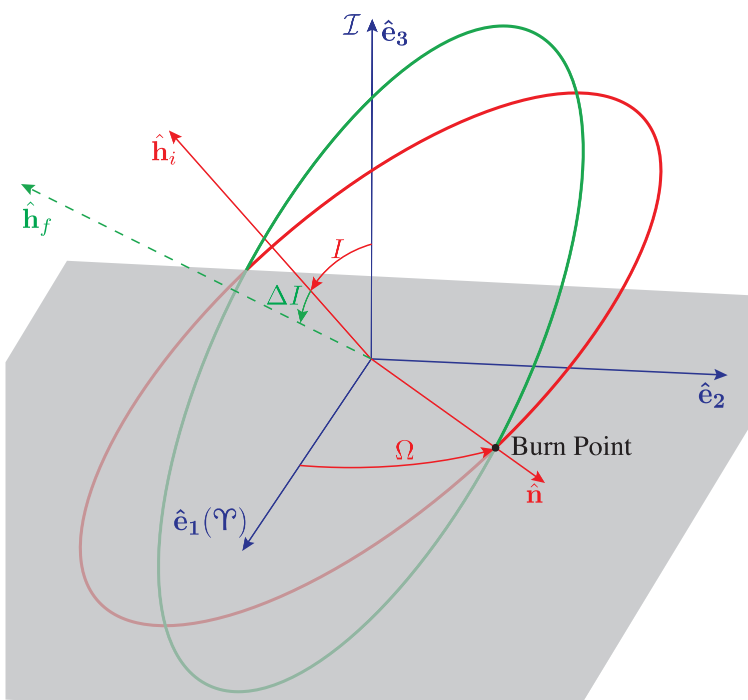
Hohmann vs. Bi-Elliptic (2)



- Hohmann maximum (for $\eta > 1$) occurs at $\eta = 15.5817$
- Hohmann and $\xi = \infty$ intersect at $\eta = 11.93876^{\pm 1}$. Hohmann is always more efficient in this range
- Bi-elliptic is typically more efficient below $\eta = 11.93876^{-1}$
- For $\eta > 15.5817$, bi-elliptic is more efficient for $\xi > \eta$

A Bi-elliptic transfer may have a lower Δv than the equivalent Hohmann transfer, but will take longer to complete.

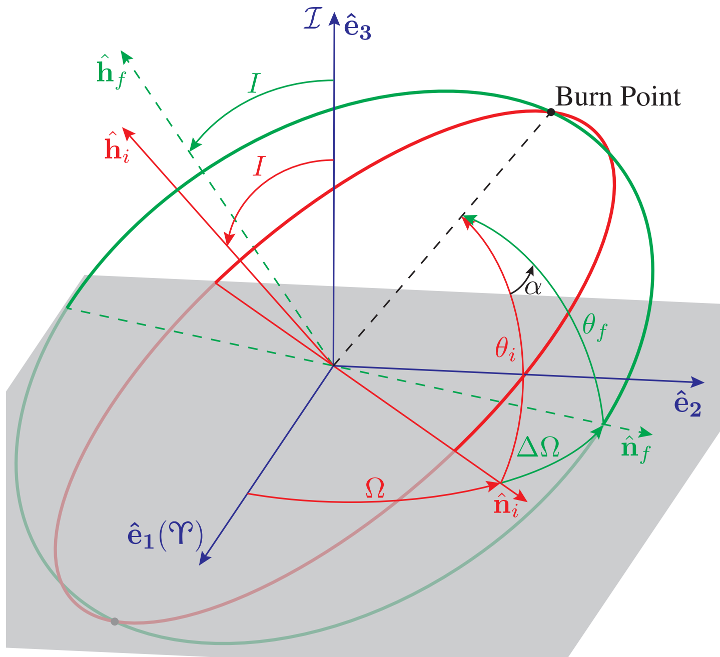
Inclination Changes (Super Costly!)



$$\boxed{\Delta v = 2 v_i \cos(\phi_{fpa}) \sin\left(\frac{\Delta I}{2}\right)}$$

- NB: Δv $\propto v_i$. For elliptical orbits, one of the two nodes will be less costly
- For $\Delta I = 60^\circ$, $\Delta v = v_i$
- To leave Ω unchanged, burn must occur on the line of nodes

Ascending Node Change



- In general, elliptical orbits require multiple burns to change **only** Ω, but circular orbits can do it in one
- The burn occurs on the original orbit at argument of latitude $\theta_i = \omega_i + \nu_i$ resulting with the spacecraft on the final orbit at θ_f, with a burn angle α

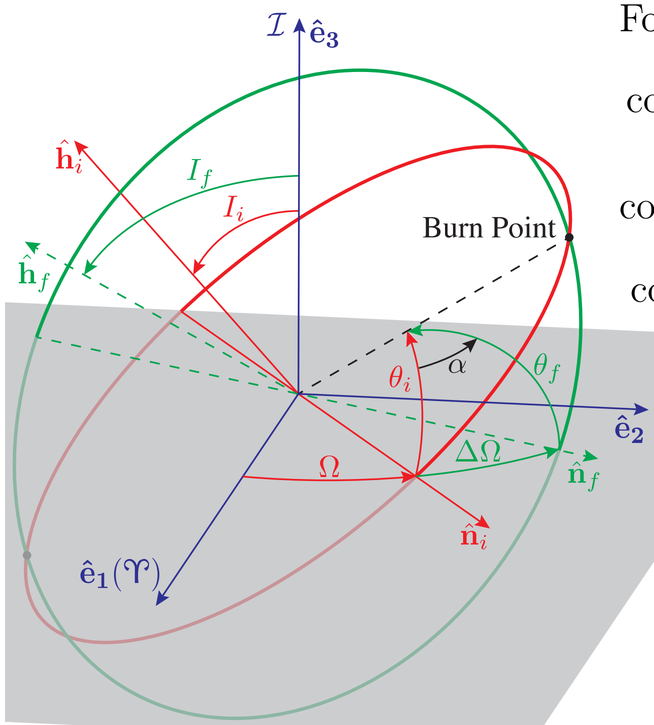
$$\cos(\theta_i) = \tan I \left(\frac{\cos(\Delta\Omega) - \cos\alpha}{\sin\alpha} \right)$$

$$\cos(\theta_f) = \cos I \sin I \left(\frac{1 - \cos(\Delta\Omega)}{\sin\alpha} \right)$$

$$\cos(\alpha) = \cos^2 I + \sin^2 I \cos(\Delta\Omega)$$

$$\Delta v^{\text{circ}} = 2 v_i \sin\left(\frac{\alpha}{2}\right)$$

Ascending Node and Inclination Change



For circular orbits:

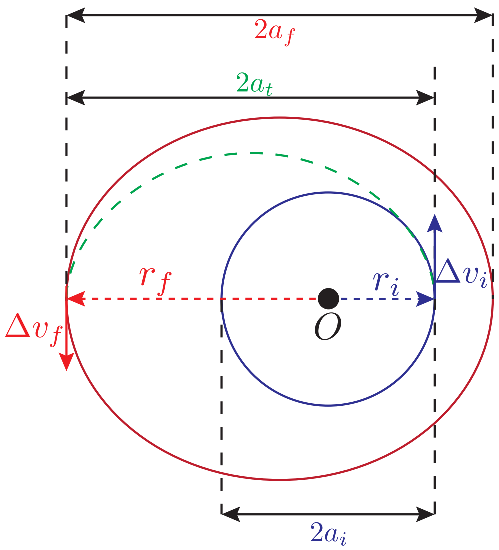
$$\cos(\theta_i) = \frac{\sin(I_f)\cos(\Delta\Omega) - \cos(\alpha)\sin(I_i)}{\sin(\alpha)\cos(I_i)}$$

$$\cos(\theta_f) = \frac{\cos(I_i)\sin(I_f) - \sin(I_i)\cos(I_f)\cos(\Delta\Omega)}{\sin(\alpha)}$$

$$\cos(\alpha) = \cos(I_i)\cos(I_f) + \sin(I_i)\sin(I_f)\cos(\Delta\Omega)$$

$$\boxed{\Delta v^{\text{circ}} = 2 v_i \sin\left(\frac{\alpha}{2}\right)}$$

Hohmann Transfer + Inclination Change



For a total inclination change of ΔI:

- Change by $x\Delta I$ on initial burn
- Change by $(1-x)\Delta I$ on final burn

- Select x to minimize total Δv:

$$\sin(x\Delta I) = \frac{\Delta v_i v_f v_{t_f} \sin((1-x)\Delta I)}{\Delta v_f v_i v_{t_i}}$$

- A good approximation is:

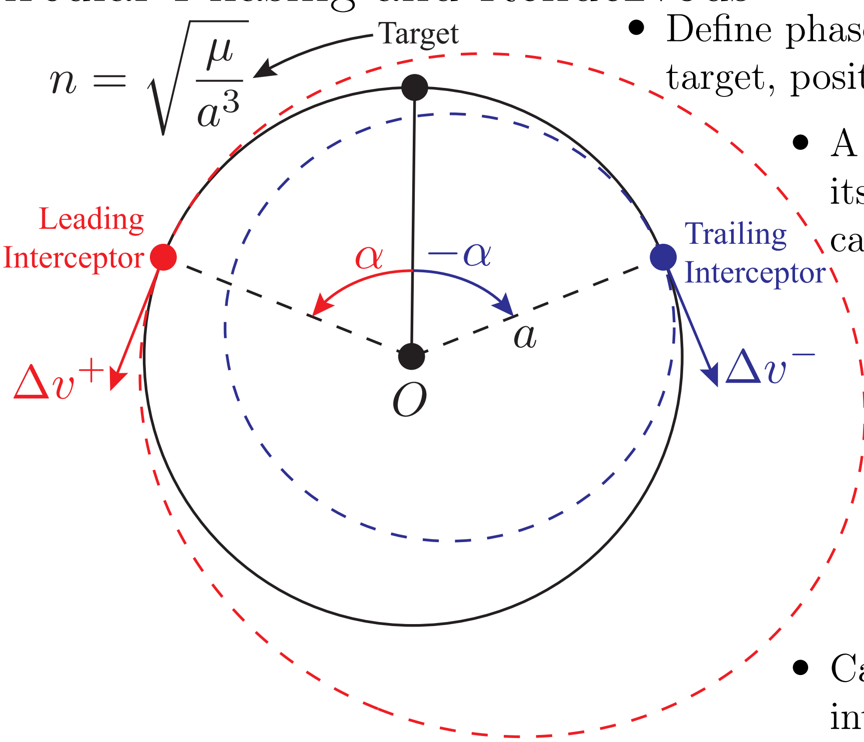
$$x \approx \frac{1}{\Delta I} \tan^{-1}\left(\frac{\sin(\Delta I)}{\frac{v_i v_{t_i}}{v_f v_{t_f}} + \cos(\Delta I)}\right)$$

- Δvs for the combined maneuvers are:

$$\Delta v_i = \sqrt{v_i^2 + v_{t_i}^2 - 2v_i v_{t_i} \cos(x\Delta I)}$$

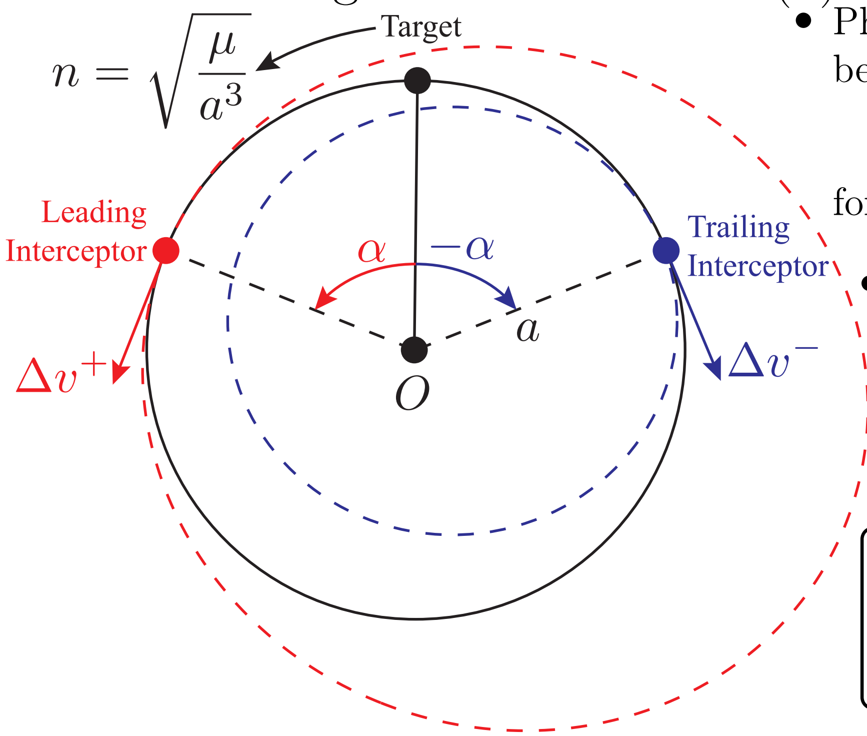
$$\Delta v_f = \sqrt{v_f^2 + v_{t_f}^2 - 2v_f v_{t_f} \cos((1-x)\Delta I)}$$

Circular Phasing and Rendezvous



- Define phase angle α between interceptor and target, positive in direction of orbital motion
  - A trailing interceptor needs to put itself onto a shorter-period orbit to catch up to the target
    - A leading interceptor needs to put itself onto a longer-period orbit to allow the target to catch up
      - Phasing orbit perigee must be greater than the central body radius (plus atmosphere)
  - Can utilize multiple target and interceptor orbits to rendezvous

Circular Phasing and Rendezvous (2)



- Phasing orbit period is equal to time between interceptor and target:
$$T_{\text{phase}} = \frac{2\pi j + \alpha}{n}$$
for j target revolutions

- Phasing orbit semi-major axis is:
$$a_{\text{phase}} = \left(\mu \left(\frac{T_{\text{phase}}}{2\pi k}\right)^2\right)^{\frac{1}{3}}$$
for k interceptor revolutions

$$\boxed{\Delta v = 2\left|\sqrt{\frac{2\mu}{a} - \frac{\mu}{a_{\text{phase}}}} - \sqrt{\frac{\mu}{a}}\right|}$$

General Impulsive Maneuvers

- Remember: $a, e, I, \omega, \Omega, \nu(t) \Longleftrightarrow \mathbf{r}(t), \mathbf{v}(t)$
- Before Burn: $\left.\begin{array}{l}\mathbf{r}_i\\ \mathbf{v}_i\end{array}\right\} a_i, e_i, I_i, \omega_i, \Omega_i, \nu_i(t)$
- After Burn: $\left.\begin{array}{l}\mathbf{r}_f \equiv \mathbf{r}_i\\ \mathbf{v}_f = \mathbf{v}_i + \mathbf{\Delta v}\end{array}\right\} a_f, e_f, I_f, \omega_f, \Omega_f, \nu_f(t)$
- You can always solve for the $\mathbf{\Delta v}$ to produce the desired change in orbital elements as long as the initial and final orbits intersect at the burn location
- These maneuvers are not guaranteed to be feasible or optimal
- Typical approach is numerical optimization

Continuous Thrust Trajectories

- All impulsive maneuvers are an idealization, but this model becomes completely inapplicable in cases of low-thrust propulsion where burns are continuous (or nearly continuous) throughout the trajectory
- The only way to solve such problems is via optimal control and numerical optimization
- Resulting trajectories resemble spirals more than the partial conic sections of impulsive trajectories

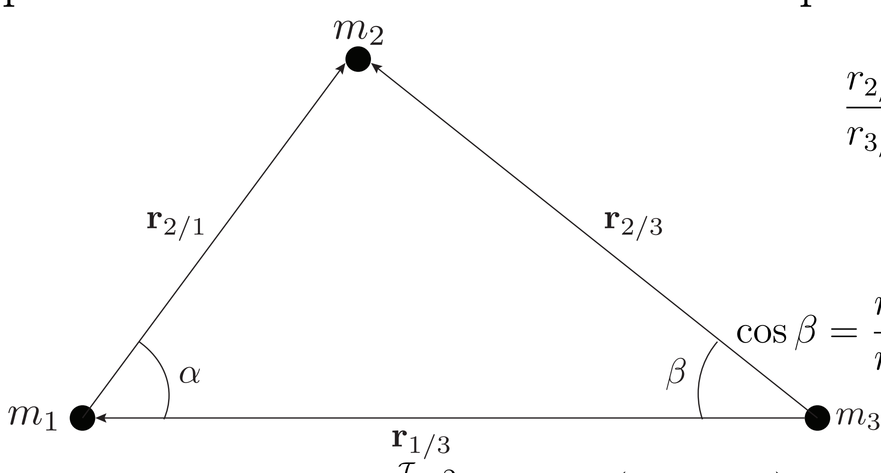
Optimization Resources

- EMTG (`https://github.com/nasa/EMTG`)
- GMAT (`https://sourceforge.net/projects/gmat/`)
- PAGMO/PyGMO (`https://github.com/esa/pagmo2`)
- SNOPT (`https://web.stanford.edu/group/SOL/snopt.htm`)
- scipy.optimize (`https://docs.scipy.org/doc/scipy/reference/optimize.html`)
- MATLAB Optimization Toolbox (`https://www.mathworks.com/help/optim/index.html`)

Sphere of Influence

In our discussion of perturbations, we saw that we can frequently describe the gravitational effects of two different major bodies on a spacecraft by treating one body as the central body and the other as a perturber. The central and perturbing body may change depending on the location of the spacecraft—when a spacecraft is in Earth orbit, clearly Earth is the central body and the moon is a perturber, but when the spacecraft is in orbit about the moon, the relationship is reversed. In order to determine which body is which, we need to find which body is exerting the greatest force on the spacecraft at a given point in time. We do so by comparing the gravitational forces of two bodies acting on the spacecraft, which allows us to define a radius about one of the bodies within which it is the central body. This radius defines the sphere of influence.

Sphere of Influence Derivation Setup



$$\frac{r_{2/3}}{r_{3/1}} = \left[1 - 2\frac{r_{2/1}}{r_{3/1}}\cos\alpha + \left(\frac{r_{2/1}}{r_{3/1}}\right)^2\right]^{\frac{1}{2}}$$

$$\cos\beta = \frac{r_{3/1}}{r_{2/3}} - \frac{r_{2/1}}{r_{2/3}}\cos\alpha$$

$$m_1 \text{ dominant}: \quad {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/1} + \frac{G(m_1+m_2)}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = -Gm_3\left(\frac{\mathbf{r}_{3/1}}{\|\mathbf{r}_{3/1}\|^3} + \frac{\mathbf{r}_{2/3}}{\|\mathbf{r}_{2/3}\|^3}\right)$$

$$m_3 \text{ dominant}: \quad {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/3} + \underbrace{\frac{G(m_2+m_3)}{\|\mathbf{r}_{2/3}\|^3}\mathbf{r}_{2/3}}_{\text{Central}} = \underbrace{-Gm_1\left(\frac{\mathbf{r}_{2/1}}{\|\mathbf{r}_{2/1}\|^3} + \frac{\mathbf{r}_{1/3}}{\|\mathbf{r}_{1/3}\|^3}\right)}_{\text{Perturbing}}$$

Sphere of Influence

$$\left(\frac{\|\mathbf{f}_{\text{perturbing}}\|}{\|\mathbf{f}_{\text{central}}\|}\right)_{m_1} = \frac{Gm_3\left[\left(\frac{\mathbf{r}_{2/3}}{r_{2/3}^3}+\frac{\mathbf{r}_{3/1}}{r_{3/1}^3}\right)\cdot\left(\frac{\mathbf{r}_{2/3}}{r_{2/3}^3}+\frac{\mathbf{r}_{3/1}}{r_{3/1}^3}\right)\right]^{\frac{1}{2}}}{G(m_1+m_2)r_{2/1}^{-2}}$$

$$= \frac{m_3}{m_2+m_1}\frac{\left(r_{2/1}/r_{3/1}\right)^2}{\left(r_{2/3}/r_{3/1}\right)^2}\left[1+\left(\frac{r_{2/3}}{r_{3/1}}\right)^4-2\left(\frac{r_{2/3}}{r_{3/1}}\right)\left(1-\frac{r_{2/1}}{r_{3/1}}\cos\alpha\right)\right]^{\frac{1}{2}}$$

$$\left(\frac{\|\mathbf{f}_{\text{perturbing}}\|}{\|\mathbf{f}_{\text{central}}\|}\right)_{m_3} = \frac{m_1}{m_2+m_3}\left(\frac{r_{2/1}}{r_{3/1}}\right)^{-2}\left(\frac{r_{2/3}}{r_{3/1}}\right)^2\left[1+\left(\frac{r_{2/1}}{r_{3/1}}\right)^4-2\left(\frac{r_{2/1}}{r_{3/1}}\right)^2\cos\alpha\right]^{\frac{1}{2}}$$

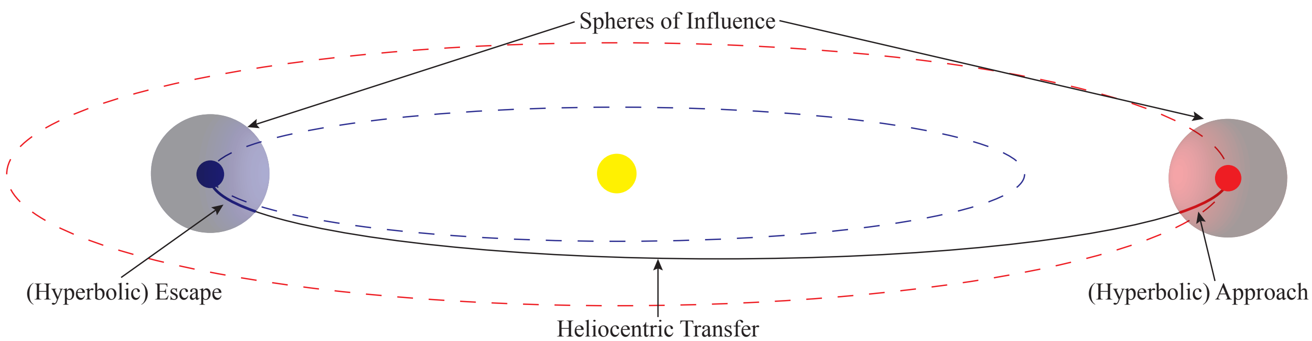
$$\text{Intersection}: \left(\frac{r_{2/1}}{r_{3/1}}\right)^4 = \frac{m_1(m_1+m_2)}{m_3(m_2+m_3)}\left(\frac{r_{2/3}}{r_{3/1}}\right)^4\left[\frac{1+\left(\frac{r_{2/1}}{r_{3/1}}\right)^4-2\left(\frac{r_{2/1}}{r_{3/1}}\right)^2\cos\alpha}{1+\left(\frac{r_{2/3}}{r_{3/1}}\right)^4-2\left(\frac{r_{2/3}}{r_{3/1}}\right)\left(1-\frac{r_{2/1}}{r_{3/1}}\cos\alpha\right)}\right]^{\frac{1}{2}}$$

$$\boxed{\left(\frac{r_{2/1}}{r_{3/1}}\right)\approx\left(\frac{m_1}{m_3}\right)^{\frac{2}{5}} \quad\Rightarrow\quad r_{\text{SOI}}\approx a_{\text{planet}}\left(\frac{m_{\text{planet}}}{m_{\text{sun}}}\right)^{\frac{2}{5}}}$$

The Patched Conic Approximation

The concept of the sphere of influence allows us to model more complex trajectories involving multiple central bodies. This leads us to the patched conic approximation, where we stitch together multiple two-body orbits (conic sections). In the case of interplanetary transfers, we typically consider three orbits—a hyperbolic escape orbit (typically originating from the Earth), a heliocentric transfer orbit (which may be a portion of an open or closed orbit about the sun), and a hyperbolic approach orbit to whatever body we wish to go to. The first and last orbits occur within the spheres of influence of the origin and destination bodies, while the middle portion is an orbit about the sun. The three are patched together at the edges of the relevant spheres of influence. It is important to remember that our two-body model is explicitly wrong exactly at the edge of a sphere of influence, as the gravitational forces acting on the spacecraft from two different bodies are nearly equal at this point. Therefore, the patched conic is only a starting point, which must then be refined within a full-force model. Nevertheless, it is an incredibly useful starting calculation, and is very often the first step in any interplanetary trajectory design.

The Patched Conic Approximation



$$r_{\text{SOI}} = a_p \left(\frac{m_p}{m_\odot}\right)^{\frac{2}{5}}$$

Departure

- Departure is modeled as a hyperbolic orbit (with Earth as the central body)
- We define $v_\infty = \|\mathbf{v}_\infty\|$ as the velocity at an infinitely large orbital radius
$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = \frac{v_\infty^2}{2} - \frac{\mu}{\infty} \quad \implies \quad v_\infty^2 = 2\mathcal{E}$$
- Recall that escape velocity corresponds to zero specific energy (parabolic orbit):
$$0 = \frac{v_{\text{esc}}^2}{2} - \frac{\mu}{r} \quad \implies \quad v_{\text{esc}} = \sqrt{\frac{2\mu}{r}}$$
- We define the **Characteristic Energy**: $C_3 = 2\mathcal{E} = v_\infty^2$
$$C_3 = 2\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = v^2 - v_{\text{esc}}^2$$
- The characteristic energy is a measure of **excess velocity** (or specific energy) above what is required to escape.
- The typical v_∞ for interplanetary missions is ~ 5 km/s

Arrival

- Arrival is just departure in reverse (incoming hyperbolic orbit about the target body)
- If you want to place the spacecraft into a circular orbit about the target, the insertion Δv is calculated the same as the injection Δv

$$\Delta v_{\text{insertion}} = \sqrt{\frac{2\mu_{\text{target}}}{r_{\text{target}}} + \left(v_\infty^{\text{target}}\right)^2} - \sqrt{\frac{\mu_{\text{target}}}{r_{\text{target}}}}$$

- If the target body has an atmosphere, you can use **aerobraking** to save fuel

Parking Orbits and Departure Design

- Assume an initial (circular) parking orbit about the Earth and an impulsive thrust to put the spacecraft on the escape hyperbola

$$\Delta v_{\text{injection}} = \sqrt{2\left(\frac{\mu}{r} + \mathcal{E}\right)} - \sqrt{\frac{\mu}{r}} = \sqrt{\frac{2\mu_\oplus}{r_{\text{park}}} + v_\infty^2} - \sqrt{\frac{\mu_\oplus}{r_{\text{park}}}}$$

- For an initial 200 km altitude, $\Delta v_{\text{injection}} = 3.22$ km/s to achieve escape, and 4.31 for $v_\infty = 5$ km/s
- Define the **right ascension** and **declination of the launch asymptote**:

$$\text{RLA} \triangleq \tan^{-1}\left(\frac{\mathbf{v}_\infty \cdot \hat{\mathbf{e}}_2'}{\mathbf{v}_\infty \cdot \hat{\mathbf{e}}_1'}\right) \qquad \text{DLA} \triangleq \sin^{-1}\left(\frac{\mathbf{v}_\infty \cdot \hat{\mathbf{e}}_3'}{v_\infty}\right)$$

  for $\hat{\mathbf{e}}_1', \hat{\mathbf{e}}_2', \hat{\mathbf{e}}_3'$ defining an ECI equatorial reference frame
- Designing a departure trajectory is designing the $\mathbf{v}_\infty$ vector: selecting its magnitude and direction

Oberth Effect

- The largest energy change is obtained by thrusting at the highest velocity:

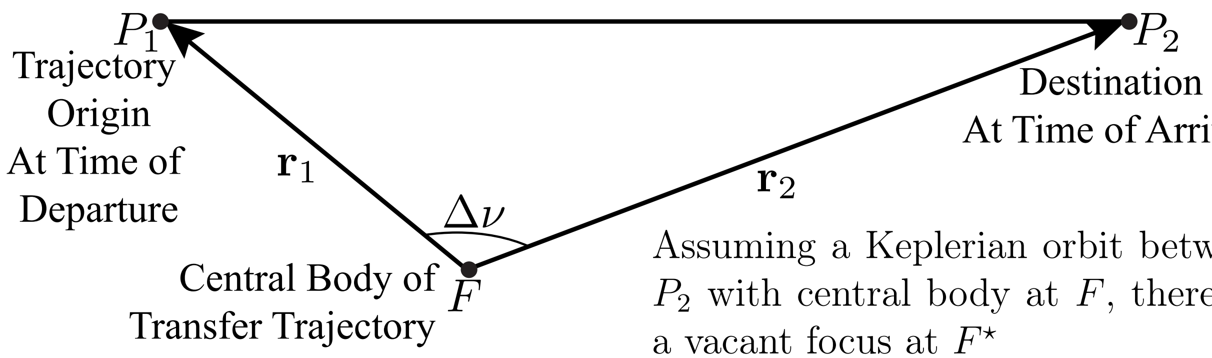
$$
\begin{aligned}
\mathcal{E} &= \frac{v^2}{2} - \frac{\mu}{r} \\
\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}v} &= v \\
\mathrm{d}\mathcal{E} &= v\,\mathrm{d}v
\end{aligned}
$$

- This is known as the **Oberth Effect**
- Biggest energy gain will occur at the lowest altitudes (but beware drag forces)

Initial and Boundary Value Problems

So far, all of the orbital propagations that we've considered have been initial value problems (IVP): we have some starting (or ending) condition and we propagate our orbits forward (or backward) in time. However, the middle portion of our patched conic trajectory must be constrained on both ends: we have to start at the sphere of influence of our originating body, and end up at the sphere of influence of our destination body. This is a boundary value problem (BVP), and the particular BVP associated with finding a two-body orbit connecting two specific points in space is called Lambert's problem. It is crucial to remember that Lambert's problem is not only about connecting two points in space, but also in time. If we are going from Earth to Mars, then the location of Mars at the time of our departure is not where we need to go, as Mars will have moved along its own orbit during our flight. Thus, the problem becomes that of finding a connecting trajectory between origin and destination, as well as optimal times when to leave and arrive. We will first consider Lambert's problem under the assumption that we know the location of our origin and destination points (i.e., we know when we wish to leave and arrive), and will then study how to use the solutions to Lambert's problem in order to actually set those departure and arrival times.

Lambert's Problem



Assuming a Keplerian orbit between P_1 and P_2 with central body at F, there must exist a vacant focus at $F^\star$

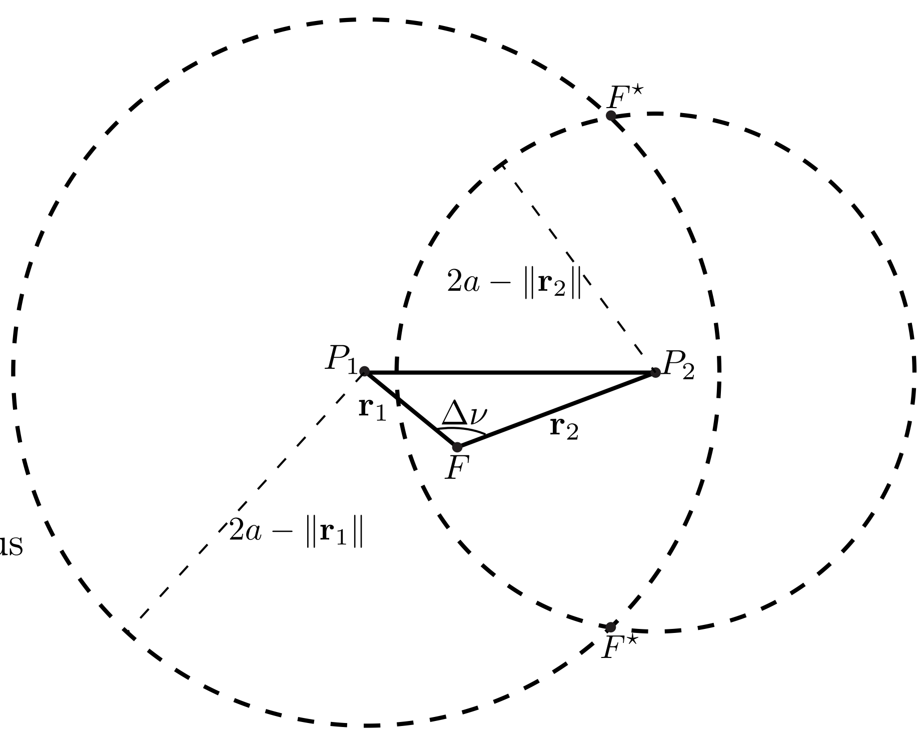
If the transfer between P_1 and P_2 is an ellipse, we know $r' + r = 2a$ so:

$$\left.\begin{matrix} \overline{P_1F} + \overline{P_1F^\star} \\ \overline{P_2F} + \overline{P_2F^\star} \end{matrix}\right\} = 2a \quad \Longrightarrow \quad \begin{matrix} \overline{P_1F^\star} = 2a - \|\mathbf{r}_1\| \\ \overline{P_2F^\star} = 2a - \|\mathbf{r}_2\| \end{matrix}$$

Lambert's Problem: Location of the Vacant Focus

- Vacant focus must be at intersection of two circles centered at P_1 and P_2 with radii of $2a - \|\mathbf{r}_1\|$ and $2a - \|\mathbf{r}_2\|$, respectively

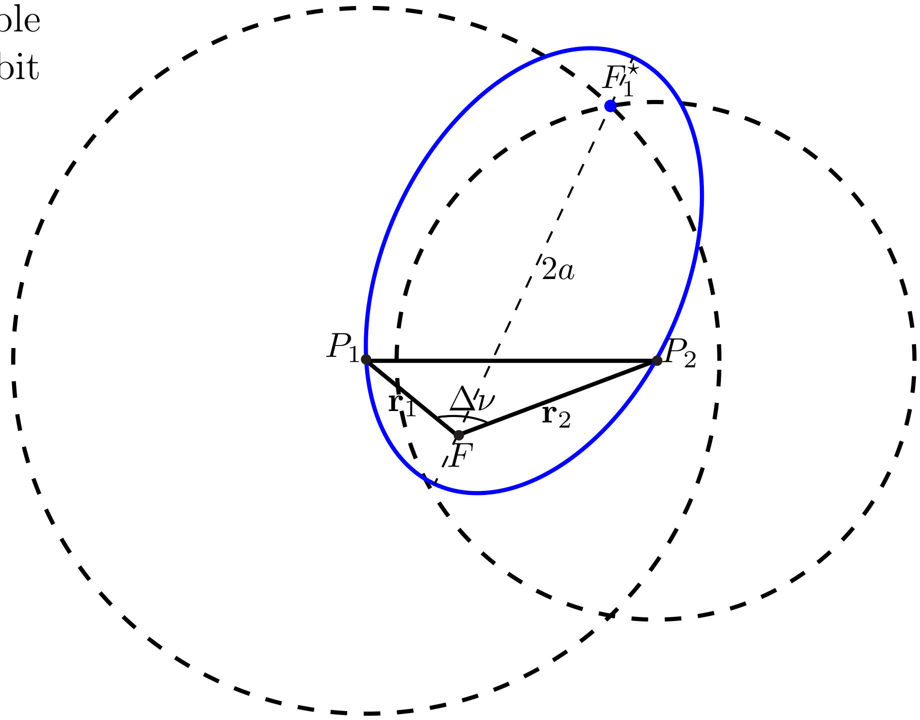
- Selecting transfer orbit a determines the possible locations of the vacant focus and sets the transfer orbit specific energy ($\mathcal{E} = -\dfrac{\mu}{2a}$) and period



Lambert's Problem: Transfer Orbit Eccentricity

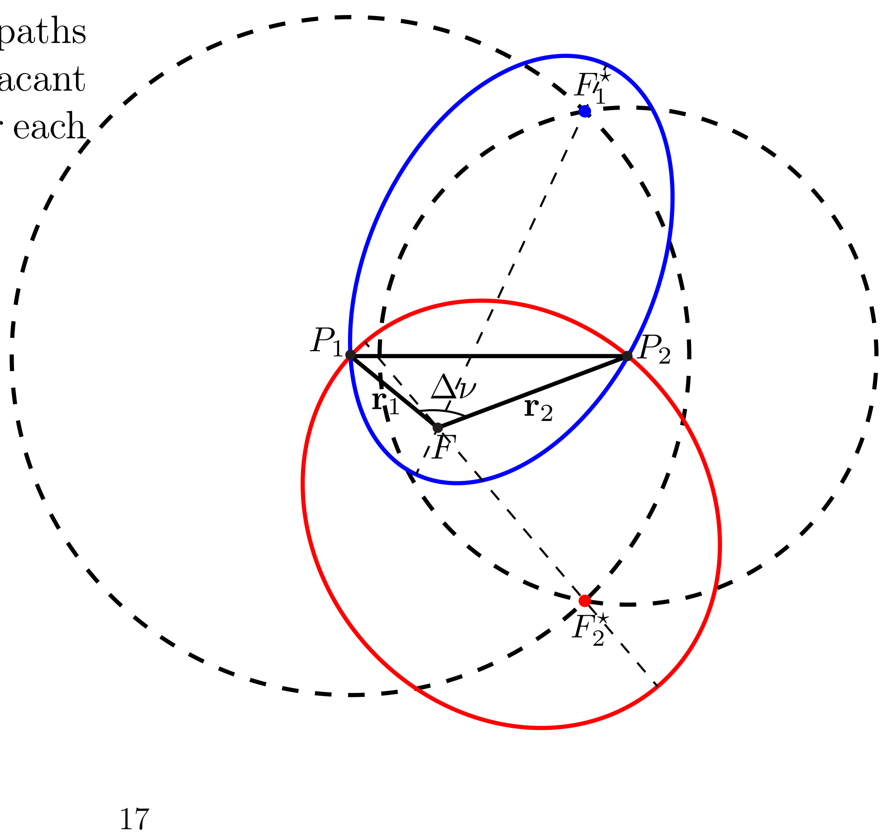
Selecting one of the two possible vacant foci sets the transfer orbit eccentricity:

$$\overline{FF^{\star}} = 2c = 2ae$$



Lambert's Problem: Closed Transfer Orbits

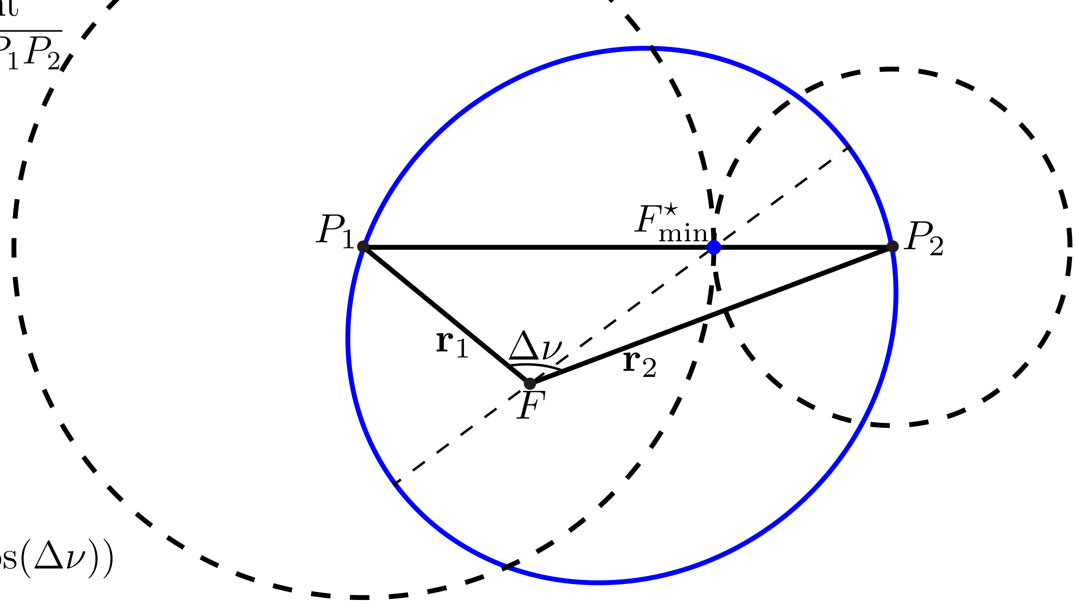
There are four possible transfer paths for each semi-major axis: 2 vacant foci, and 2 directions of travel for each



Lambert's Problem: Minimum Energy Transfer

There is always a **unique** elliptical orbit minimizing energy, where the vacant focus lies on the chord $\overline{P_1P_2}$

$$s \triangleq \frac{\|\mathbf{r}_1\| + \|\mathbf{r}_2\| + \|\mathbf{r}_{1/2}\|}{2}$$



$$a_{\min} = \frac{s}{2}$$

$$e_{\min} = \sqrt{1 - \frac{2\ell_{\min}}{s}}$$

$$\ell_{\min} = \frac{\|\mathbf{r}_1\|\|\mathbf{r}_2\|}{\|\mathbf{r}_{1/2}\|}(1 - \cos(\Delta\nu))$$

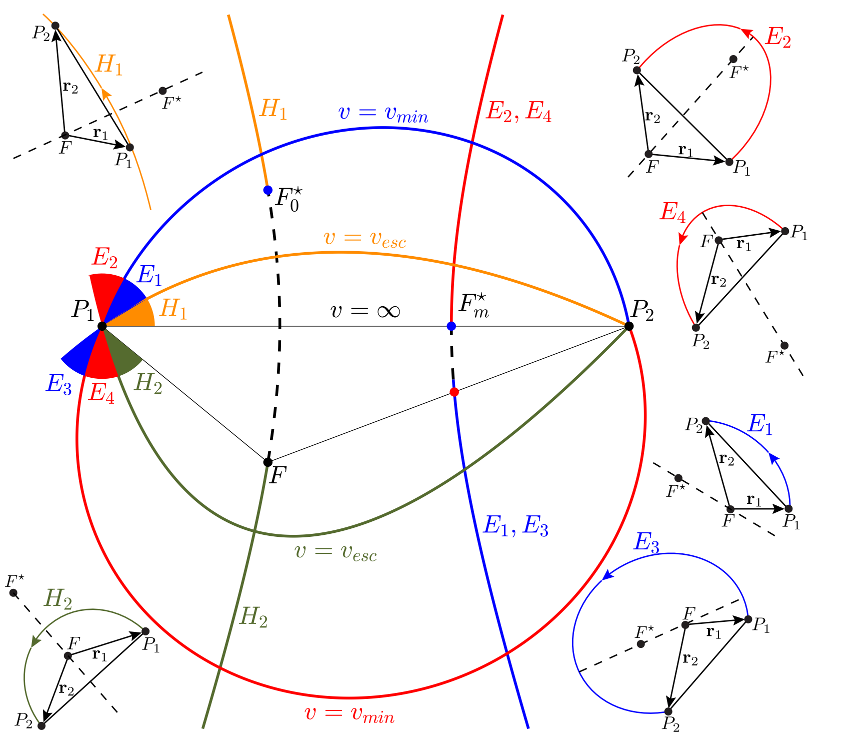
Lambert's Problem: All Possible Transfers

Location of the vacant focus is given by the hyperbola:

$$a_F = \frac{\|\mathbf{r}_1\| - \|\mathbf{r}_2\|}{2}$$

$$e_F = \left|\frac{\|\mathbf{r}_{1/2}\|}{\|\mathbf{r}_1\| - \|\mathbf{r}_2\|}\right|$$

NB: This is **not** a transfer orbit itself.



Based on Kaplan (1976)

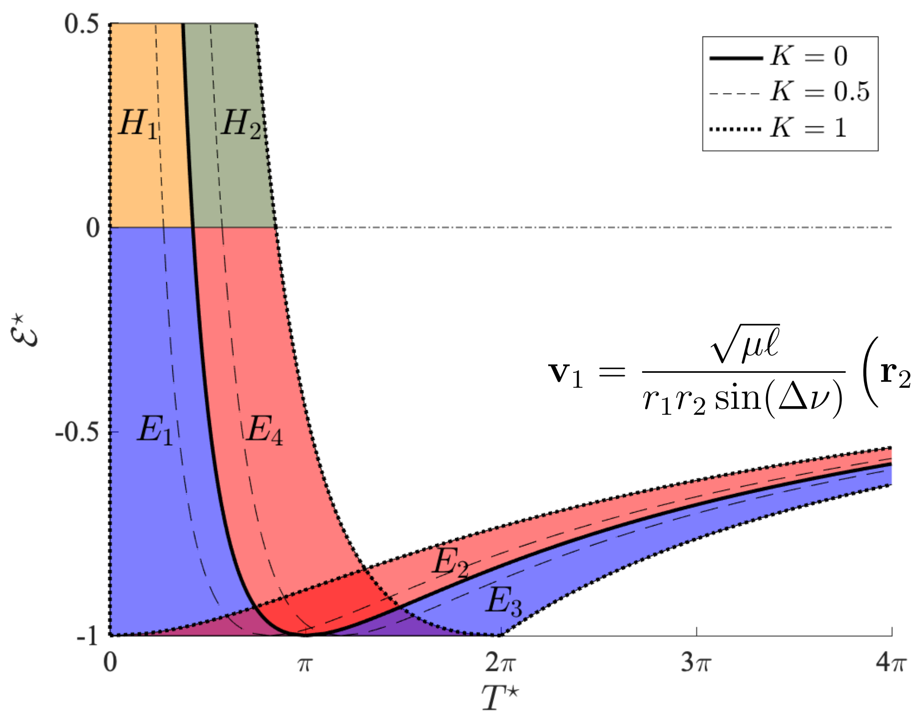
Lambert's Time of Flight Theorem

	$t = \dfrac{1}{\sqrt{\mu}} \displaystyle\int_{s-c}^{s} \dfrac{r}{\sqrt{2r - r^2/a}}\,\mathrm{d}r$	ℓ
$E_{1/4}$	$\dfrac{T_P}{2\pi}\left[(\alpha - \sin\alpha) \mp (\beta - \sin\beta)\right]$	$\dfrac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\dfrac{\alpha\pm\beta}{2}\right)$
$E_{2/3}$	$T_P - \dfrac{T_P}{2\pi}\left[(\alpha - \sin\alpha) \pm (\beta - \sin\beta)\right]$	$\dfrac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\dfrac{\alpha\mp\beta}{2}\right)$
Parabolas	$\dfrac{1}{3}\sqrt{\dfrac{2}{\mu}}\left[s^{\frac{3}{2}} \mp (s-c)^{\frac{3}{2}}\right]$	$\dfrac{4(s-r_1)(s-r_2)}{c^2}\left[\sqrt{\dfrac{s}{2}} \pm \sqrt{\dfrac{s-c}{2}}\right]^2$
$H_{1/2}$	$\sqrt{\dfrac{-a^3}{\mu}}\left[(\sinh\gamma - \gamma) \mp (\sinh\delta - \delta)\right]$	$\dfrac{-4a(s-r_1)(s-r_2)}{c^2}\sinh^2\left(\dfrac{\gamma\pm\delta}{2}\right)$

$$c = \|\mathbf{r}_{1/2}\| \qquad r_1 = \|\mathbf{r}_1\| \qquad r_2 = \|\mathbf{r}_2\| \qquad 2s = \|\mathbf{r}_1\| + \|\mathbf{r}_2\| + \|\mathbf{r}_{1/2}\|$$

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}} \quad \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{s-c}{2a}} \quad \sinh\left(\frac{\gamma}{2}\right) = \sqrt{\frac{s}{-2a}} \quad \sinh\left(\frac{\delta}{2}\right) = \sqrt{\frac{s-c}{-2a}}$$

Lambert's Problem Non-Dimensionalized



$$\mathcal{E}^\star \triangleq -\frac{a_{\min}}{a}$$

$$T^\star \triangleq \sqrt{\frac{\mu}{a_{\min}^3}}t$$

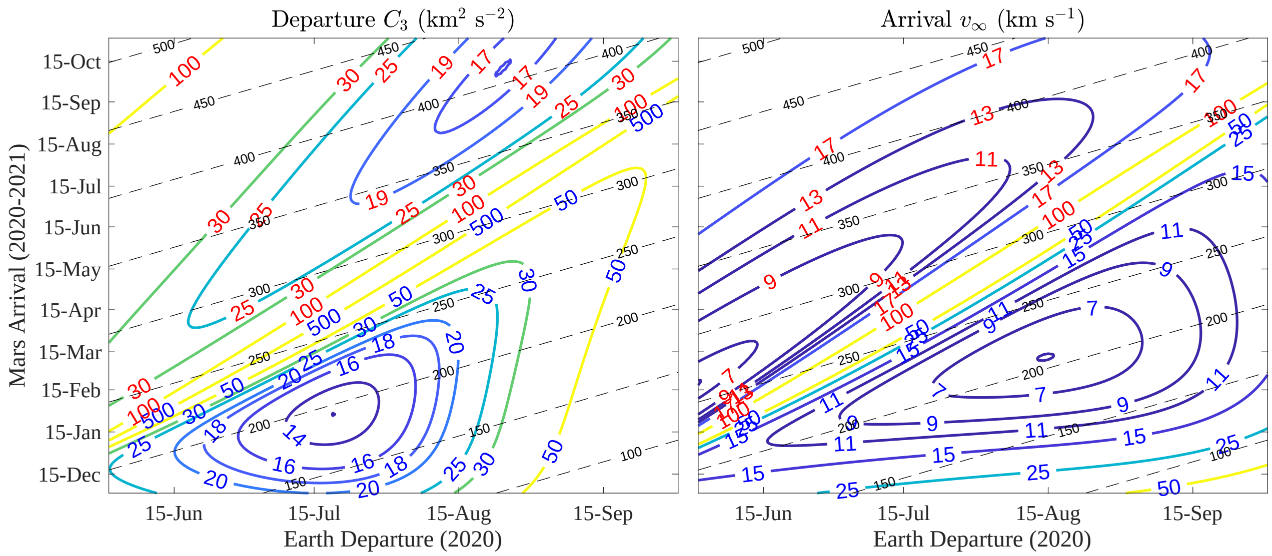
$$K \triangleq 1 - \frac{\|\mathbf{r}_{1/2}\|}{s}$$

$$\mathbf{v}_1 = \frac{\sqrt{\mu\ell}}{r_1 r_2 \sin(\Delta\nu)}\left(\mathbf{r}_2 - \left(1 - \frac{r_2}{\ell}\left(1 - \cos(\Delta\nu)\right)\right)\mathbf{r}_1\right)$$

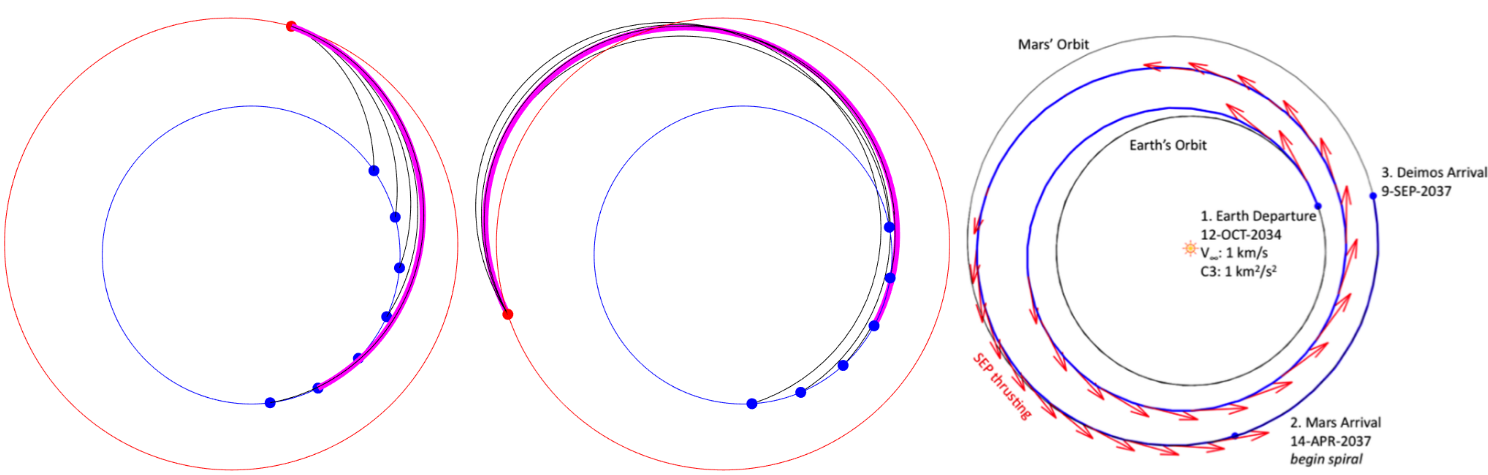
Based on Kaplan (1976)

Porkchop Plots

Porkchop plots are a fundamental trajectory design tool, plotting contours of Δv (or associated values) as functions of departure and arrival times for a single transfer. The dashed lines represent constant transfer times.



Transfer Types



Type I
$\Delta\nu < 180°$

Type II
$180° < \Delta\nu < 360°$

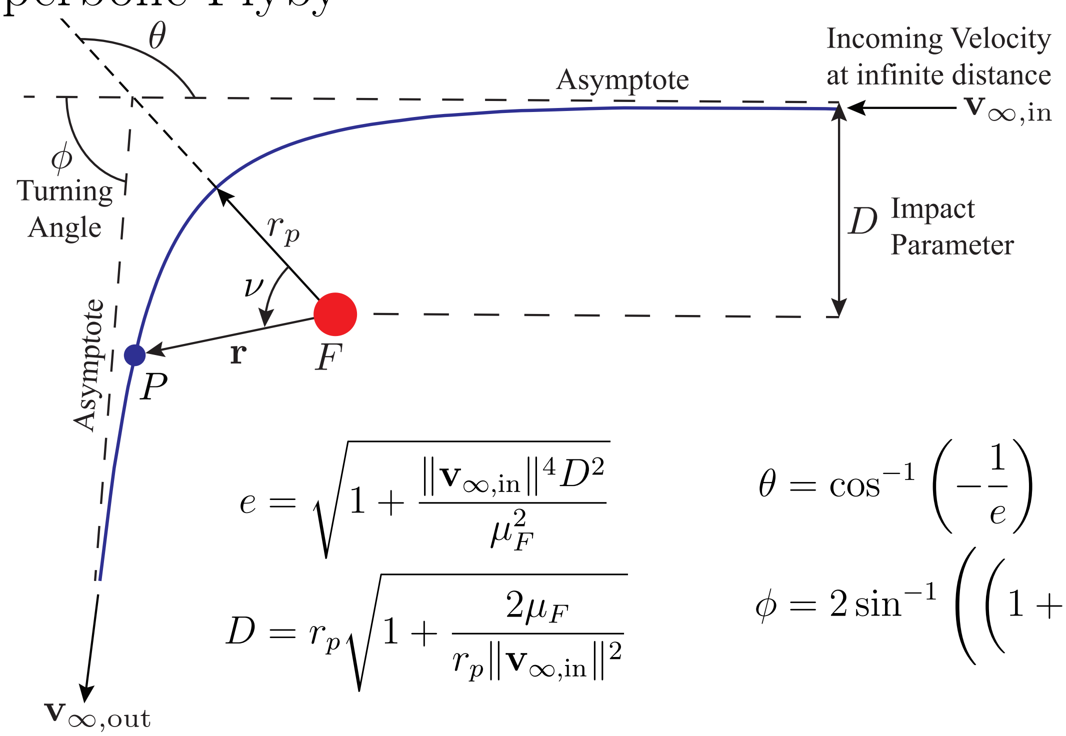
Type III
$\Delta\nu > 360°$

From: Strange et al. (2013)

Gravity Assists

An incredibly important tool in interplanetary and deep space trajectories is the gravitational assist, or hyperbolic flyby. We can use the gravitational interaction of our spacecraft with a solar system body (typically one of the planets) in order to modify its heliocentric trajectory. We again base our analysis on the patched conic approximation: the hyperbolic trajectory occurs within the assisting body's sphere of influence and is treated as a purely two-body orbit of the spacecraft about the assisting body. As such, this means that energy is conserved, and therefore the magnitude of the spacecraft's velocity with respect to the assisting body remains **unchanged** throughout the assist. The purpose of the assist is to change the direction of the velocity, which changes the spacecraft's heliocentric velocity vector, and can therefore produce a change in the magnitude of the heliocentric velocity.

Hyperbolic Flyby



$$e = \sqrt{1 + \frac{\|\mathbf{v}_{\infty,\mathrm{in}}\|^4 D^2}{\mu_F^2}}$$

$$D = r_p \sqrt{1 + \frac{2\mu_F}{r_p \|\mathbf{v}_{\infty,\mathrm{in}}\|^2}}$$

$$\theta = \cos^{-1}\left(-\frac{1}{e}\right)$$

$$\phi = 2\sin^{-1}\left(\left(1 + \frac{r_p \|\mathbf{v}_{\infty,\mathrm{in}}\|^2}{\mu_F}\right)^{-1}\right)$$

Gravity Assist Effects

- The $\mathbf{v}_\infty$ vectors are with respect to the flyby body ($\mathbf{v}_\infty \equiv \mathbf{v}_{\infty/F}$)
- For a flyby entering and exiting a body's SOI at times t_1 and t_2:

$$\mathbf{v}_{\infty,\text{in}/\odot} = \mathbf{v}_{\infty,\text{in}} + \mathbf{v}_{F/\odot}(t_1) \quad \text{and} \quad \mathbf{v}_{\infty,\text{out}/\odot} = \mathbf{v}_{\infty,\text{out}} + \mathbf{v}_{F/\odot}(t_2)$$

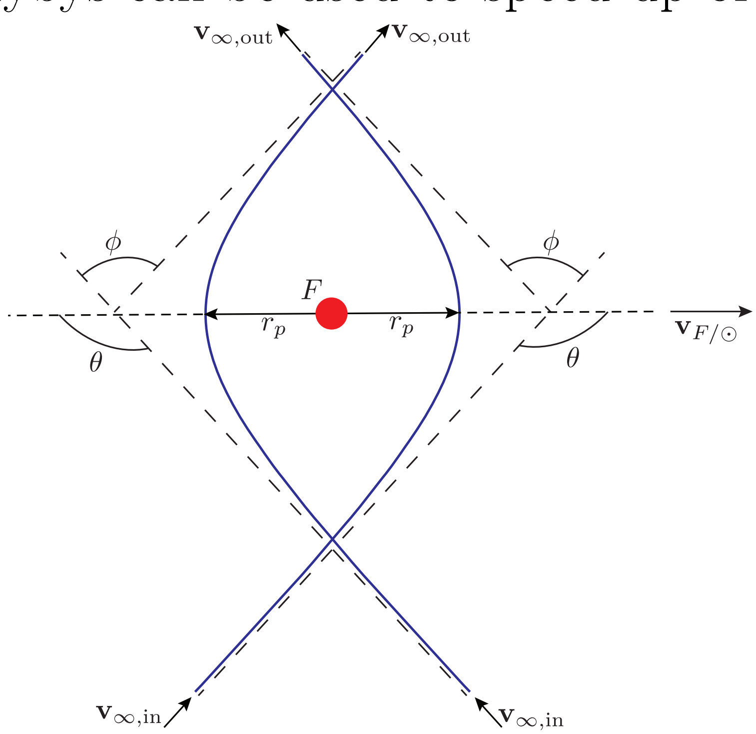
- Assuming $\mathbf{v}_{F/\odot}(t_1) \approx \mathbf{v}_{F/\odot}(t_2)$, the heliocentric Δv is:

$$\Delta v \triangleq \|\mathbf{v}_{\infty,\text{out}/\odot} - \mathbf{v}_{\infty,\text{in}/\odot}\| = 2\|\mathbf{v}_\infty\| \sin\left(\frac{\phi}{2}\right) = \frac{2\|\mathbf{v}_\infty\|}{1 + r_p \|\mathbf{v}_\infty\|^2/\mu_F}$$

- Maximum Δv will be when ${}^{\mathrm{d}\Delta v}/_{\mathrm{d}\|\mathbf{v}_\infty\|} = 0 \implies \phi = 60^\circ$ and $\|\mathbf{v}_\infty\| = \sqrt{\mu_F/r_p}$
- r_p must be greater than the flyby body's radius (R_F) therefore:

$$\boxed{\Delta v_{\max} = \sqrt{\frac{\mu_F}{R_F}} = \frac{v_{\text{esc},F}}{\sqrt{2}}}$$

Flybys can be used to speed up or slow down

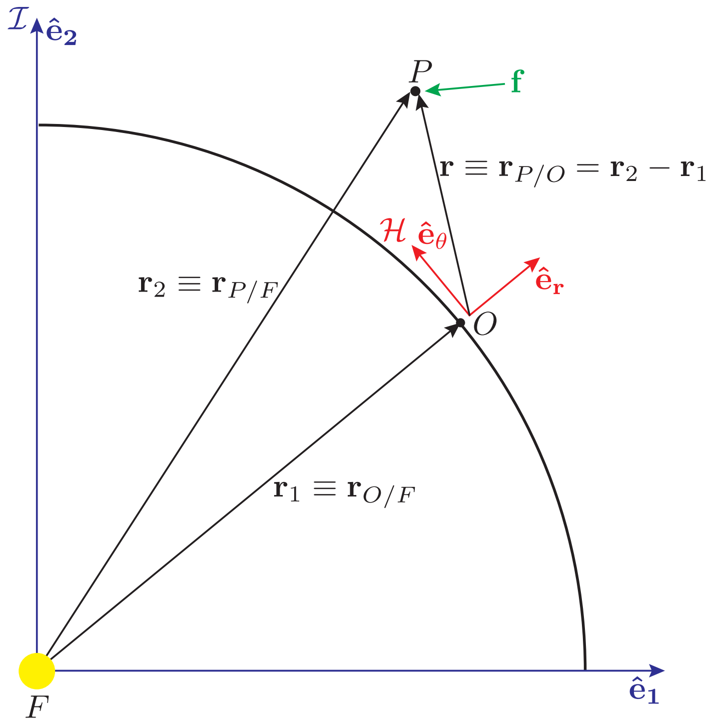


- Passing **behind** the flyby body (with respect to its heliocentric velocity) **increases** your heliocentric velocity and specific energy
- Passing **in front** of the flyby body (with respect to its heliocentric velocity) **decreases** your heliocentric velocity and specific energy

Relative Motion

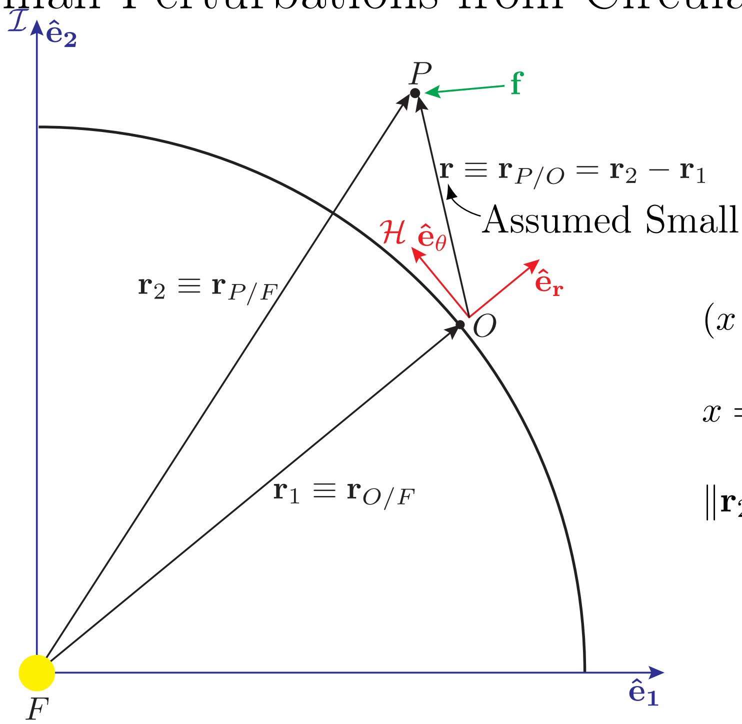
Frequently, we wish to analyze the relative motion between two objects on orbit. This is particularly useful when treating rendezvous and docking problems, for example between an Earth-launched vehicle and a space station. Here, we will consider one approach to this problem: the Clohessy-Wiltshire (or equivalently Euler-Hill) equations. In this formulation, we define a rotating frame based on a circular orbit, with a coordinate origin at a particular point on this orbit (e.g., the location of a space station). We can write the equations of motion of another object in orbit about the same central body with respect to this point and rotating frame. Further, under the assumption that the separation of our spacecraft and the reference point is small, we can linearize the equations of motion to get a fully linear system. This last is important if you are interested in applying control to this system, as the analysis of feedback control for linear systems is significantly simpler than that of nonlinear ones.

The Euler-Hill Frame



$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \frac{\mu}{\|\mathbf{r}_1\|^3}\left(\mathbf{r} - \left(\frac{\|\mathbf{r}_1\|}{\|\mathbf{r}_2\|}\right)^3 \mathbf{r}_2\right) + \mathbf{f}$$

Small Perturbations from Circular Orbits



$$ {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \underbrace{\frac{\mu}{\|\mathbf{r}_1\|^3}}_{n^2}\left(\mathbf{r} - \left(\frac{\|\mathbf{r}_1\|}{\|\mathbf{r}_2\|}\right)^3 \mathbf{r}_2\right) + \mathbf{f} $$

$$ \|\mathbf{r}_2\|^{-3} = \left[(\mathbf{r}_1 + \mathbf{r}) \cdot (\mathbf{r}_1 + \mathbf{r})\right]^{-\frac{3}{2}} $$

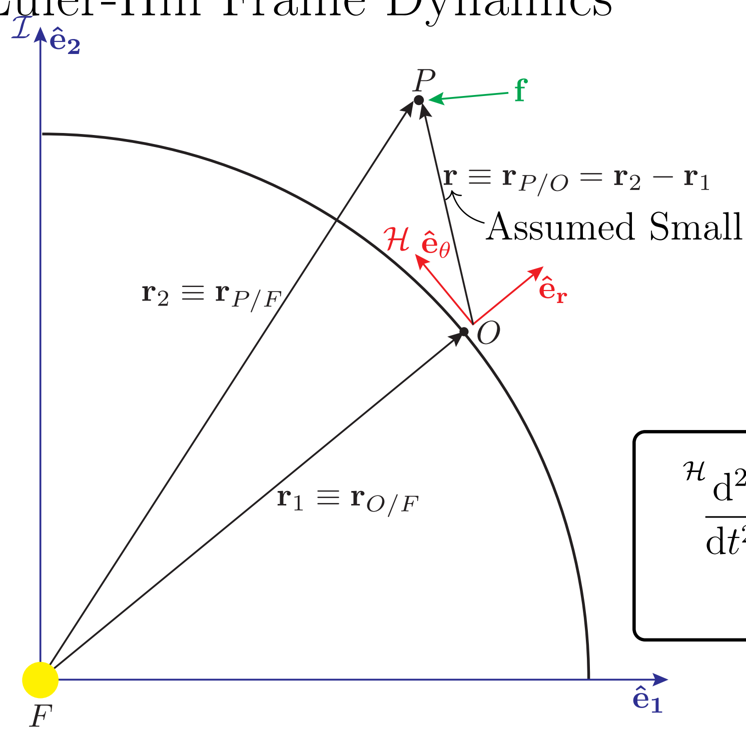
$$ (x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k = x^r + r x^{r-1} y + \ldots $$

$$ x = \|\mathbf{r}_1\|^2 \qquad y = 2\mathbf{r}\cdot\mathbf{r}_1 + \|\mathbf{r}\|^2 \qquad r = -\frac{3}{2} $$

$$ \|\mathbf{r}_2\|^{-3} = \|\mathbf{r}_1\|^{-3}\left(1 - \frac{3}{2}\left(\frac{2\mathbf{r}\cdot\mathbf{r}_1}{\|\mathbf{r}_1\|^2}\right) + \mathcal{O}\left(\mathbf{r}^2\right)\right) $$

$$ \boxed{ {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} \approx n^2\left(-\mathbf{r} + 3\frac{\mathbf{r}_1\cdot\mathbf{r}}{\|\mathbf{r}_1\|^2}\mathbf{r}_1\right) + \mathbf{f} } $$

Euler-Hill Frame Dynamics



$$ \begin{aligned} {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} &= n^2\left(\mathbf{r} - \left(\frac{\|\mathbf{r}_1\|}{\|\mathbf{r}_2\|}\right)^3 \mathbf{r}_2\right) + \mathbf{f} \\ &\approx n^2\left(-\mathbf{r} + 3\frac{\mathbf{r}_1\cdot\mathbf{r}}{\|\mathbf{r}_1\|^2}\mathbf{r}_1\right) + \mathbf{f} \end{aligned} $$

$$ \boxed{ \begin{aligned} {}^{\mathcal{H}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = &- 2n\hat{\mathbf{e}}_3 \times {}^{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} - n^2\left(\hat{\mathbf{e}}_3 \times \left(\hat{\mathbf{e}}_3 \times \mathbf{r}\right)\right) \\ &- n^2\left(\mathbf{r} - 3\left(\hat{\mathbf{e}}_\mathbf{r}\cdot\mathbf{r}\right)\hat{\mathbf{e}}_\mathbf{r}\right) + \mathbf{f} \end{aligned} } $$

Euler-Hill/Clohessy-Wiltshire Equations

$$[\mathbf{r}]_{\mathcal{H}} \triangleq \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{H}} \implies \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}_{\mathcal{H}} = \underbrace{\begin{bmatrix} 2n\dot{y} \\ -2n\dot{x} \\ 0 \end{bmatrix}_{\mathcal{H}} + \begin{bmatrix} n^2 x \\ n^2 y \\ 0 \end{bmatrix}_{\mathcal{H}}}_{\text{Rotating Frame}} - \underbrace{\begin{bmatrix} n^2 x \\ n^2 y \\ n^2 z \end{bmatrix}_{\mathcal{H}} + \begin{bmatrix} 3n^2 x \\ 0 \\ 0 \end{bmatrix}_{\mathcal{H}}}_{\text{Gravity}} + \underbrace{[\mathbf{f}]_{\mathcal{H}}}_{\substack{\text{Other} \\ \text{Perturbations}}}$$

$$\boxed{\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= \mathbf{f} \cdot \hat{\mathbf{e}}_{\mathbf{r}} \triangleq f_x \\ \ddot{y} + 2n\dot{x} &= \mathbf{f} \cdot \hat{\mathbf{e}}_{\theta} \triangleq f_y \\ \ddot{z} + n^2 z &= \mathbf{f} \cdot \hat{\mathbf{e}}_{\mathbf{3}} \triangleq f_z \end{aligned}}$$

Natural Motion

$$\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= 0 \\ \ddot{y} + 2n\dot{x} &= 0 \\ \ddot{z} + n^2 z &= 0 \end{aligned} \qquad \begin{aligned} X(s) &\triangleq \mathcal{L}\{x(t)\} \\ \dot{Y}(s) &\triangleq \mathcal{L}\{\dot{y}(t)\} \end{aligned}$$

$$\mathcal{L}\left\{\begin{bmatrix} \ddot{x} - 2n\dot{y} - 3n^2 x \\ \ddot{y} + 2n\dot{x} \end{bmatrix} = 0\right\} \implies \underbrace{\begin{bmatrix} s^2 - 3n^2 & -2n \\ 2ns & s \end{bmatrix}}_{\triangleq A} \begin{bmatrix} X(s) \\ \dot{Y}(s) \end{bmatrix} = 0 - \text{Initial Conditions}$$

$$\det A = s(s^2 - 3n^2) + 4n^2 s = 0 \implies s = 0, \pm in$$

$$\boxed{\begin{aligned} x(t) &= 4x_0 - 3x_0 \cos(nt) + \frac{\dot{x}_0}{n}\sin(nt) + 2\frac{\dot{y}_0}{n} - 2\frac{\dot{y}_0 \cos(nt)}{n} \\ y(t) &= -6x_0 nt + 6x_0 \sin(nt) + 2\cos(nt)\frac{\dot{x}_0}{n} - 2\frac{\dot{x}_0}{n} + \frac{\dot{y}_0}{n}\left(4\sin(nt) - 3nt\right) + y_0 \\ z(t) &= z_0 \cos(nt) + \frac{\dot{z}_0}{n}\sin(nt) \end{aligned}}$$

Mode 1: $s = 0$

$$A = \begin{bmatrix} s^2 - 3n^2 & -2n \\ 2ns & s \end{bmatrix} = \begin{bmatrix} -3n^2 & -2n \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_0 &= \text{arbitrary} \\ y_0 &= \text{arbitrary} \\ \dot{x}_0 &= \text{arbitrary (often set to 0)} \\ \dot{y}_0 &= \frac{-3nx_0}{2} \end{aligned} \quad \xRightarrow{\dot{x}_0 = 0} \quad \begin{aligned} x(t) &= x_0 \\ y(t) &= -\frac{3}{2}x_0 nt + y_0 \end{aligned}$$

Body is on a circular orbit of radius $\|\mathbf{r}_1\| + x_0$

Modes 2/3: $s = \pm in$

$$A = \begin{bmatrix} s^2 - 3n^2 & -2n \\ 2ns & s \end{bmatrix} = \begin{bmatrix} -n^2 - 3n^2 & -2n \\ \pm 2in^2 & \pm in \end{bmatrix}$$

$$\begin{aligned} x_0 &= \text{arbitrary} \\ y_0 &= \text{arbitrary} \\ \dot{x}_0 &= \text{arbitrary (often set to 0)} \\ \dot{y}_0 &= -2nx_0 \end{aligned} \quad \xRightarrow{\dot{x}_0 = 0} \quad \begin{aligned} x(t) &= x_0 \cos(nt) \\ y(t) &= -2x_0 \sin(nt) + y_0 \end{aligned}$$

Oscillatory motion about O in the rotating frame

Propulsion and Launch

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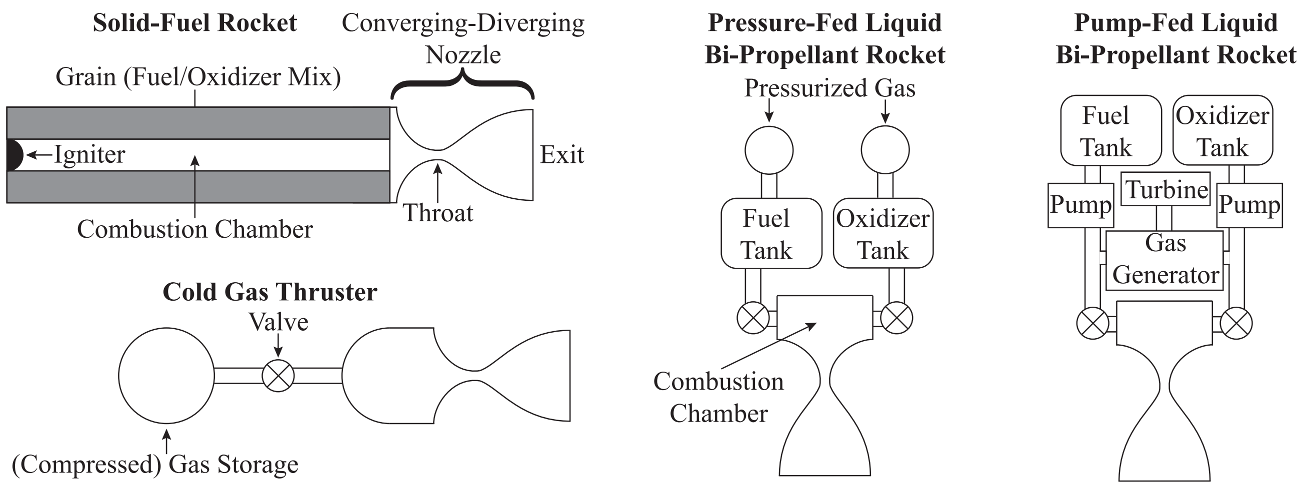
MAE 4060/5065, Fall 2021

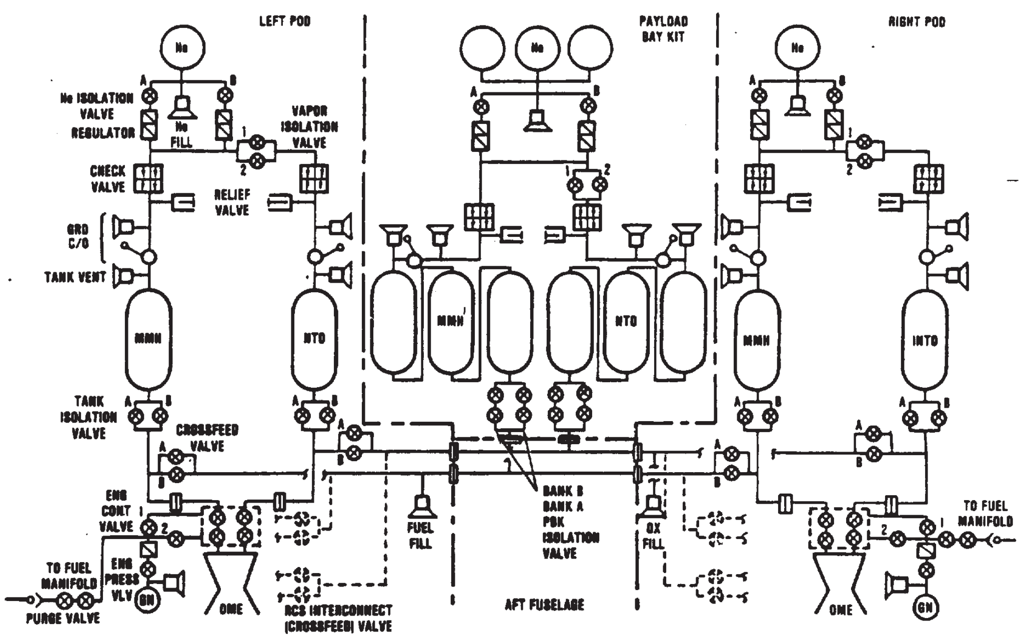
Propulsion and Launch

While the focus of this course is primarily dynamics, it is crucial for us to understand the basic capabilities of the hardware used to enable the trajectories we design, and see whether these orbital solutions are feasible given current capabilities. The topic of space propulsion easily fills multiple whole courses, so our focus here will be to develop a basic understanding of the operating principles of space propulsion systems, along with their current and near-future capabilities. Similarly, as effectively all space missions start from the Earth, it is important for us to understand the constraints and capabilities of launch vehicles, and the impact of launch site location. Current space propulsion can be roughly split into chemical and electric (although other types do exist). In general, chemical propulsion is less fuel-mass efficient but produces higher thrust than electric propulsion. This may change in the future as we learn to build more powerful (i.e., MW-class) power systems for our spacecraft. For now, however, effectively all operating launch systems are chemical-based, while more and more in-space propulsion is utilizing electric systems.

Chemical Rockets

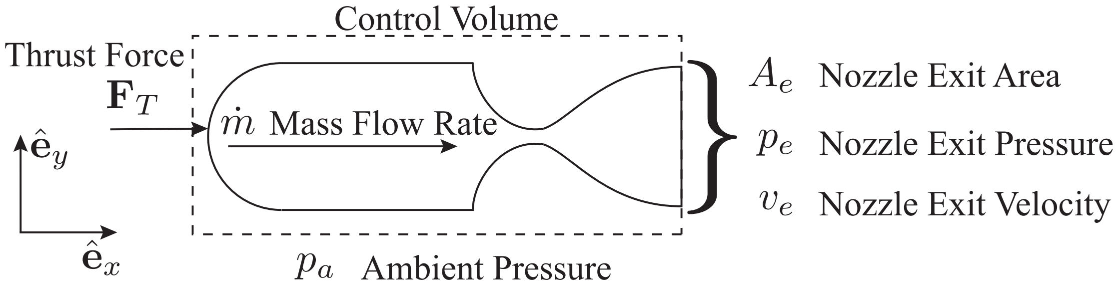


Example: Space Shuttle Orbital Maneuvering System



From: https://ntrs.nasa.gov/citations/19850008634

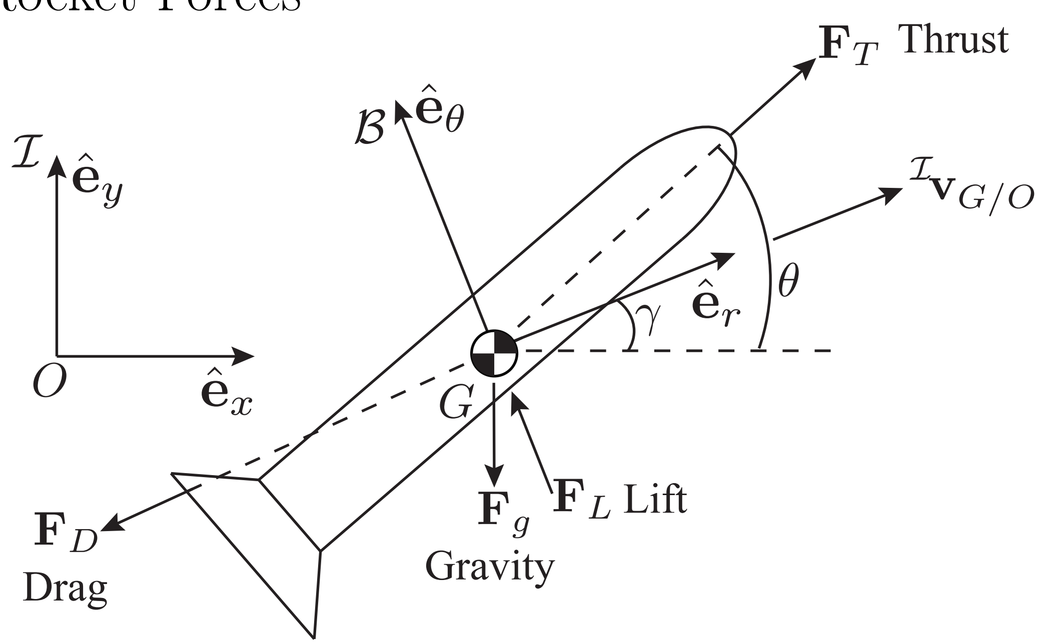
Rocket Propulsion



$$\left.\begin{aligned}\sum \mathbf{F}\cdot\hat{\mathbf{e}}_x &= \underbrace{\|\mathbf{F}_T\|}_{\triangleq F_T} - A_e(p_e - p_a)\\ \text{Change in Momentum: } \boldsymbol{\Delta}\mathbf{p} &= \int \dot{m} v_e \hat{\mathbf{e}}_x \,\mathrm{d}t\end{aligned}\right\} F_T = \dot{m} v_e + A_e(p_e - p_a)$$

$$\boxed{\text{Effective Exhaust Velocity: } v_{\text{eff}} \triangleq \frac{F_T}{\dot{m}} = v_e + \frac{A_e}{\dot{m}}(p_e - p_a)}$$

Rocket Forces



NB: Rocket mass (m) is **not** constant

$$\begin{aligned} m\,{}^{\mathcal{I}}\mathbf{a}_{G/O} &= \sum F = \mathbf{F}_g + \mathbf{F}_L + \mathbf{F}_D + \mathbf{F}_T \\ &= -mg\hat{\mathbf{e}}_y + F_L\hat{\mathbf{e}}_\theta - F_D\hat{\mathbf{e}}_r + F_T\left(\cos(\theta-\gamma)\hat{\mathbf{e}}_r + \sin(\theta-\gamma)\hat{\mathbf{e}}_\theta\right)\end{aligned}$$

The Tsiolkovsky (Ideal) Rocket Equation

$$^{\mathcal{I}}\mathbf{a}_{G/O} = \frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} \underbrace{^{\mathcal{I}}\mathbf{v}_{G/O}}_{\equiv v\hat{\mathbf{e}}_r} + \underbrace{^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}}_{\equiv \dot{\gamma}\hat{\mathbf{e}}_3} \times \underbrace{^{\mathcal{I}}\mathbf{v}_{G/O}}_{\equiv v\hat{\mathbf{e}}_r} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{\mathbf{e}}_r + v\frac{\mathrm{d}\gamma}{\mathrm{d}t}\hat{\mathbf{e}}_\theta$$

$$^{\mathcal{I}}\mathbf{a}_{G/O} \cdot \hat{\mathbf{e}}_r = \frac{\mathrm{d}v}{\mathrm{d}t} = -g\sin\gamma - \frac{F_D}{m} + \frac{F_T}{m}\cos(\theta - \gamma)$$

$$^{\mathcal{I}}\mathbf{a}_{G/O} \cdot \hat{\mathbf{e}}_\theta = v\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -g\cos\gamma + \frac{F_L}{m} + \frac{F_T}{m}\sin(\theta - \gamma)$$

$$\Delta v \triangleq v(t_f) - v(t_0) = \int_{t_0}^{t_f} \left[-g\sin\gamma - \frac{F_D}{m} + \frac{F_T}{m}\cos(\theta-\gamma)\right] \mathrm{d}t$$

Assuming gravity and drag are negligible:
$g = 0, F_D = 0, \theta = \gamma$, and v_{eff} is constant:

$$\Delta v = \int_{t_0}^{t_f} \frac{F_T}{m}\,\mathrm{d}t = v_{\text{eff}} \int_{t_0}^{t_f} \frac{\dot{m}}{m}\,\mathrm{d}t = -v_{\text{eff}} \int_{m_0}^{m_f} \frac{\mathrm{d}m}{m}$$

$$\boxed{\Delta v = v_{\text{eff}} \ln\left(\frac{m_0}{m_f}\right)}$$

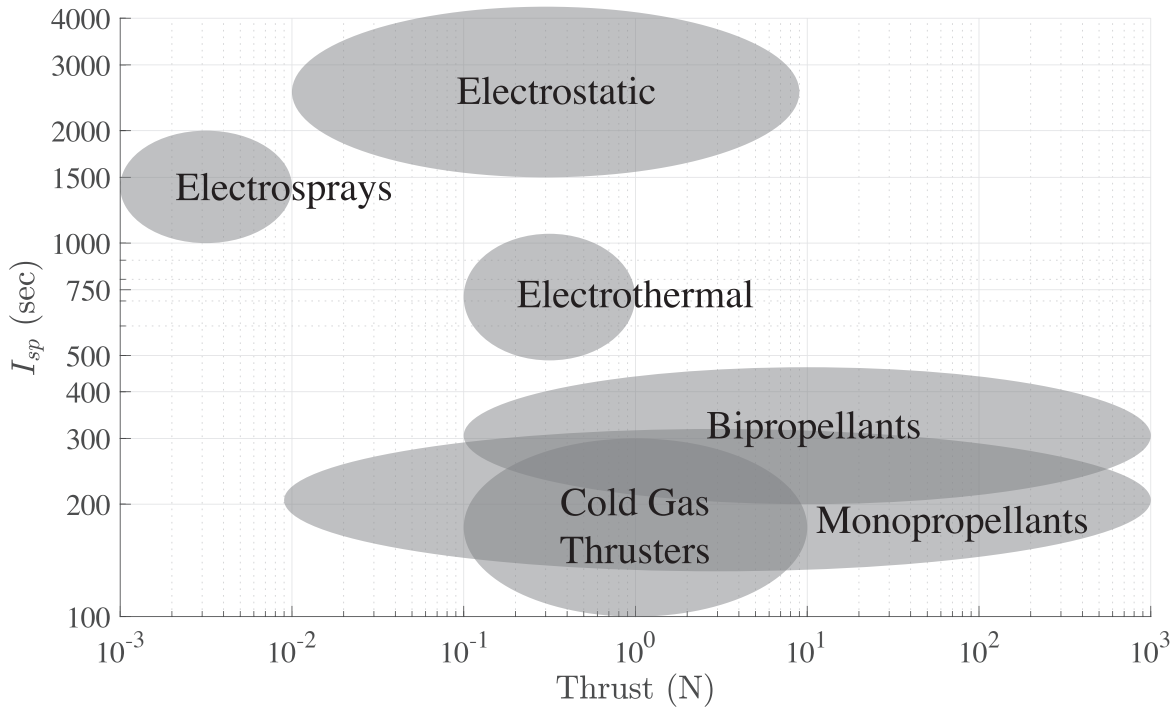
Specific Impulse

$$\text{Specific Impulse} \triangleq I_{sp} = \underbrace{\frac{1}{w_p}}_{\text{Propellant Weight}} \overbrace{\int_{t_0}^{t_f} F_T\,\mathrm{d}t}^{\text{Total Impulse}} \overset{\text{Assuming Constant Thrust and Mass Flow Rate}}{=} \frac{F_T}{\dot{m}g_0} = \frac{v_{\text{eff}}}{g_0}$$

Standard Gravity (g_0)
Gravity at Earth's Surface. $g_0 = 9.80665\ \text{m/s}^2$

$$\boxed{\begin{aligned} \Delta v &= I_{sp} g_0 \ln\left(\frac{m_0}{m_f}\right) \\ m_0 - m_f &= m_0 \left(1 - \exp\left[-\frac{\Delta v}{I_{sp} g_0}\right]\right) \end{aligned}}$$

Efficiency vs. Thrust



An Illustrative Example

- You wish to launch an H_2-O_2 rocket ($v_{\text{eff}} = 4000$ m/s) to a 600 km circular orbit:

$$\Delta v = v_{\text{circ}} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{3.986 \times 10^{14}\,\text{m}^3\text{s}^{-2}}{600\text{km} + 6371\text{km}}} \approx 7.5\text{km/s}$$

- Typically require an additional 1.5 km/s for atmospheric drag and gravity compensation for a total of 9 km/s

$$\Delta v = v_{\text{eff}} \ln\left(\frac{m_0}{m_f}\right) \Rightarrow \frac{m_0}{m_f} = \exp\left[\Delta v / v_{\text{eff}}\right] \approx 9.5$$

- Your rocket must be 89.5% fuel by mass (and we still haven't even included all other losses)

Staging

$$\boxed{\Delta v = \sum_{i=1}^{n} \Delta v_i = \sum_{i=1}^{n} v_{\text{eff}_i} \ln\left(\frac{m_{0_i}}{m_{f_i}}\right)}$$

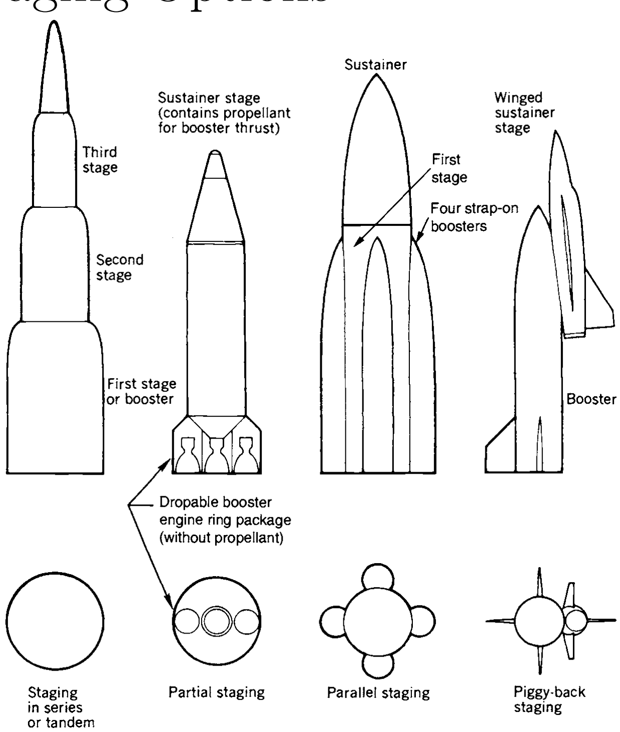
- m_{0_i}: Total mass before stage i ignition
- m_{f_i}: Total mass after stage i fuel expended but **before** stage i separation

If all stages have the same effective exhaust velocity:

$$\exp\left(\Delta v / v_{\text{eff}}\right) = \prod_{i=1}^{n} \frac{m_{0_i}}{m_{f_i}}$$

Stages can be optimized for thrust, or efficiency, or maximized payload mass

Staging Options



Sutton and Biblarz (2001) Fig. 4-14

Another Illustrative Example

Consider a 2-stage rocket with a 6000 m/s Δv requirement and 4500 kg total launch mass. Both stages have the same thrusters: 300 s I_{sp}, 3000 m/s v_{eff}, and 0.88 fuel mass fraction (ξ). m_1, m_2 are the stage wet masses (with fuel).

$$m_{0_1} = m_1 + m_2 + m_{\text{payload}} \qquad m_{0_2} = m_2 + m_{\text{payload}}$$

$$m_{f_1} = m_1(1-\xi) + m_2 + m_{\text{payload}} \qquad m_{f_2} = m_2(1-\xi) + m_{\text{payload}}$$

Option 1: Equal Mass stages

$$m_1 = m_2 \triangleq m \text{ and } m_{\text{tot}} = 2m + m_{\text{payload}}$$

$$c \triangleq \exp\left(\Delta v / v_{\text{eff}}\right) = \frac{m_{\text{tot}}}{m(1-\xi) + m + m_{\text{payload}}}\left(\frac{m + m_{\text{payload}}}{m(1-\xi) + m_{\text{payload}}}\right)$$

$$m_{\text{payload}} = \frac{m_{\text{tot}}}{c\xi\left(\xi+1\right)}\left(c\left(\xi^2 - \xi - 1\right) + \sqrt{c^2 + 4c\xi^2 - 2c + 1} + 1\right)$$

$$\approx 300\,\text{kg}$$

Option 2: Equal Mass-Ratio stages

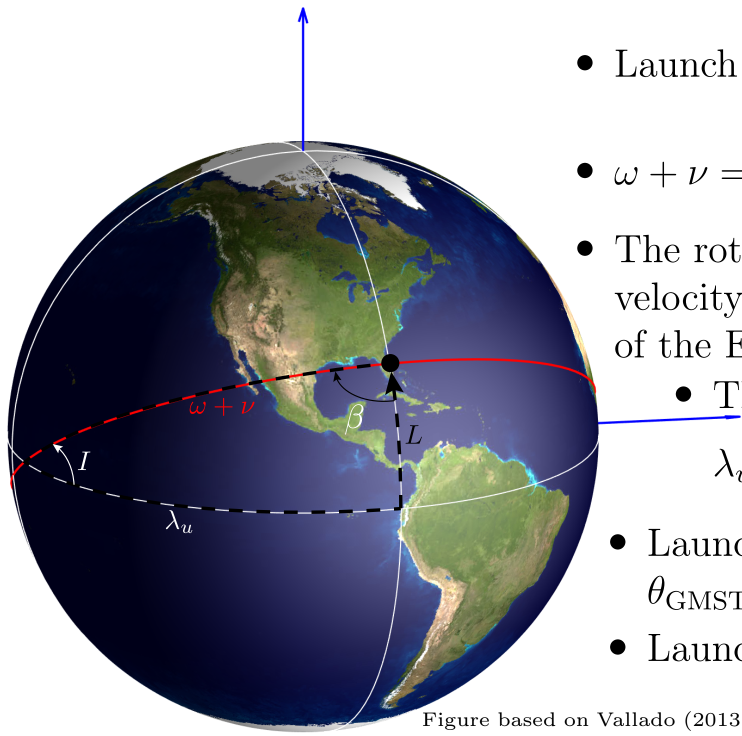
$$\frac{m_{0_1}}{m_{f_1}} = \frac{m_{0_2}}{m_{f_2}} \text{ and } m_{\text{tot}} = m_1 + m_2 + m_{\text{payload}}$$

$$\frac{m_{\text{tot}}}{m_{\text{tot}} - \xi m_1} = \frac{m_2 + m_{\text{payload}}}{m_2(1-\xi) + m_{\text{payload}}} \qquad c \triangleq \exp\left(\Delta v / v_{\text{eff}}\right) = \left(\frac{m_{\text{tot}}}{m_{\text{tot}} - m_1\xi}\right)^2$$

$$m_{\text{payload}} = \frac{m_{\text{tot}}}{c^2\xi^2}\left(c^2\left(\xi^2 - 2\xi + 1\right) + c + 2\sqrt{c^3}\left(\xi - 1\right)\right)$$

$$\approx 357\,\text{kg} \quad \Longrightarrow \quad \sim 20\%\text{ Gain!}$$

Launch Sites



- Launch azimuth $\beta = \sin^{-1}\left(\frac{\cos I}{\cos L}\right) \Rightarrow I \geq L$
- $\omega + \nu = \sin^{-1}\left(\frac{\sin L}{\sin I}\right)$
- The rotation of the Earth provides an Eastward velocity of $\omega_\oplus \times \mathbf{r}_{O/G}$ for launch site O and center of the Earth G.
  - The launch orbit crosses the equator $\lambda_u = \cos^{-1}\left(\frac{\cos \beta}{\cos I}\right)$ west of the launch site
- Launch time can be found from $\theta_{\text{GMST}} = \Omega + \lambda_u - \lambda_e$
- Launch sites have different azimuth restrictions

Figure based on Vallado (2013)

Flavors of Electric Propulsion

Electrothermal

Electromagnetic fields generate plasma to heat propellants expelled via a nozzle

- Arcjet
- Resistojet
- Laser ablative

Electrostatic

Static electric fields accelerate charged particles (Coulomb force)

- Ion thrusters
- Hall Effect thrusters
- Field Emission
- Nano-particle Field Extraction

Electromagnetic

Electromagnetic fields accelerate charged particles (Lorentz force)

- Plasma Thrusters
- Magnetoplasmodynamic
- VASIMR

Electrodynamic

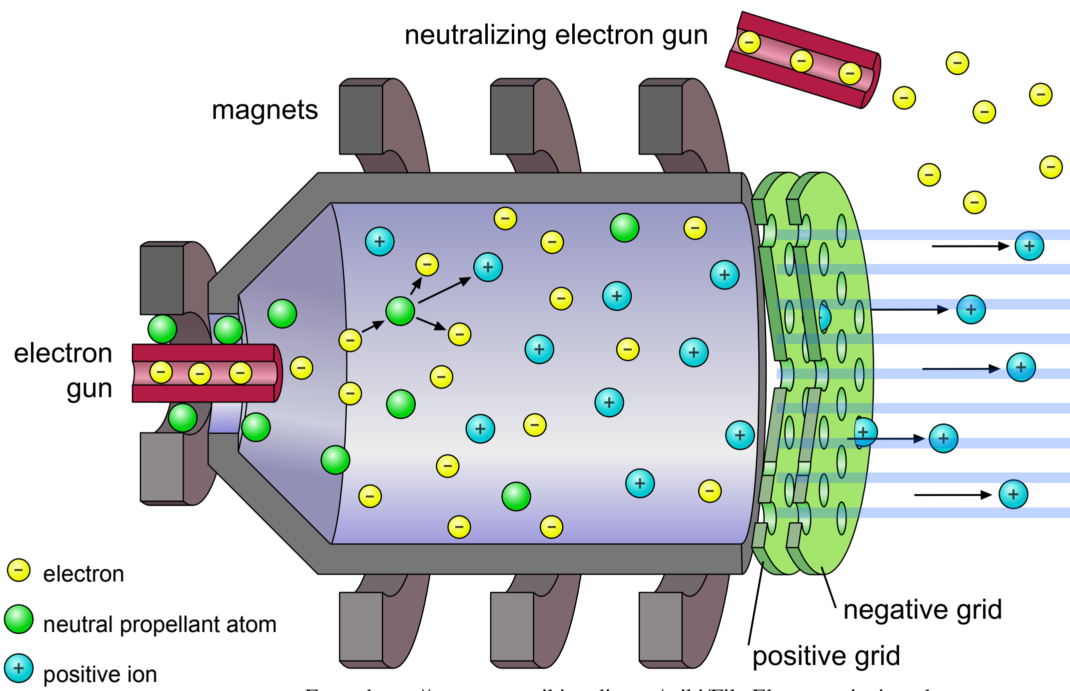
Electric potential generated by motion through natural magnetic field and converted to kinetic energy

- Electrodynamic Tether

Coulomb Force

- Inverse-square force governing attraction between two stationary, electrically charged particles with charge magnitudes q_1 and q_2: $\mathbf{F}_{1,2} = k_e \frac{q_1 q_2}{\|\mathbf{r}_{1/2}\|^3} \mathbf{r}_{1/2}$
- Equivalently, the magnitude of the electric field created by a point charge q at a distance r: $\|\mathbf{E}\| = k_e \frac{|q|}{r^2}$
- k_e is Coulomb's constant: 8.99×10^9 N m^2 C^{-2}

Gridded Ion Thurster



From: https://commons.wikimedia.org/wiki/File:Electrostatic_ion_thruster-en.svg

Solar Sails

- Recall that solar radiation pressure can act as a perturbing force via momentum transfer to spacecraft
- This effect can instead by used for propulsion
- Two primary mechanisms: absorption and reflection: $P_{\text{absorb}} = \dfrac{F_\odot}{r^2}\cos^2\alpha$ and $P_{\text{reflect}} = 2P_{\text{absorb}}$ for heliocentric distance r^2, solar flux pressure at 1 AU $F_\odot$ and incidence angle α
- For an efficiency ε and sail area A, the magnitude of force on the sail is $F_{\text{sail}} = 2\varepsilon\dfrac{F_\odot}{r^2}\cos^2\alpha A$
- More complex models exist. See: Heaton and Artusio-Glimpse (2015) and Dachwald (2005)

Solar Sailing Geometry

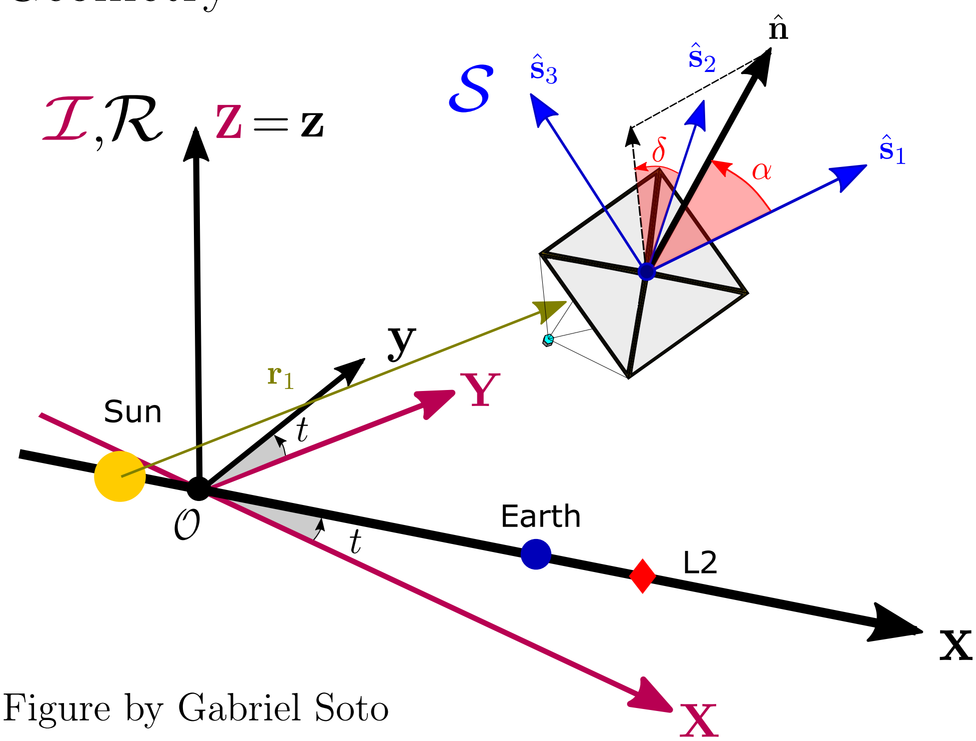
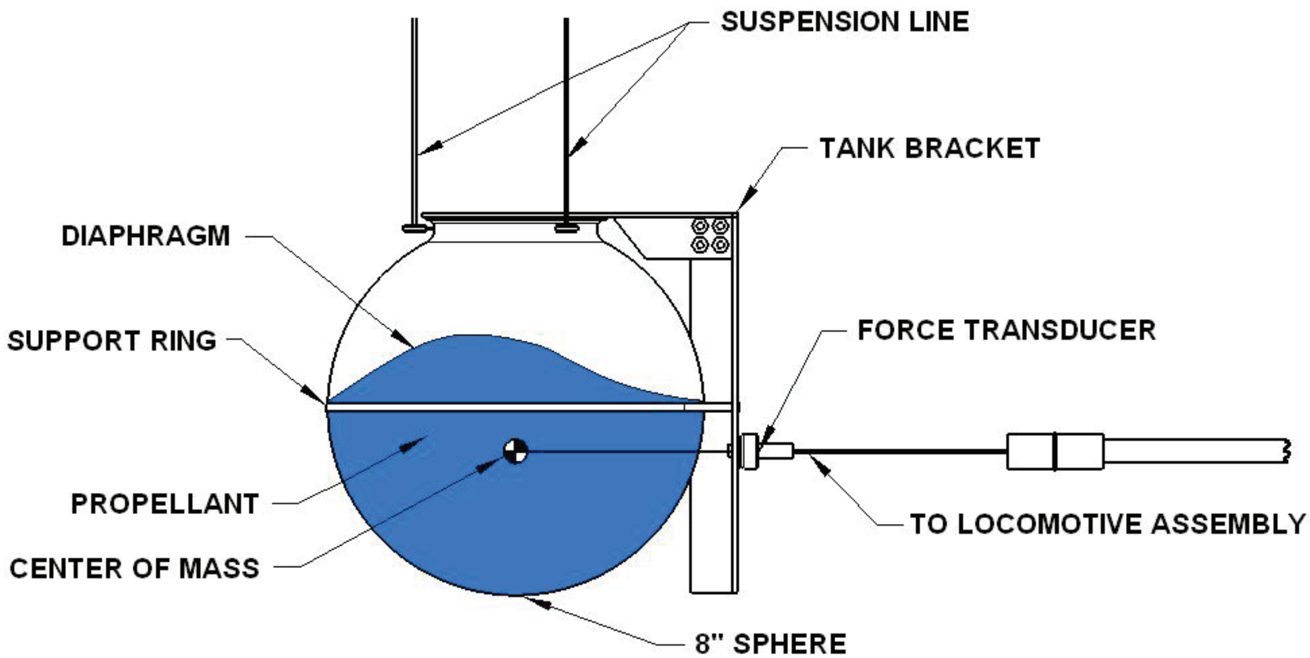


Figure by Gabriel Soto

Propellant Tanks

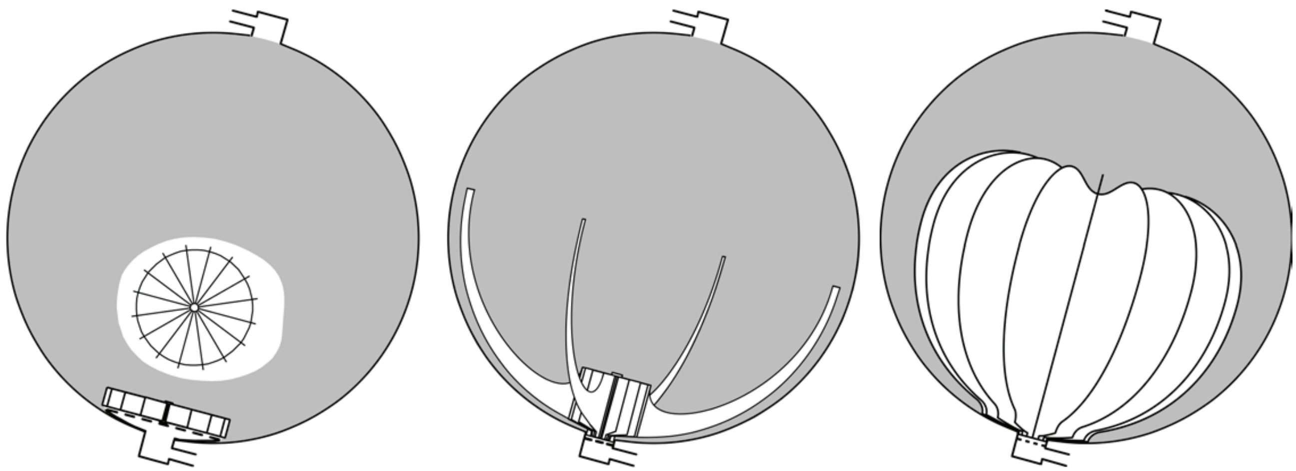
- Propellant tanks are typically designed with low ballistic coefficient (high drag relative to mass)—typically tanks are spherical or cylindrical
- Fluid slosh induces attitude disturbances and represents unmodeled dynamics
- Slosh-induced disturbances can mask other important dynamics (deployments, etc.)
- Liquid motion dissipates energy (this can destabilize spacecraft)
- Slosh can be controlled via diaphragm tanks, tank shape, or propellant management devices

Diaphragm Tanks



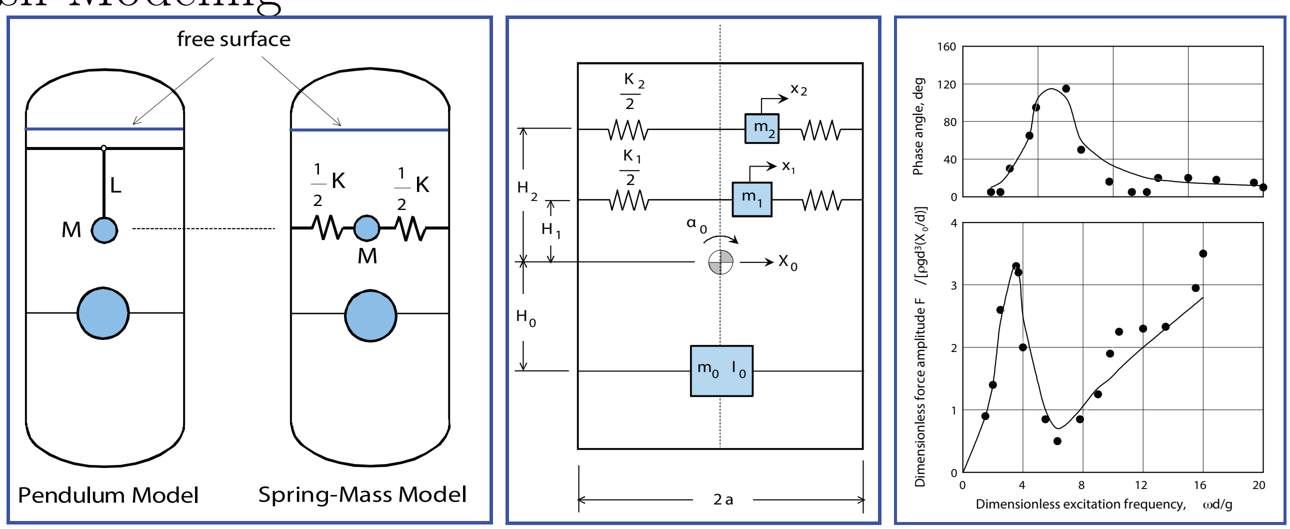
Chatman et al. (2007) Fig. 12

Propellant Management Devices



From: Jaekle (1993). See also: http://www.pmdtechnology.com/Index.html

Slosh Modeling



Dodge et al. (2002) Figs. 3.1, 3.2, 3.6

$$F = -\left(m_0 \ddot{X}_0 + \sum_i m_i \left(m_0 \ddot{X}_0 + \ddot{x}_i\right)\right) \qquad M = -\left(\left(I_0 + m_0 H_0^2\right) \ddot{\alpha}_0 + \sum_i \left(m_i H_i \left(\ddot{x}_i + H_i \ddot{\alpha}_0\right)\right) - g \sum_i m_i x_i\right)$$

Attitude Dynamics

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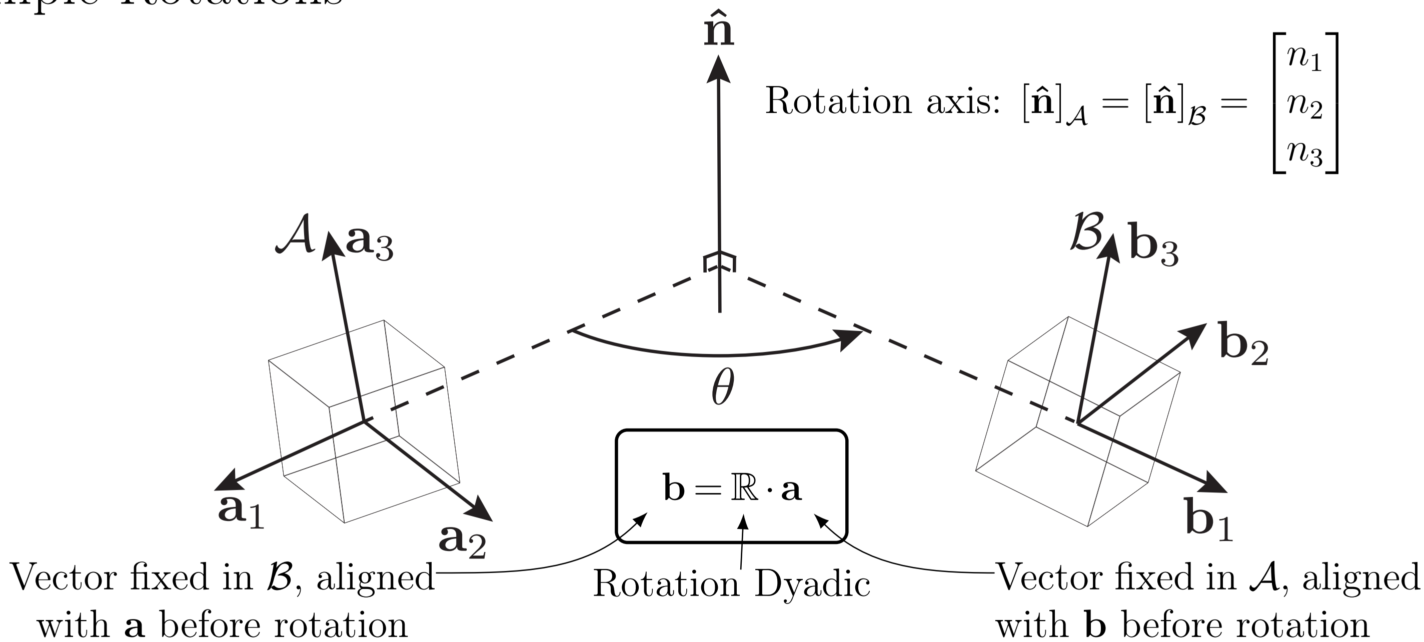
MAE 4060/5065, Fall 2021

Attitude Dynamics

The ability to measure and control a spacecraft's orientation (or attitude) is equally important to being able to measure and control its position. Our starting model for an arbitrary spacecraft is a rigid body—one that experiences no deformation or changes in mass distribution. Of course, real spacecraft do experience various types of flexure, and changes in mass, especially when fuel is consumed. As usual, our model is only a starting point, which must be validated with additional analyses. To consider the effects of thermal and structural deformation, we typically utilize finite element analysis (FEA). Nevertheless, in most cases, the majority of spacecraft orientation dynamics will be well predicted by the rigid body model. Here we review the basics of rigid body kinematics and dynamics in three dimensions, and introduce some new formalism that will help us in our continued study of spacecraft attitude.

Simple Rotations



$$\mathbf{b} = \hat{\mathbf{n}}\otimes\hat{\mathbf{n}}\cdot\mathbf{a}+\cos\theta\,(\mathbf{a}-\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{a}))+\sin\theta\,\hat{\mathbf{n}}\times\mathbf{a}$$

Vector and Tensor Products

(Scalar) Dot Product

- $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$
- $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
- $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$
- $x\mathbf{a}\cdot y\mathbf{b} = xy(\mathbf{a}\cdot\mathbf{b})$

(Vector) Cross Product

- $\mathbf{a}\times\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta\hat{\mathbf{c}}$ where $\mathbf{c}\perp\mathbf{a},\mathbf{b}$
- $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
- $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
- $y\mathbf{a}\times\mathbf{b} = y\,(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times y\mathbf{b}$

(Tensor) Outer Product

- $(\mathbf{a}+\mathbf{b})\otimes\mathbf{c} = \mathbf{a}\otimes\mathbf{c}+\mathbf{b}\otimes\mathbf{c}$
- $\mathbf{c}\otimes(\mathbf{a}+\mathbf{b}) = \mathbf{c}\otimes\mathbf{a}+\mathbf{c}\otimes\mathbf{b}$
- $x(\mathbf{a}\otimes\mathbf{b}) = (x\mathbf{a})\otimes\mathbf{b} = \mathbf{a}\otimes(x\mathbf{b})$
- $(\mathbf{a}\otimes\mathbf{b})\otimes\mathbf{c} = \mathbf{a}\otimes(\mathbf{b}\otimes\mathbf{c})$

Remember: All Vector **and Tensor** Operations Can Be Written as Matrix Multiplications

$$\mathcal{I} = (O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{cases} \mathbf{a} = \sum_i a_i \mathbf{e}_i \Rightarrow a_i = \mathbf{a}\cdot\mathbf{e}_i \qquad \mathbf{b} = \sum_i b_i \mathbf{e}_i \Rightarrow b_i = \mathbf{b}\cdot\mathbf{e}_i \\ \mathbb{T} = \mathbf{a}\otimes\mathbf{b} = \sum_i\sum_j T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j \Rightarrow T_{ij} = \mathbf{e}_i\cdot\mathbb{T}\cdot\mathbf{e}_j = a_i b_j \end{cases}$$

$$[\mathbf{a}]_{\mathcal{I}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{I}} \qquad [\mathbf{b}]_{\mathcal{I}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}} \qquad [\mathbb{T}]_{\mathcal{I}} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}_{\mathcal{I}}$$

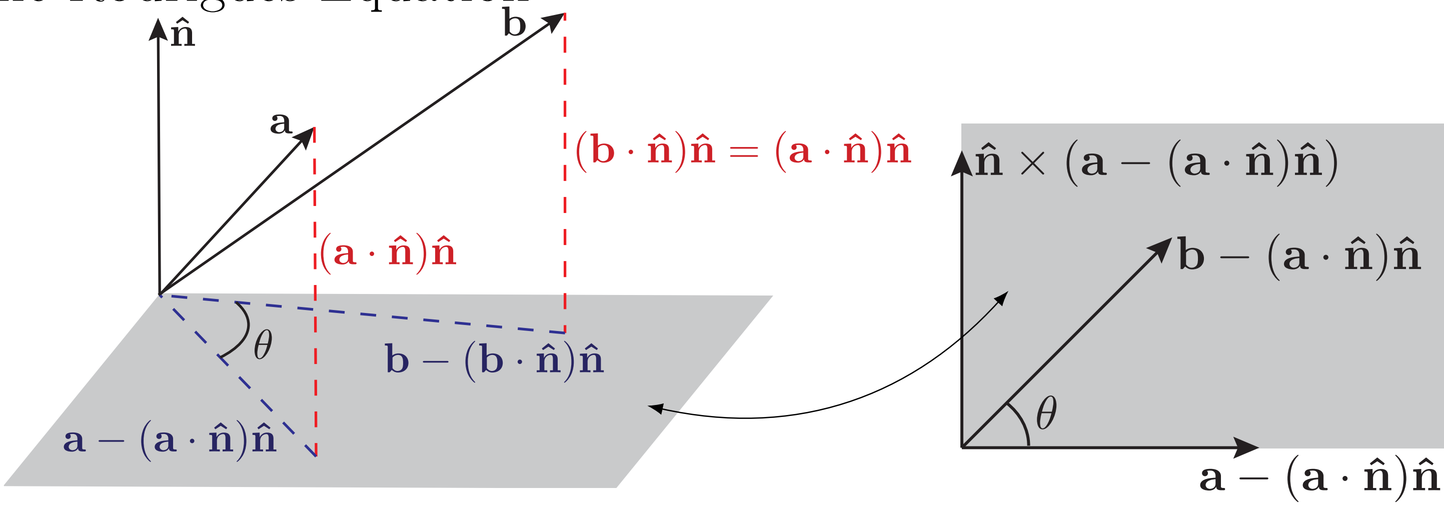
$$\begin{aligned} [\mathbf{a}\cdot\mathbb{T}]_{\mathcal{I}} &= [\mathbf{a}]_{\mathcal{I}}^T[\mathbb{T}]_{\mathcal{I}} \\ [\mathbb{T}\cdot\mathbf{a}]_{\mathcal{I}} &= [\mathbb{T}]_{\mathcal{I}}[\mathbf{a}]_{\mathcal{I}} \\ [\mathbf{a}\times\mathbb{T}]_{\mathcal{I}} &= [\mathbf{a}\times]_{\mathcal{I}}[\mathbb{T}]_{\mathcal{I}} \\ [\mathbb{T}\times\mathbf{a}]_{\mathcal{I}} &= -[\mathbb{T}]_{\mathcal{I}}[\mathbf{a}\times]_{\mathcal{I}} \end{aligned}$$

where

$$[\mathbb{T}]_{\mathcal{I}} = [\mathbf{a}\otimes\mathbf{b}]_{\mathcal{I}} = [\mathbf{a}]_{\mathcal{I}}[\mathbf{b}]_{\mathcal{I}}^T$$

$$[\mathbf{a}\times]_{\mathcal{I}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_{\mathcal{I}}$$

The Rodrigues Equation



$$\mathbf{b} = \mathbb{R}\cdot\mathbf{a} \Longrightarrow \boxed{\begin{aligned} \mathbb{R} &= \cos\theta\mathbb{U} + \sin\theta\hat{\mathbf{n}}_\times + (1-\cos\theta)\hat{\mathbf{n}}\otimes\hat{\mathbf{n}} \\ [\mathbb{R}]_{\mathcal{A}} \equiv {}^{\mathcal{A}}C^{\mathcal{B}} &= I\cos\theta + \sin\theta\,[\hat{\mathbf{n}}\times]_{\mathcal{A}} + (1-\cos\theta)\,[\hat{\mathbf{n}}]_{\mathcal{A}}\,[\hat{\mathbf{n}}]_{\mathcal{A}}^T \end{aligned}}$$

Direction Cosine Matrices Revisited

$$[\hat{\mathbf{n}}]_{\mathcal{A}} = [\hat{\mathbf{n}}]_{\mathcal{B}} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$${}^{\mathcal{B}}C^{\mathcal{A}} = I\cos\theta - \sin\theta\,[\hat{\mathbf{n}}\times]_{\mathcal{A}} + (1-\cos\theta)\,[\hat{\mathbf{n}}]_{\mathcal{A}}\,[\hat{\mathbf{n}}]_{\mathcal{A}}^T =$$

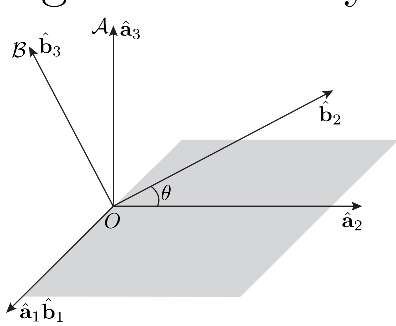
$$\begin{bmatrix} n_1^2\left(1-\cos\left(\theta\right)\right)+\cos\left(\theta\right) & -n_1n_2\left(\cos\left(\theta\right)-1\right)+n_3\sin\left(\theta\right) & -n_1n_3\left(\cos\left(\theta\right)-1\right)-n_2\sin\left(\theta\right) \\ -n_1n_2\left(\cos\left(\theta\right)-1\right)-n_3\sin\left(\theta\right) & n_2^2\left(1-\cos\left(\theta\right)\right)+\cos\left(\theta\right) & n_1\sin\left(\theta\right)-n_2n_3\left(\cos\left(\theta\right)-1\right) \\ -n_1n_3\left(\cos\left(\theta\right)-1\right)+n_2\sin\left(\theta\right) & -n_1\sin\left(\theta\right)-n_2n_3\left(\cos\left(\theta\right)-1\right) & n_3^2\left(1-\cos\left(\theta\right)\right)+\cos\left(\theta\right) \end{bmatrix}$$

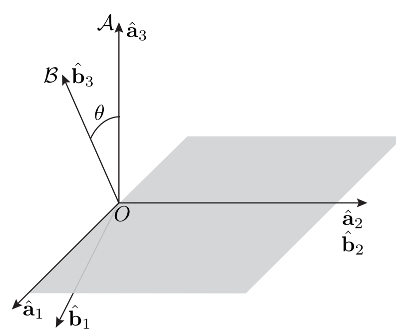
$$\left[{}^{\mathcal{B}}C^{\mathcal{A}}\right]_{ij} = \delta_{ij}\cos\theta + \underbrace{\epsilon_{ijk}}_{k\neq i,j} n_k\sin\theta + n_in_j(1-\cos\theta)$$

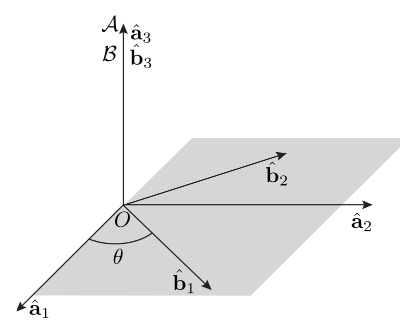
Kronecker Delta *Levi-Civita Symbol*

$$\delta_{ij} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \qquad \epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) = \begin{cases} 1 & \text{Even permutations} \\ -1 & \text{Odd permutations} \\ 0 & \text{Repeated indices} \end{cases}$$

Euler Angles and Body Rotations



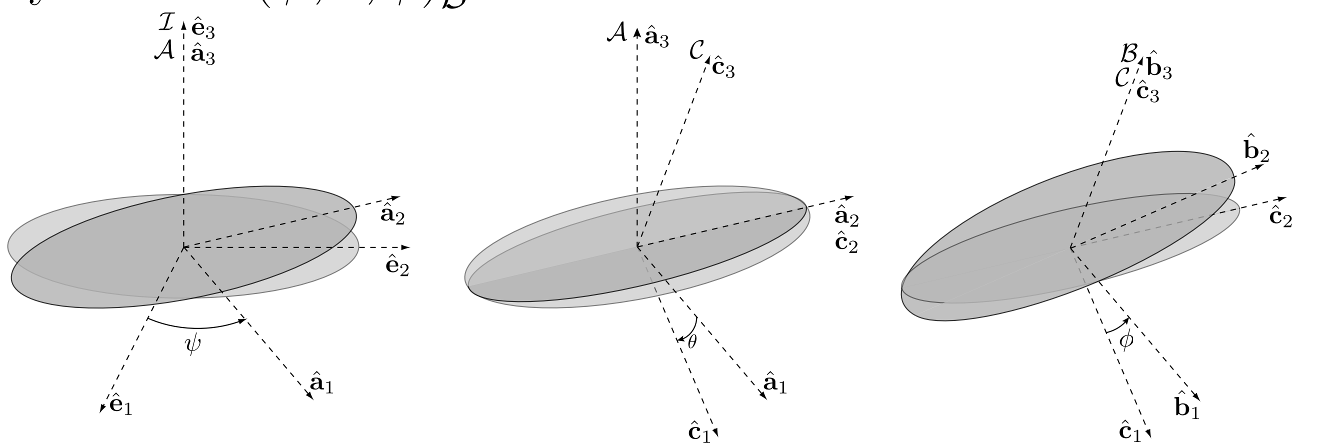




$${}^{\mathcal{B}}C^{\mathcal{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\triangleq C_1(\theta)} \quad \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{\triangleq C_2(\theta)} \quad \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\triangleq C_3(\theta)}$$

Any DCM can be decomposed into three rotations about non-repeating frame axes.

Body-2 3-2-3 $(\psi, \theta, \phi)^{\mathcal{I}}_{\mathcal{B}}$ rotation



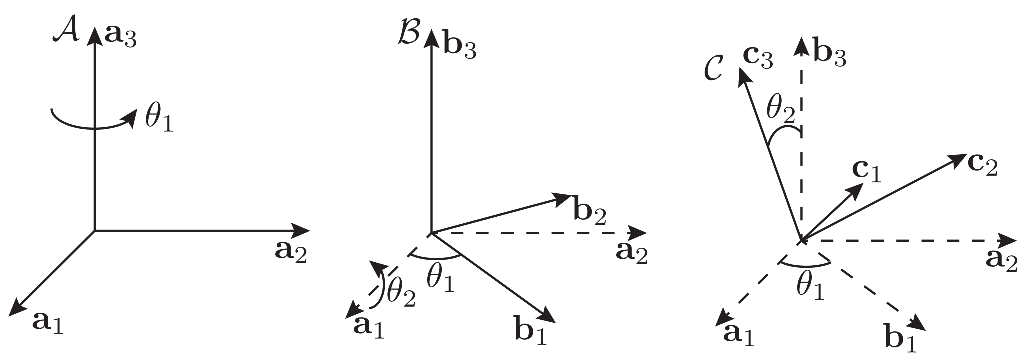
$$
{}^{\mathcal{A}}C^{\mathcal{I}} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{\mathcal{C}}C^{\mathcal{A}} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad {}^{\mathcal{B}}C^{\mathcal{C}} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{\mathcal{B}}C^{\mathcal{I}} = {}^{\mathcal{B}}C^{\mathcal{C}}\,{}^{\mathcal{C}}C^{\mathcal{A}}\,{}^{\mathcal{A}}C^{\mathcal{I}} =
$$

$$
\begin{bmatrix} -\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\cos(\theta) & \sin(\phi)\cos(\psi) + \sin(\psi)\cos(\phi)\cos(\theta) & -\sin(\theta)\cos(\phi) \\ -\sin(\phi)\cos(\psi)\cos(\theta) - \sin(\psi)\cos(\phi) & -\sin(\phi)\sin(\psi)\cos(\theta) + \cos(\phi)\cos(\psi) & \sin(\phi)\sin(\theta) \\ \sin(\theta)\cos(\psi) & \sin(\psi)\sin(\theta) & \cos(\theta) \end{bmatrix}
$$

Space Rotations

Rather than rotating about body-fixed axes (body rotations), we can choose to always rotate about the axes of our original (presumably inertial) reference frame. These are called space rotations. Consider a rotation about the third axis of some frame $\mathcal{A}$, followed by a rotation about the first axis of the original $\mathcal{A}$ frame (producing intermediate frame $\mathcal{B}$ and $\mathcal{C}$, respectively):



$$
{}^{\mathcal{B}}C^{\mathcal{A}} = \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\hat{\mathbf{n}}_2]_{\mathcal{B}} = {}^{\mathcal{B}}C^{\mathcal{A}}\,[\hat{\mathbf{n}}_2]_{\mathcal{A}} = \begin{bmatrix} \cos\theta_1 \\ -\sin\theta_1 \\ 0 \end{bmatrix}_{\mathcal{B}}
$$

Space Rotations Continued

$$
{}^{\mathcal{C}}C^{\mathcal{B}} = \begin{bmatrix} (-\cos(\theta_2)+1)\cos^2(\theta_1)+\cos(\theta_2) & (\cos(\theta_2)-1)\sin(\theta_1)\cos(\theta_1) & \sin(\theta_1)\sin(\theta_2) \\ (\cos(\theta_2)-1)\sin(\theta_1)\cos(\theta_1) & (-\cos(\theta_2)+1)\sin^2(\theta_1)+\cos(\theta_2) & \sin(\theta_2)\cos(\theta_1) \\ -\sin(\theta_1)\sin(\theta_2) & -\sin(\theta_2)\cos(\theta_1) & \cos(\theta_2) \end{bmatrix}
$$

$$
\begin{aligned}
{}^{\mathcal{C}}C^{\mathcal{A}} = {}^{\mathcal{C}}C^{\mathcal{B}\mathcal{B}}C^{\mathcal{A}} &= \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1)\cos(\theta_2) & \sin(\theta_1)\sin(\theta_2) \\ -\sin(\theta_1) & \cos(\theta_1)\cos(\theta_2) & \sin(\theta_2)\cos(\theta_1) \\ 0 & -\sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \\
&\equiv C_3^T(-\theta_1)C_1^T(-\theta_2) = C_3(\theta_1)C_1(\theta_2)
\end{aligned}
$$

So, the DCMs for space rotations can be calculated exactly in the same way as those for body rotation, except with the order of rotations reversed!

Gimbal Lock

While Euler angles are the most efficient encoding of orientation (carrying the minimum number of required variables), they have one serious drawback. It is always possible to find a particular orientation which causes a mathematical singularity within a particular Euler angle encoding, corresponding to the loss of a degree of freedom in the system. For example, consider the case where the intermediate rotation in a 3-1-3 Euler angle set is zero. This results in two subsequent rotations about the same third axis, meaning that the first and third angles of the set cannot be disambiguated (note that we've already seen this condition when studying two-body orbits, in the case where orbital inclination is zero). The name comes from gyroscopes based on mechanical gimbals—when this condition is encountered, two of the gimbals become aligned with one another, therefore causing the gyroscope to lose its ability to track the full three-dimensional orientation. In this case, the gimbals aren't actually locked together - there is just nothing causing them to become unaligned. To deal with this, you can carry multiple Euler angle sets and switch between them as gimbal lock conditions are approached. Alternatively, you can add a fourth parameter to the encoding, which eliminates the gimbal lock condition.

Euler Parameters

$$\boxed{\boldsymbol{\epsilon} \triangleq \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{n}} \qquad \epsilon_4 \triangleq \cos\left(\frac{\theta}{2}\right)}$$

$$\begin{aligned}
{}^{\mathcal{A}}C^{\mathcal{B}} &= I\cos\theta + \sin\theta\,[\hat{\mathbf{n}}\times]_{\mathcal{A}} + (1-\cos\theta)\,[\hat{\mathbf{n}}]_{\mathcal{A}}\,[\hat{\mathbf{n}}]_{\mathcal{A}}^T \\
&= I\left(\epsilon_4^2 - [\boldsymbol{\epsilon}]_{\mathcal{A}}^T\,[\boldsymbol{\epsilon}]_{\mathcal{A}}\right) + 2\epsilon_4\,[\boldsymbol{\epsilon}\times]_{\mathcal{A}} + 2\,[\boldsymbol{\epsilon}]_{\mathcal{A}}\,[\boldsymbol{\epsilon}]_{\mathcal{A}}^T \\
&= \begin{bmatrix} \epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2 + \epsilon_4^2 & 2\epsilon_1\epsilon_2 - 2\epsilon_3\epsilon_4 & 2\epsilon_1\epsilon_3 + 2\epsilon_2\epsilon_4 \\ 2\epsilon_1\epsilon_2 + 2\epsilon_3\epsilon_4 & -\epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2 + \epsilon_4^2 & -2\epsilon_1\epsilon_4 + 2\epsilon_2\epsilon_3 \\ 2\epsilon_1\epsilon_3 - 2\epsilon_2\epsilon_4 & 2\epsilon_1\epsilon_4 + 2\epsilon_2\epsilon_3 & -\epsilon_1^2 - \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 \end{bmatrix}
\end{aligned}$$

$$\boldsymbol{\epsilon} = \frac{1}{4\epsilon_4}\begin{bmatrix} {}^{\mathcal{A}}C_{32}^{\mathcal{B}} - {}^{\mathcal{A}}C_{23}^{\mathcal{B}} \\ {}^{\mathcal{A}}C_{13}^{\mathcal{B}} - {}^{\mathcal{A}}C_{31}^{\mathcal{B}} \\ {}^{\mathcal{A}}C_{21}^{\mathcal{B}} - {}^{\mathcal{A}}C_{12}^{\mathcal{B}} \end{bmatrix}_A \qquad \epsilon_4 = \frac{1}{2}\left(1 + \mathrm{Tr}\left[{}^{\mathcal{A}}C^{\mathcal{B}}\right]\right)^{\frac{1}{2}}$$

$$\mathbf{b} = \mathbb{R}\cdot\mathbf{a} \quad \Longrightarrow \quad \mathbf{b} = \mathbf{a} + 2\left(\epsilon_4\boldsymbol{\epsilon}\times\mathbf{a} + \boldsymbol{\epsilon}\times(\boldsymbol{\epsilon}\times\mathbf{a})\right)$$

Rodrigues Parameters

$$\boxed{\boldsymbol{\rho} \triangleq \tan\left(\frac{\theta}{2}\right)\hat{\mathbf{n}}}$$

$$\begin{aligned}
\rho_i &\equiv \frac{\epsilon_i}{\epsilon_4} \\
{}^{\mathcal{A}}C^{\mathcal{B}} &= (I + [\boldsymbol{\rho}\times])\,(I - [\boldsymbol{\rho}\times])^{-1} \\
\mathbf{a} - \mathbf{b} &= (\mathbf{a} + \mathbf{b})\times\boldsymbol{\rho}
\end{aligned}$$

Quaternions

Define a new basis set, $\mathbf{i}, \mathbf{j}, \mathbf{k}$, s. t. $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$. Products of basis elements are anti-commutative: $\begin{array}{lll} \mathbf{ij} = \mathbf{k} & \mathbf{jk} = \mathbf{i} & \mathbf{ki} = \mathbf{j} \\ \mathbf{ji} = -\mathbf{k} & \mathbf{kj} = -\mathbf{i} & \mathbf{ik} = -\mathbf{j} \end{array}$

A quaternion is a vector in this basis and a scalar:

$$\mathbf{q} \triangleq \begin{bmatrix} \mathbf{v} \\ r \end{bmatrix} = \begin{bmatrix} v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \\ r \end{bmatrix}$$

Quaternion product:

$$\mathbf{q}_1 = \begin{bmatrix} \mathbf{v}_1 \\ r_1 \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} \mathbf{v}_2 \\ r_2 \end{bmatrix} \implies \mathbf{q}_1\mathbf{q}_2 = \begin{bmatrix} r_1\mathbf{v}_2 + r_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \\ r_1 r_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \end{bmatrix}$$

Markely & Crassidis call this $\mathbf{q}_1 \odot \mathbf{q}_2$ and define:

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = \begin{bmatrix} r_1\mathbf{v}_2 + r_2\mathbf{v}_1 - \mathbf{v}_1 \times \mathbf{v}_2 \\ r_1 r_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \end{bmatrix}$$

Not to be confused with outer product

Quaternion Products

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = [\mathbf{q}_1 \otimes]\, \mathbf{q}_2 \qquad\qquad \mathbf{q}_1 \odot \mathbf{q}_2 = [\mathbf{q}_1 \odot]\, \mathbf{q}_2$$

$$[\mathbf{q}\otimes] \triangleq \left[\underbrace{\begin{bmatrix} rI - [\mathbf{v}\times] \\ -\mathbf{v}^T \end{bmatrix}}_{\triangleq\, \boldsymbol{\Psi}(\mathbf{q})} \;\; \underbrace{\begin{bmatrix} \mathbf{v} \\ r \end{bmatrix}}_{\mathbf{q}} \right] \qquad [\mathbf{q}\odot] \triangleq \left[\underbrace{\begin{bmatrix} rI + [\mathbf{v}\times] \\ -\mathbf{v}^T \end{bmatrix}}_{\triangleq\, \boldsymbol{\Xi}(\mathbf{q})} \;\; \underbrace{\begin{bmatrix} \mathbf{v} \\ r \end{bmatrix}}_{\mathbf{q}} \right]$$

$$\boxed{\mathbf{q}_1 \otimes \mathbf{q}_2 = \mathbf{q}_2 \odot \mathbf{q}_1}$$

Quaternion Representation of Rotations

$$\boxed{\mathbf{q}(\hat{\mathbf{n}}, \theta) = \begin{bmatrix} \sin\left(\dfrac{\theta}{2}\right)\hat{\mathbf{n}} \\ \\ \cos\left(\dfrac{\theta}{2}\right) \end{bmatrix}}$$

$$\begin{aligned} \mathbf{b} = \mathbb{R} \cdot \mathbf{a} \quad \Longrightarrow \quad \mathbf{b} &= \mathbf{q} \otimes \mathbf{a} \otimes \mathbf{q}^{\star} \\ &= [\mathbf{q}\odot]^T [\mathbf{q}\otimes] \begin{bmatrix} \mathbf{a} \\ 0 \end{bmatrix} \end{aligned}$$

$$\mathbf{q}^{\star} = \begin{bmatrix} -\mathbf{v} \\ r \end{bmatrix} \quad \text{for} \quad \mathbf{q} = \begin{bmatrix} \mathbf{v} \\ r \end{bmatrix}$$

See Markely & Crassidis (2014), Sec. 2.7 & 2.9.3 for lots more details

Small Rotations

$$\text{Recall: } \mathbf{b} = \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} \cdot \mathbf{a} + \cos\theta \left(\mathbf{a} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{a})\right) + \sin\theta \hat{\mathbf{n}} \times \mathbf{a}$$

Assume $\theta \ll 1$:

$$\mathbf{b} \approx \mathbf{a} + \theta\hat{\mathbf{n}} \times \mathbf{a} \quad \Rightarrow \quad {}^{\mathcal{A}}C^{\mathcal{B}} \approx I + \theta\,[\hat{\mathbf{n}}\times]_{\mathcal{A}}$$

$$\mathbf{q} \approx \begin{bmatrix} \dfrac{\theta}{2}\hat{\mathbf{n}} \\ 1 \end{bmatrix} \quad \Rightarrow \quad {}^{\mathcal{A}}C^{\mathcal{B}} \approx I + 2\,[\mathbf{q}_{1:3}\times]_{\mathcal{A}}$$

$$\boldsymbol{\rho} \approx \frac{\theta}{2}\hat{\mathbf{n}} \quad \Rightarrow \quad {}^{\mathcal{A}}\boldsymbol{\rho}^{\mathcal{B}} \approx {}^{\mathcal{A}}\boldsymbol{\rho}^{\mathcal{C}} + {}^{\mathcal{C}}\boldsymbol{\rho}^{\mathcal{B}}$$

The Angular Velocity Matrix

$$\boxed{\widetilde{\omega} \triangleq {}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}}$$

$${}^{\mathcal{A}}\dot{C}^{\mathcal{B}} \triangleq \frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{A}}C^{\mathcal{B}}$$

1 $$\left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{-1}\widetilde{\omega} = \left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{-1}{}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}} = {}^{\mathcal{A}}\dot{C}^{\mathcal{B}} \quad \Longrightarrow \quad {}^{\mathcal{A}}\dot{C}^{\mathcal{B}} = {}^{\mathcal{A}}C^{\mathcal{B}}\widetilde{\omega}$$

2 $$\widetilde{\omega}^T + \widetilde{\omega} = \underbrace{\left({}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}\right)^T + \left({}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}\right)}_{\equiv \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}C^{\mathcal{B}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(I)} = 0$$

The angular velocity matrix must be skew-symmetric.

Components of Angular Velocity Matrix

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} {}^{\mathcal{A}}C^{\mathcal{B}}_{11} & {}^{\mathcal{A}}C^{\mathcal{B}}_{21} & {}^{\mathcal{A}}C^{\mathcal{B}}_{31} \\ {}^{\mathcal{A}}C^{\mathcal{B}}_{12} & {}^{\mathcal{A}}C^{\mathcal{B}}_{22} & {}^{\mathcal{A}}C^{\mathcal{B}}_{32} \\ {}^{\mathcal{A}}C^{\mathcal{B}}_{13} & {}^{\mathcal{A}}C^{\mathcal{B}}_{23} & {}^{\mathcal{A}}C^{\mathcal{B}}_{33} \end{bmatrix} \begin{bmatrix} {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{11} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{12} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{13} \\ {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{21} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{22} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{23} \\ {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{31} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{32} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{33} \end{bmatrix}$$

$$\left.\begin{aligned} \omega_1 &= {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{12}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{13} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{22}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{23} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{32}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{33} \\ \omega_2 &= {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{13}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{11} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{23}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{21} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{33}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{31} \\ \omega_3 &= {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{11}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{12} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{21}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{22} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{31}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{32} \end{aligned}\right\} \omega_i = \frac{1}{2}\epsilon_{igh}(\epsilon_{igh}+1)\,{}^{\mathcal{A}}C^{\mathcal{B}}_{jh}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{jg}$$

Poisson's Kinematic Equations

$${}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{ij} = \epsilon_{ghj}\omega_h\,{}^{\mathcal{A}}C^{\mathcal{B}}_{ig}$$

The Angular Velocity Vector

$$\left.\begin{aligned} \hat{\mathbf{b}}_i &= \sum_{j=1}^{3} (\hat{\mathbf{b}}_i \cdot \hat{\mathbf{a}}_j)\hat{\mathbf{a}}_j \\ \frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{b}}_i &= \sum_{j=1}^{3} \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\left(\hat{\mathbf{b}}_i \cdot \hat{\mathbf{a}}_j\right)}_{\equiv\, {}^{\mathcal{B}}C^{\mathcal{A}}_{ij}} \hat{\mathbf{a}}_j \end{aligned}\right\} \omega_1 = \frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{b}}_3 \qquad \omega_2 = \frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{b}}_1 \qquad \omega_3 = \frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2$$

$$\text{Recall:} \quad \begin{aligned} {}^{\mathcal{A}}C^{\mathcal{B}} &= I\cos\theta + \sin\theta\,[\hat{\mathbf{n}}\times]_{\mathcal{B}} + (1-\cos\theta)\,[\hat{\mathbf{n}}]_{\mathcal{B}}\,[\hat{\mathbf{n}}]_{\mathcal{B}}^T \\ \omega_i &= \frac{1}{2}\epsilon_{igh}(\epsilon_{igh}+1)\,{}^{\mathcal{A}}C^{\mathcal{B}}_{jh}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{jg} \end{aligned}$$

$$\Longrightarrow \left.\begin{aligned} \omega_1 &= n_1\dot{\theta} \\ \omega_2 &= n_2\dot{\theta} \\ \omega_3 &= n_3\dot{\theta} \end{aligned}\right\} \boxed{\begin{aligned} {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} &\triangleq \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3 \\ &= \dot{\theta}\hat{\mathbf{n}} \end{aligned}}$$

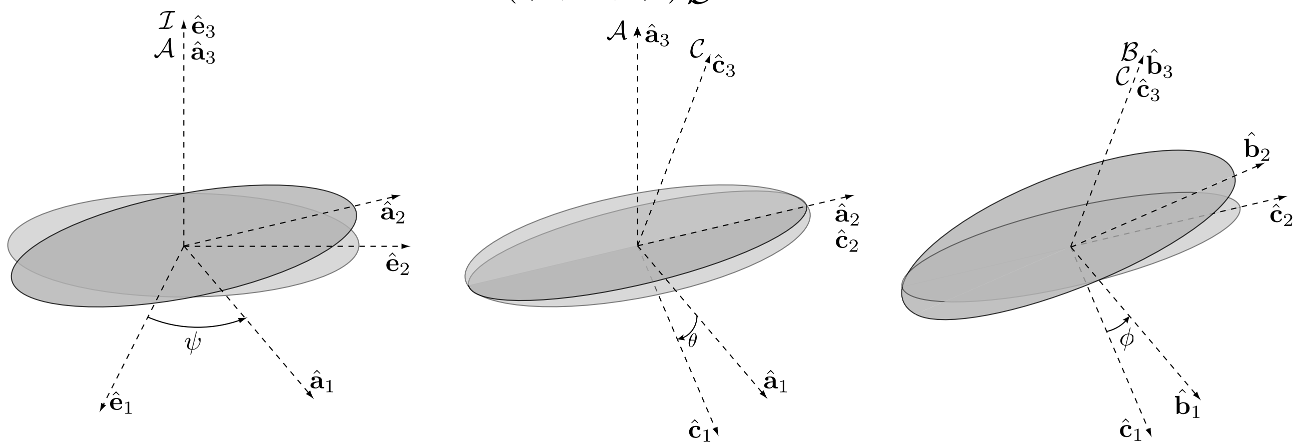
The Transport Equation

$$\mathbf{r} = [\mathbf{r}]_{\mathcal{A}}^T \begin{bmatrix}\hat{\mathbf{a}}_1\\ \hat{\mathbf{a}}_2\\ \hat{\mathbf{a}}_3\end{bmatrix} = [\mathbf{r}]_{\mathcal{B}}^T \begin{bmatrix}\hat{\mathbf{b}}_1\\ \hat{\mathbf{b}}_2\\ \hat{\mathbf{b}}_3\end{bmatrix} \qquad \begin{bmatrix}\hat{\mathbf{a}}_1\\ \hat{\mathbf{a}}_2\\ \mathbf{a}_3\end{bmatrix} = {}^{\mathcal{A}}C^{\mathcal{B}} \begin{bmatrix}\hat{\mathbf{b}}_1\\ \hat{\mathbf{b}}_2\\ \mathbf{b}_3\end{bmatrix} \qquad [\mathbf{r}]_{\mathcal{A}} = {}^{\mathcal{A}}C^{\mathcal{B}}\,[\mathbf{r}]_{\mathcal{B}}$$

Unit Vectors of Frame $\mathcal{A}$ and Frame $\mathcal{B}$

$$\begin{aligned} \frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left([\mathbf{r}]_{\mathcal{A}}^T\right) \begin{bmatrix}\hat{\mathbf{a}}_1\\ \hat{\mathbf{a}}_2\\ \hat{\mathbf{a}}_3\end{bmatrix} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left([\mathbf{r}]_{\mathcal{B}}^T\right){}^{\mathcal{B}}C^{\mathcal{A}} + [\mathbf{r}]_{\mathcal{B}}^T\,{}^{\mathcal{B}}\dot{C}^{\mathcal{A}}\right) \begin{bmatrix}\hat{\mathbf{a}}_1\\ \hat{\mathbf{a}}_2\\ \hat{\mathbf{a}}_3\end{bmatrix} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left([\mathbf{r}]_{\mathcal{B}}^T\right) \begin{bmatrix}\hat{\mathbf{b}}_1\\ \hat{\mathbf{b}}_2\\ \hat{\mathbf{b}}_3\end{bmatrix} + [\mathbf{r}]_{\mathcal{B}}^T\,\tilde{\omega}^T \begin{bmatrix}\hat{\mathbf{b}}_1\\ \hat{\mathbf{b}}_2\\ \hat{\mathbf{b}}_3\end{bmatrix} \Rightarrow \frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} \end{aligned}$$

Kinematics of the 3-2-3 $(\psi, \theta, \phi)^{\mathcal{I}}_{\mathcal{B}}$ rotation



$$\begin{aligned} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} &= \dot{\psi}\hat{\mathbf{a}}_3 + \dot{\theta}\hat{\mathbf{c}}_2 + \dot{\phi}\hat{\mathbf{b}}_3 \\ &= \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3 \end{aligned}$$

$$\begin{aligned} \omega_1 &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \omega_2 &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \omega_3 &= \dot{\phi} + \dot{\psi}\cos\theta \end{aligned}$$

$$\begin{aligned} \dot{\psi} &= (-\omega_1\cos\phi + \omega_2\sin\phi)\csc\theta \\ \dot{\theta} &= \omega_1\sin\phi + \omega_2\cos\phi \\ \dot{\phi} &= (\omega_1\cos\phi - \omega_2\sin\phi)\cot\theta + \omega_3 \end{aligned}$$

Kinematics of Euler Parameters (and Quaternions)

$${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = 2\left(\epsilon_4 \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\epsilon} - \dot{\epsilon}_4\boldsymbol{\epsilon} - \boldsymbol{\epsilon}\times\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\epsilon}\right)$$

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\epsilon} = \frac{1}{2}\left(\epsilon_4\,{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} + \boldsymbol{\epsilon}\times{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}\right) \qquad \dot{\epsilon}_4 = -\frac{1}{2}\,{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}\cdot\boldsymbol{\epsilon}$$

$$\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\\0\end{bmatrix} = 2\underbrace{\begin{bmatrix}\epsilon_4 & \epsilon_3 & -\epsilon_2 & -\epsilon_1\\ -\epsilon_3 & \epsilon_4 & \epsilon_1 & -\epsilon_2\\ \epsilon_2 & -\epsilon_1 & \epsilon_4 & -\epsilon_3\\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4\end{bmatrix}}_{\text{Inverse } \equiv \text{ Transpose}}\begin{bmatrix}\dot{\epsilon}_1\\\dot{\epsilon}_2\\\dot{\epsilon}_3\\\dot{\epsilon}_4\end{bmatrix}$$

$$\boxed{\mathbf{q} = \begin{bmatrix}\boldsymbol{\epsilon}\\\epsilon_4\end{bmatrix} \quad\Longrightarrow\quad \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{q} = \frac{1}{2}\mathbf{q}\odot\begin{bmatrix}{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}\\0\end{bmatrix}}$$

Kinematics of Rodrigues Parameters

$$
\begin{aligned}
{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} &= \frac{2}{1+\boldsymbol{\rho}\cdot\boldsymbol{\rho}}\left(\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho} - \boldsymbol{\rho}\times\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho}\right) \\
\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho} &= \frac{1}{2}\left({}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} + \boldsymbol{\rho}\times{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} + \boldsymbol{\rho}\otimes\boldsymbol{\rho}\cdot{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}\right)
\end{aligned}
$$

Attitude Dynamics

So far, we have only been looking at attitude (orientation) kinematics, but now we will turn to the dynamics side of things and introduce Euler's Laws. At first glance, it appears that Euler's laws are just a restatement of Newton's second law, which equivalently gives you a linear momentum relationship and an angular momentum relationship. Recall, however, that Newton's second law applies explicitly to ideal particles, whereas Euler's laws describe extended, rigid bodies. It is true that you can derive Euler's first law by applying Newton's second law to an infinite collection of infinitesimal particles, and so it is not, strictly speaking, encoding any new physics. Euler's second law, however, has hidden within it a very specific postulate about rigid bodies, as we shall see shortly.

Euler's Laws

I. The product of the inertial acceleration of the center of mass of a rigid body and its total mass is equal to the total external force applied to the body

$$\mathbf{F}_G = m_G {}^{\mathcal{I}}\mathbf{a}_{G/O}$$

II. The rate of change of the inertial angular momentum of a rigid body about a fixed point O in the inertial frame is equal to the total external moment applied to the body about O

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\mathbf{h}_O = \mathbf{M}_O$$

Center of Mass (A Quick Reminder)

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \sum_{i=1}^{N} \mathbf{r}_{i/O} m_i$$

- As $N \to \infty$: $m_i \to 0$

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/O} \,\mathrm{d}m$$

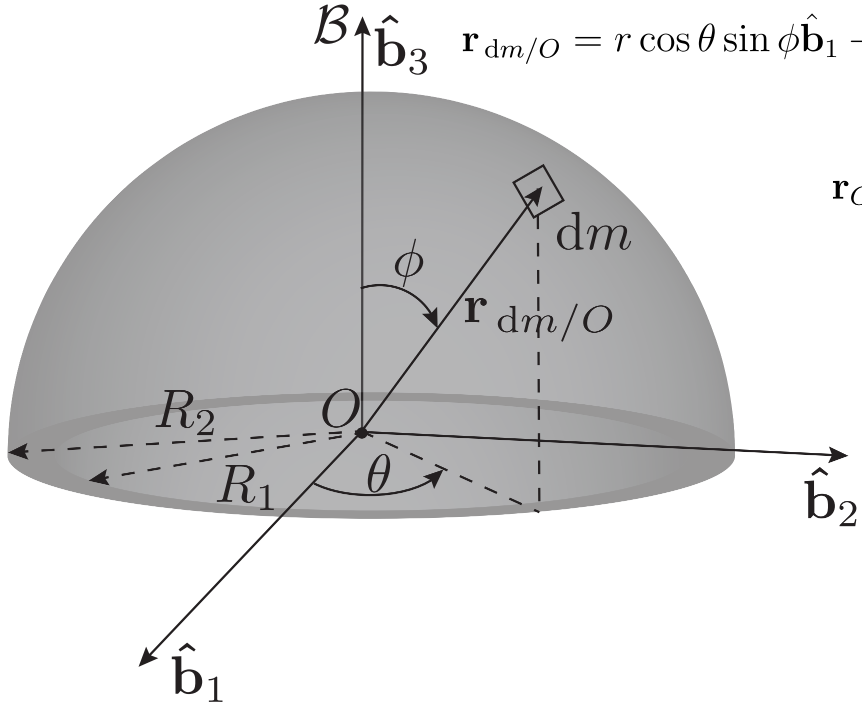
- If the density is given by $\rho(\mathbf{r}_{\,\mathrm{d}m/O})$:

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/O} \rho(\mathbf{r}_{\,\mathrm{d}m/O}) \,\mathrm{d}V$$

- The center of mass corollary becomes:

$$\int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/G} \rho(\mathbf{r}_{\,\mathrm{d}m/G}) \,\mathrm{d}V = 0$$

Example: Center of Mass of a Hemispherical Shell



$$\mathbf{r}_{\mathrm{d}m/O} = r\cos\theta\sin\phi\hat{\mathbf{b}}_1 + r\sin\theta\sin\phi\hat{\mathbf{b}}_2 + r\cos\phi\hat{\mathbf{b}}_3$$

$$\mathbf{r}_{G/O} = \frac{\rho}{m_G}\iiint_{\mathscr{B}} \mathbf{r}_{\mathrm{d}m/O}\,\mathrm{d}V$$

$$m_G = \rho V = \rho \iiint_{\mathscr{B}} dV$$

$$\mathbf{r}_{G/O} = \frac{3\left(R_1^4 - R_2^4\right)}{8\left(R_1^3 - R_2^3\right)}\hat{\mathbf{b}}_3$$

Angular Momentum of a Rigid Body

$${}^{\mathcal{I}}\mathbf{h}_O = \sum_{i=1}^{N} m_i \mathbf{r}_{i/O} \times {}^{\mathcal{I}}\mathbf{v}_{i/O} \quad \xRightarrow[N\to\infty]{m_i \to \mathrm{d}m} \quad {}^{\mathcal{I}}\mathbf{h}_O = \int_{\mathscr{B}} \mathbf{r}_{\mathrm{d}m/O} \times {}^{\mathcal{I}}\mathbf{v}_{\mathrm{d}m/O}\,\mathrm{d}m$$

Internal forces between i and j

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_O\right) = \underbrace{\sum_{i=1}^{N} \mathbf{r}_{i/O} \times \mathbf{F}_i^{(\mathrm{ext})}}_{\triangleq\, \mathbf{M}_O^{(\mathrm{ext})}} + \underbrace{\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\mathbf{r}_{i/O} - \mathbf{r}_{j/O}\right) \times \mathbf{F}_{i,j}}_{\text{Equals zero for rigid bodies}}$$

This is known as the internal moment assumption

NB: The internal moment assumption for rigid bodies is effectively the only additional postulate added by Euler's laws to Newton's laws. It is possible to define internal forces within a collection of particles that violate this assumption, therefore, we take its applicability to rigid bodies as a new law.

The Separation Principle

$$
{}^{\mathcal{I}}\mathbf{h}_O = \underbrace{\text{Angular Momentum of COM about an Inertially Fixed Point}}_{{}^{\mathcal{I}}\mathbf{h}_{G/O}} + \underbrace{\text{Angular Momentum of Body about its COM}}_{{}^{\mathcal{I}}\mathbf{h}_G}
$$

$$
{}^{\mathcal{I}}\mathbf{h}_{G/O} \triangleq m_G \mathbf{r}_{G/O} \times {}^{\mathcal{I}}\mathbf{v}_{G/O} \quad \Longrightarrow \quad \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_{G/O}\right) = \mathbf{M}_{G/O} \triangleq \mathbf{r}_{G/O} \times \mathbf{F}_G
$$

$$
{}^{\mathcal{I}}\mathbf{h}_G \triangleq \begin{cases} \displaystyle\sum_{i=1}^{N} m_i \mathbf{r}_{i/G} \times {}^{\mathcal{I}}\mathbf{v}_{i/G} & \text{Particles} \\ \displaystyle\int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/G} \times {}^{\mathcal{I}}\mathbf{v}_{\,\mathrm{d}m/G}\,\mathrm{d}m & \text{Continuous Bodies} \end{cases}
$$

$$
\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_G\right) = \mathbf{M}_G \triangleq \begin{cases} \displaystyle\sum_{i=1}^{N} \mathbf{r}_{i/G} \times \mathbf{F}_i^{(\text{ext})} & \text{Contact Forces } (N = \#\text{ contacts for rigid bodies}) \\ \displaystyle\int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/G} \times \mathbf{f}_{\mathrm{d}m}\,\mathrm{d}m & \text{Field Forces} \end{cases}
$$

Moment of Inertia

$$
\boxed{{}^{\mathcal{I}}\mathbf{h}_G = \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}}
$$

$$
\mathbb{I}_G \triangleq \begin{cases} \displaystyle\sum_{i=1}^{N} m_i \left[(\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G})\mathbb{U} - (\mathbf{r}_{i/G} \otimes \mathbf{r}_{i/G})\right] & \text{Collection of Particles} \\ \displaystyle\int_{\mathscr{B}} \left[(\mathbf{r}_{dm/G} \cdot \mathbf{r}_{dm/G})\mathbb{U} - (\mathbf{r}_{dm/G} \otimes \mathbf{r}_{dm/G})\right]\,\mathrm{d}m & \text{Rigid Body} \end{cases}
$$

Matrix of Inertia

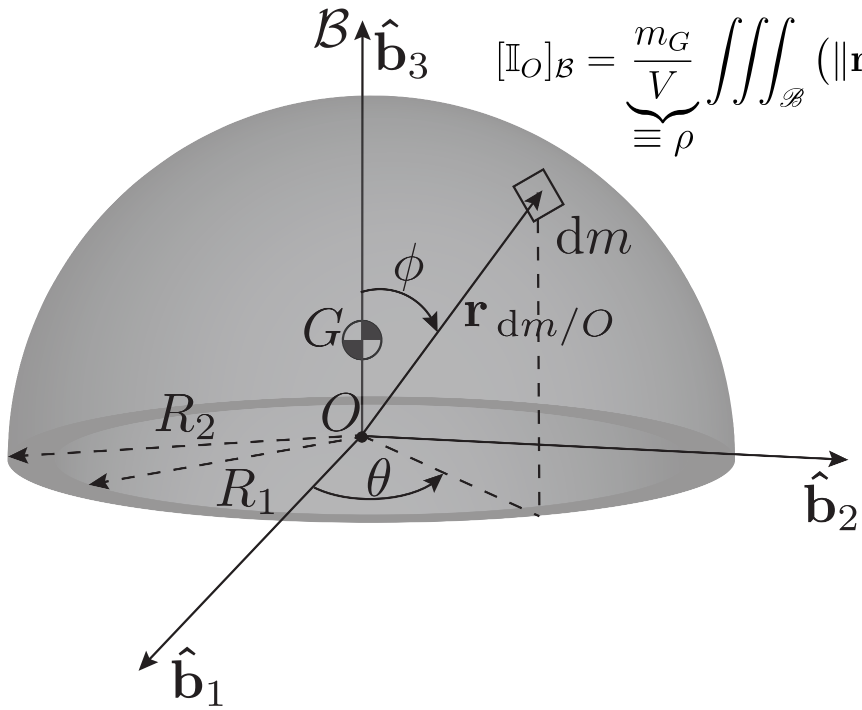
$$\mathbb{I}_G = \sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}\hat{\mathbf{b}}_i \otimes \hat{\mathbf{b}}_j$$

$$\begin{aligned}
[\mathbb{I}_G]_{\mathcal{B}} &= \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}_{\mathcal{B}} \\
&= \sum_{i=1}^{N} m_i \left(([\mathbf{r}_{i/G}]_{\mathcal{B}}^T [\mathbf{r}_{i/G}]_{\mathcal{B}}) I - [\mathbf{r}_{i/G}]_{\mathcal{B}} [\mathbf{r}_{i/G}]_{\mathcal{B}}^T \right) \\
&= \int_{\mathscr{B}} \left(\|\mathbf{r}_{dm/G}\|^2 I - [\mathbf{r}_{dm/G}]_{\mathcal{B}} [\mathbf{r}_{dm/G}]_{\mathcal{B}}^T \right) \,\mathrm{d}m \\
&= \int_{\mathscr{B}} \left(\|\mathbf{r}_{dm/G}\|^2 I - [\mathbf{r}_{dm/G}]_{\mathcal{B}} [\mathbf{r}_{dm/G}]_{\mathcal{B}}^T \right) \rho(\mathbf{r}_{dm/G}) \,\mathrm{d}V
\end{aligned}$$

Diagonal elements of the inertia matrix are called **moments** of inertia, while off-diagonal entries are known as **products** of inertia.

Example: Moment of Inertia of a Hemispherical Shell

$$[\mathbb{I}_O]_{\mathcal{B}} = \underbrace{\frac{m_G}{V}}_{\equiv \rho} \iiint_{\mathscr{B}} \left(\|\mathbf{r}_{dm/O}\|^2 I - [\mathbf{r}_{dm/O}]_{\mathcal{B}} [\mathbf{r}_{dm/O}]_{\mathcal{B}}^T \right) \,\mathrm{d}V$$



$$[\mathbb{I}_O]_{\mathcal{B}} = \frac{2m_G\left(R_1^5 - R_2^5\right)}{5\left(R_1^3 - R_2^3\right)} I$$

Moments and Angular Momentum about an Arbitrary Point Q fixed to a Rigid Body

$$\boxed{\mathbf{M}_Q = \mathbf{M}_G - \mathbf{r}_{Q/G} \times \sum_{i=1}^{N} \mathbf{F}_i^{(\mathrm{ext})}}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_Q\right) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_Q\right) + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_Q = \mathbf{M}_Q + \mathbf{r}_{Q/G} \times m_G {}^{\mathcal{I}}\mathbf{a}_{Q/O}$$

$${}^{\mathcal{I}}\mathbf{h}_Q = \mathbb{I}_Q \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \qquad \mathbb{I}_Q \triangleq \sum_{i=1}^{N} m_i \left((\mathbf{r}_{i/Q} \cdot \mathbf{r}_{i/Q}))\mathbb{U} - (\mathbf{r}_{i/Q} \otimes \mathbf{r}_{i/Q})\right) \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$$

The Parallel Axis Theorem

$$\mathbb{I}_Q = \mathbb{I}_G + m_G \left[(\mathbf{r}_{Q/G} \cdot \mathbf{r}_{Q/G})\mathbb{U} - (\mathbf{r}_{Q/G} \otimes \mathbf{r}_{Q/G})\right]$$

Rigid Body Dynamics

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\mathbf{h}_G = \mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \left(\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right) = \mathbf{M}_G$$

$$[\mathbb{I}_G]_{\mathcal{B}} \left[\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} + \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}} [\mathbb{I}_G]_{\mathcal{B}} \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} = [\mathbf{M}_G]_{\mathcal{B}}$$

Diagonalization

$$\text{For any tensor } \mathbb{T}\text{:} \quad [\mathbb{T}]_{\mathcal{B}} = {}^{\mathcal{B}}C^{\mathcal{A}}\,[\mathbb{T}]_{\mathcal{A}}\,\underbrace{{}^{\mathcal{A}}C^{\mathcal{B}}}_{\equiv\left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^T}$$

Recall the eigendecomposition of a symmetric matrix

$$A = A^T \quad \Rightarrow \quad A\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad \Rightarrow \quad \lambda_i \in \mathbb{R}\ \forall i, \quad \mathbf{v}_i \cdot \mathbf{v}_j = 0\ \forall i, j$$

$$\text{Let } P \triangleq \begin{bmatrix} \mathbf{v}_i & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \quad \Rightarrow \quad A = PDP^{-1}, \quad P^{-1} = P^T$$

$$D = \operatorname{diag}\left(\{\lambda_i\}\right) = P^T A P$$

$$\boxed{[\mathbb{I}_G]_{\mathcal{B}_P} = {}^{\mathcal{B}_\mathcal{P}}C^{\mathcal{A}}\,[\mathbb{I}_G]_{\mathcal{A}}\,{}^{\mathcal{A}}C^{\mathcal{B}_\mathcal{P}}}$$

Eigenvectors of $[\mathbb{I}_G]_{\mathcal{A}}$

Principal Axis Frame ($\mathcal{B}_p$) and Euler's Equations

$$[\mathbb{I}_G]_{\mathcal{B}_p} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p}$$

$$[\mathbb{I}_G]_{\mathcal{B}_p} \left[\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}_p} + \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}_p} [\mathbb{I}_G]_{\mathcal{B}_p} \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}_p} = [\mathbf{M}_G]_{\mathcal{B}_p} \quad \Longrightarrow$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p} \begin{bmatrix} \dot\omega_1 \\ \dot\omega_2 \\ \dot\omega_3 \end{bmatrix}_{\mathcal{B}_p} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}_{\mathcal{B}_p} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_{\mathcal{B}_p} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}_{\mathcal{B}_p}$$

Euler's Equations

$$I_1\dot\omega_1 + (I_3 - I_2)\omega_2\omega_3 = M_1$$
$$I_2\dot\omega_2 + (I_1 - I_3)\omega_1\omega_3 = M_2$$
$$I_3\dot\omega_3 + (I_2 - I_1)\omega_1\omega_2 = M_3$$

Every Rigid Body Always Has a Principal Axis Frame

NB: Moment of inertia matrices are real and symmetric, and so are always diagonizable. The eigendecomposition of the moment of inertia matrix for any arbitrary body-fixed frame gives the MOI matrix in the principal axis frame (the diagonal matrix whose non-zero entries are the eigenvalues) and the DCM rotating between the two frames (the matrix whose columns are the eigenvectors).

Torque Free Motion

$$^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3$$

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0 \end{aligned}$$

$$\underline{I_1 = I_2}$$

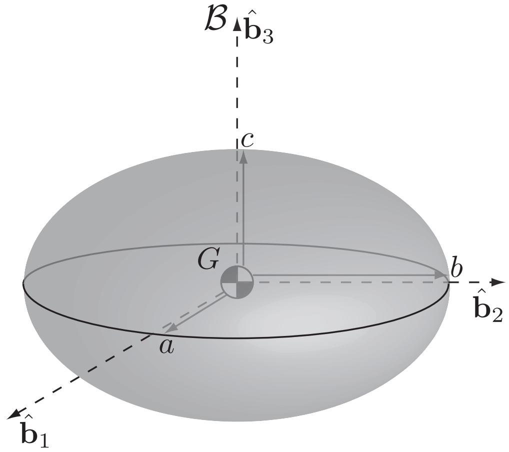
$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

$$\underline{\omega_3 \gg \omega_1, \omega_2}$$

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 &\approx 0 \end{aligned}$$

$$\left.\begin{aligned} \dot{\omega}_1 &= -\frac{I_3 - I_2}{I_1}\omega_2\omega_3 \\ \dot{\omega}_2 &= -\frac{I_1 - I_3}{I_2}\omega_1\omega_3 \end{aligned}\right\} \left.\begin{aligned} \ddot{\omega}_1 &= -\frac{I_3 - I_2}{I_1}\dot{\omega}_2\omega_3 \\ \ddot{\omega}_2 &= -\frac{I_1 - I_3}{I_2}\dot{\omega}_1\omega_3 \end{aligned}\right\} \boxed{\begin{aligned} \ddot{\omega}_1 + \omega_n^2\omega_1 &= 0 \\ \ddot{\omega}_2 + \omega_n^2\omega_2 &= 0 \\ \omega_n^2 &\triangleq \frac{(I_3 - I_2)(I_3 - I_1)\omega_3^2}{I_1 I_2} \end{aligned}}$$

Solid Ellipsoids



$$[\mathbb{I}_G]_{\mathcal{B}} = \frac{m}{5}\begin{bmatrix} (b^2+c^2) & 0 & 0 \\ 0 & (a^2+c^2) & 0 \\ 0 & 0 & (a^2+b^2) \end{bmatrix}_{\mathcal{B}}$$

$$a > b > c \Longrightarrow \underset{\text{Major Axis}}{I_3} > I_2 > \underset{\text{Minor Axis}}{I_1}$$

From this matrix, we can also see a general property of all inertia matrices: the sum of any two moments of inertia is greater than or equal to the third.

Torque Free Motion Conserved Quantities

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3 \qquad [\mathbb{I}_G]_{\mathcal{B}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}} \qquad \begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0 \end{aligned}$$

$$\boxed{\left.\begin{aligned} {}^{\mathcal{I}}\mathbf{h}_G &= \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = I_1\omega_1\hat{\mathbf{b}}_1 + I_2\omega_2\hat{\mathbf{b}}_2 + I_3\omega_3\hat{\mathbf{b}}_3 \\ T_G &= \frac{1}{2}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \cdot \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \end{aligned}\right\}\text{Constant}}$$

$$I_3 > I_2 > I_1 \begin{cases} \max T = \text{ spin about minor axis}(I_1) \\ \min T = \text{ spin about major axis}(I_3) \end{cases}$$

$$\sqrt{2TI_1} < \|{}^{\mathcal{I}}\mathbf{h}_G\| < \sqrt{2TI_3}$$

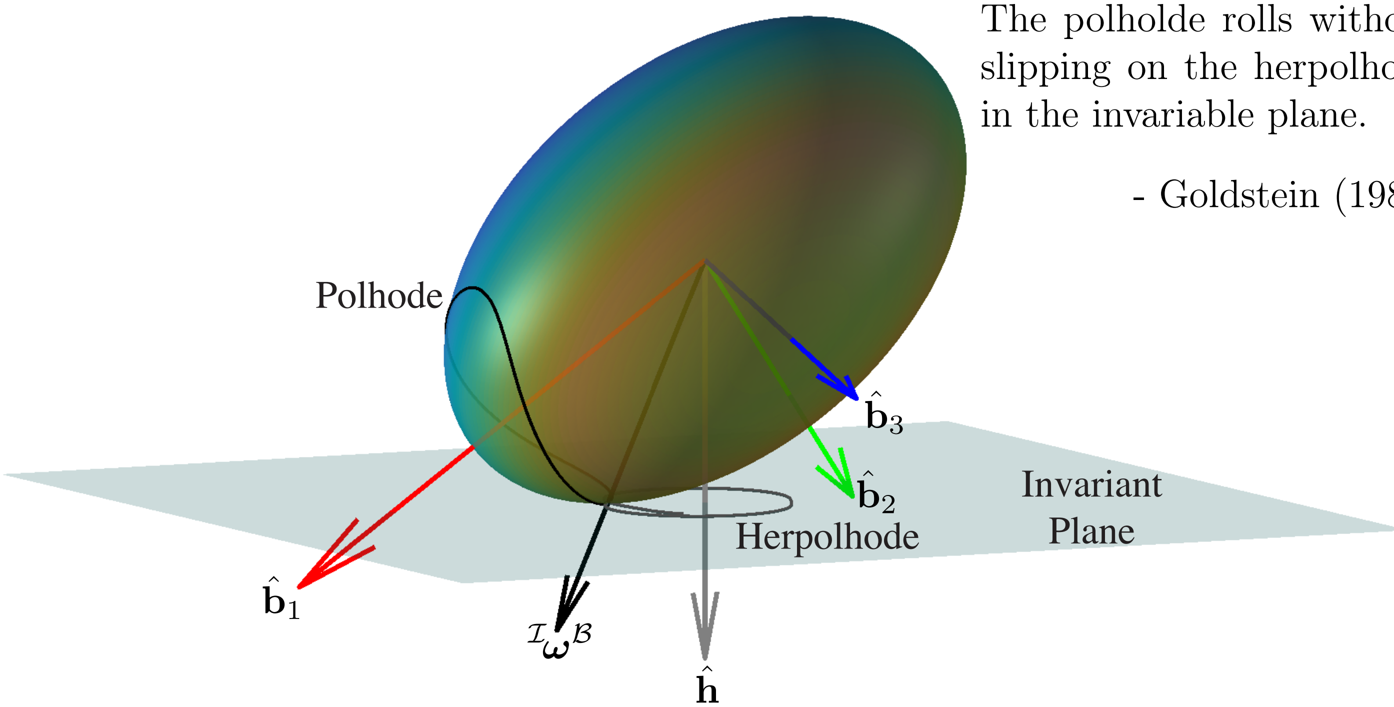
Torque Free Motion Conserved Quantities (again)

$$T_G = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \Longrightarrow \frac{\omega_1^2}{2T_G/I_1} + \frac{\omega_2^2}{2T_G/I_2} + \frac{\omega_3^2}{2T_G/I_3} = 1$$

$$\|{}^{\mathcal{I}}\mathbf{h}_G\|^2 \triangleq h_G^2 = \|\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\|^2 \Longrightarrow \frac{\omega_1^2}{h_G^2/I_1^2} + \frac{\omega_2^2}{h_G^2/I_2^2} + \frac{\omega_3^2}{h_G^2/I_3^2} = 1$$

These two equations describe two ellipsoids. The intersection of these two ellipsoids traces the path of the angular velocity vector in the body-fixed frame (polhode). The contact point between the inertia ellipsoid and an invariant plane orthogonal to the angular momentum vector traces the path of the angular velocity vector in the inertial frame (herpolohode). This is called the Poinsot construction.

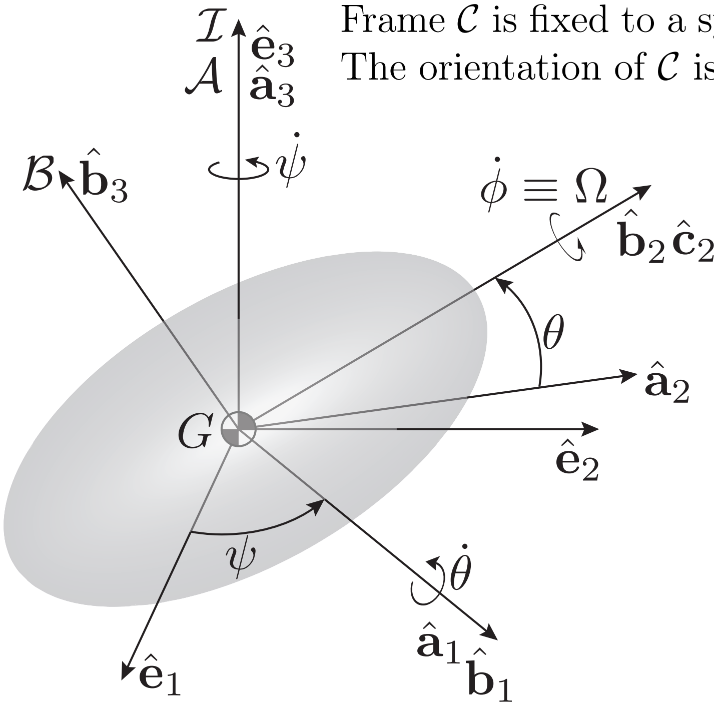
The Poinsot Construction



The polholde rolls without slipping on the herpolhode in the invariable plane.

- Goldstein (1980)

Spinning Symmetric Rigid Body (The Setup)



Frame $\mathcal{C}$ is fixed to a symmetric rigid body with symmetry axis $\hat{\mathbf{c}}_2$. The orientation of $\mathcal{C}$ is given by a 3-1-2 $(\psi, \theta, \phi)^{\mathcal{I}}_{\mathcal{C}}$ Body-3 Rotation

Intermediate frame $\mathcal{B}$ is defined with orientation 3-1 $(\psi, \theta)^{\mathcal{I}}_{\mathcal{B}}$ such that $\hat{\mathbf{b}}_2 \equiv \hat{\mathbf{c}}_2$

$$^{\mathcal{B}}C^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{aligned} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} &= \dot{\psi}\hat{\mathbf{a}}_3 + \dot{\theta}\hat{\mathbf{b}}_1 \\ &= \dot{\theta}\hat{\mathbf{b}}_1 + \dot{\psi}\sin\theta\hat{\mathbf{b}}_2 + \dot{\psi}\cos\theta\hat{\mathbf{b}}_3 \\ {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{C}} &= \Omega\hat{\mathbf{b}}_2 \\ {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{C}} &= {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{C}} \end{aligned}$$

NB: $\hat{\mathbf{b}}_2 \equiv \hat{\mathbf{c}}_2$ is the symmetry axis of the body, which means that $[\mathbb{I}_G]_{\mathcal{B}} = [\mathbb{I}_G]_{\mathcal{C}}$.

Spinning Symmetric Rigid Body (The Dynamics)

$$\left[{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}} = [\mathbb{I}_G]_{\mathcal{B}} \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{C}}\right]_{\mathcal{B}} = \begin{bmatrix} I_1\dot{\theta} \\ I_2\left(\Omega + \dot{\psi}\sin(\theta)\right) \\ I_1\dot{\psi}\cos(\theta) \end{bmatrix}_{\mathcal{B}}$$

NB: This is **not** a typo

$$\begin{aligned} \left[\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}} &= \left[\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}} + \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}}\left[{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}} \\ &= \begin{bmatrix} I_1\dot{\psi}^2\sin(\theta)\cos(\theta) + I_1\ddot{\theta} - I_2\dot{\psi}\left(\Omega + \dot{\psi}\sin(\theta)\right)\cos(\theta) \\ I_2\left(\ddot{\psi}\sin(\theta) + \dot{\psi}\dot{\theta}\cos(\theta)\right) \\ I_1\ddot{\psi}\cos(\theta) - 2I_1\dot{\psi}\dot{\theta}\sin(\theta) + I_2\dot{\theta}\left(\Omega + \dot{\psi}\sin(\theta)\right) \end{bmatrix}_{\mathcal{B}} \end{aligned}$$

The first line defines the inertial angular momentum, and so the angular velocity used must be of the body-fixed frame ($\mathcal{C}$). The second line is the application of the transport equation for frame $\mathcal{B}$, so the angular velocity used there is ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$.

Spinning Symmetric Rigid Body (The Solution)

Note: $({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_G) \cdot \hat{\mathbf{b}}_2 = 0$

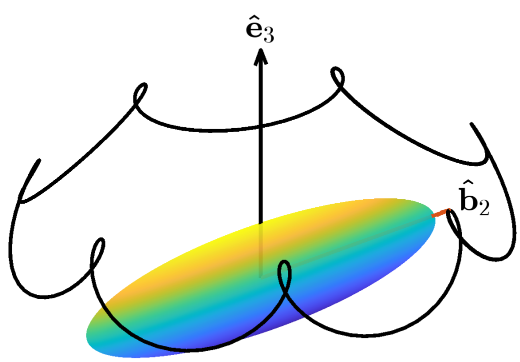
Therefore: ${}^{\mathcal{B}}\mathbf{h}_G \cdot \hat{\mathbf{b}}_2 = 0 \quad \Longrightarrow \quad I_2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\Omega + \dot{\psi}\sin\theta\right) = 0$

$$\Longrightarrow \quad \Omega + \dot{\psi}\sin\theta = C\,(\text{constant})$$

Assume: $\mathbf{M}_G = -M_1\hat{\mathbf{b}}_1$. Then:

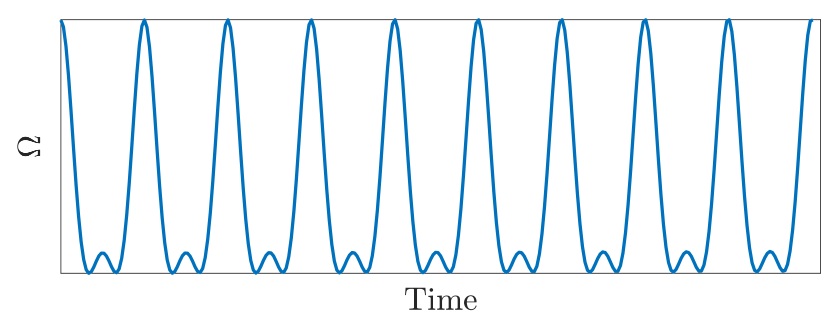
$$\boxed{\begin{aligned} \ddot{\theta} &= \frac{1}{I_1}\left(CI_2\dot{\psi}\cos(\theta) - \frac{I_1\dot{\psi}^2}{2}\sin(2\theta) - M_1\right) \\ \ddot{\psi} &= \frac{\dot{\theta}}{I_1}\left(-\frac{CI_2}{\cos(\theta)} + 2I_1\dot{\psi}\tan(\theta)\right) \end{aligned}}$$

Torqued, Spinning, Symmetric Rigid Body Motion



In general, a torqued spinning, symmetric, rigid body will display two bulk types of motion: **Precession**: the (relatively) slow rotation of the spin/symmetry axis about the inertially fixed axis orthogonal to the axis about which torque is applied ($\hat{\mathbf{e}}_3$ in the preceding example); and **nutation**: the (relatively) fast secondary rotation of the spin/symmetry axis along its precessing trajectory.

Remember that Ω (the spin rate) is not constant in time. Only the quantity $\Omega + \dot{\psi}\sin\theta$ is conserved.



Attitude Determination and Control

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MAE 4060/5065, Fall 2021

Attitude Determination and Control

Now that we have established the basics of attitude dynamics, we can turn our attention to the control of the orientation of our spacecraft. Just as we did with propulsion, we will briefly look at the hardware that is available to us for controlling spacecraft orientation, and then look into specific approaches to attitude control. We will discuss both passive and active control methods and will look at some basic attitude control laws. While the full study of feedback control is beyond the scope of this class, it is a vital component in the attitude control toolbox, and anyone wishing to work in this area must study booth classical feedback control, as well as modern and optimal control (the latter is also required for trajectory planning and optimization). Finally, it is equally important to be able to measure a spacecraft's attitude as to control it, and, in fact, you cannot control attitude until you have a way of estimating it. Therefore, we will also take a quick look at some basic attitude estimation algorithms, as well as the hardware typically used for attitude determination.

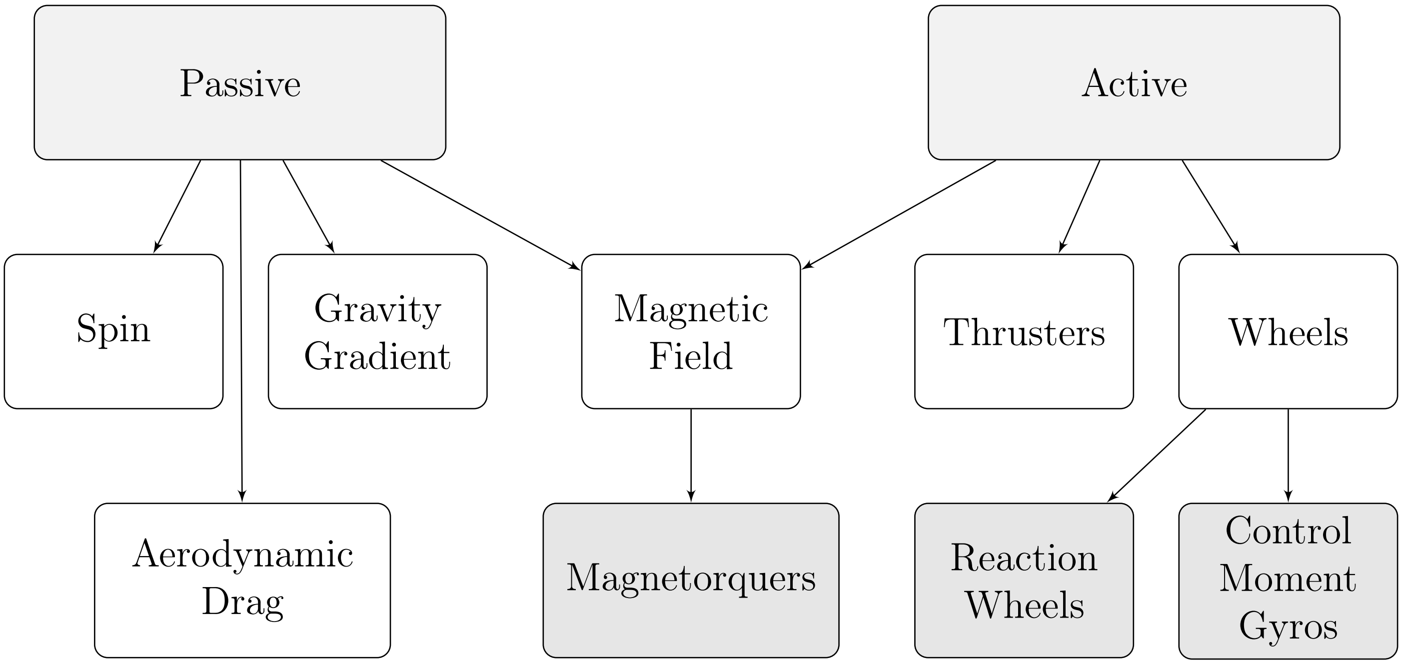
Attitude Alphabet Soup

ACS	Attitude Control System
ADCS	Attitude Determination and Control System
ADCNS	Attitude Determination, Control, and Navigation System
GNC	Guidance, Navigation, and Control

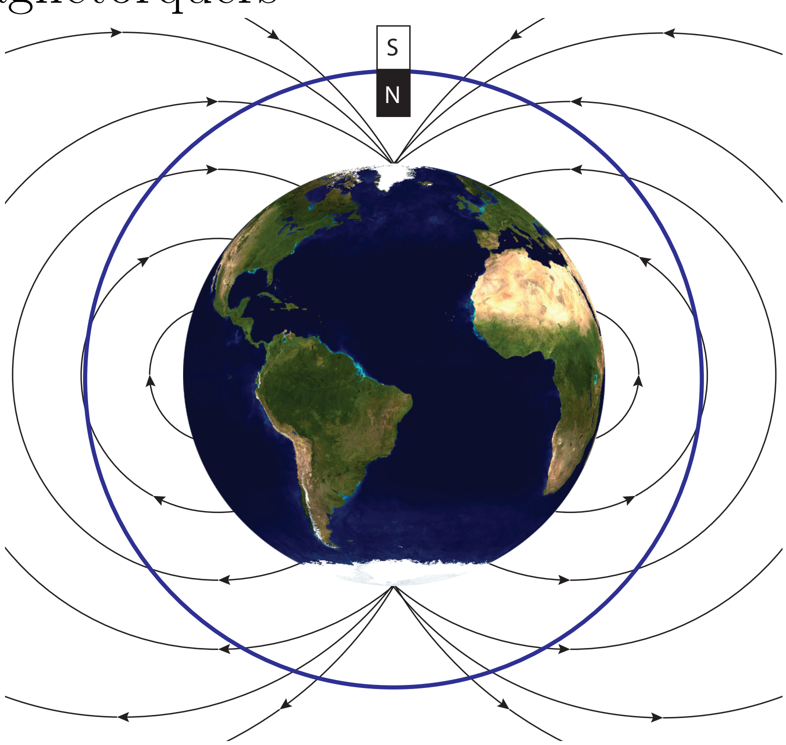
Attitude Perturbations

- Atmospheric Drag - Typically dominant for Low Earth Orbits
- Solar Radiation Pressure - Only significant perturbation when operating away from planetary masses
- Magnetic Field Torque - Typically only significant below GEO
- Gravity Gradient - Typically only significant at LEO/MEO

Attitude Control Technologies



Magnetorquers



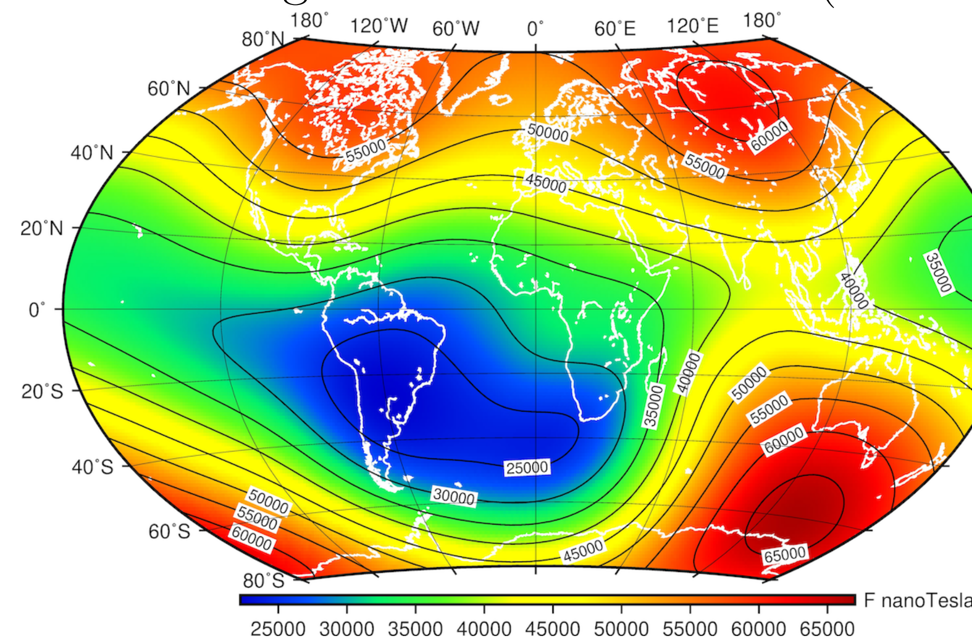
Magnetic Control Torque

Geomagnetic Field Vector

$$\mathbf{M} = -\mathbf{B} \times \mathbf{N}$$

Magnetic Dipole Moment due to Magnetorquer

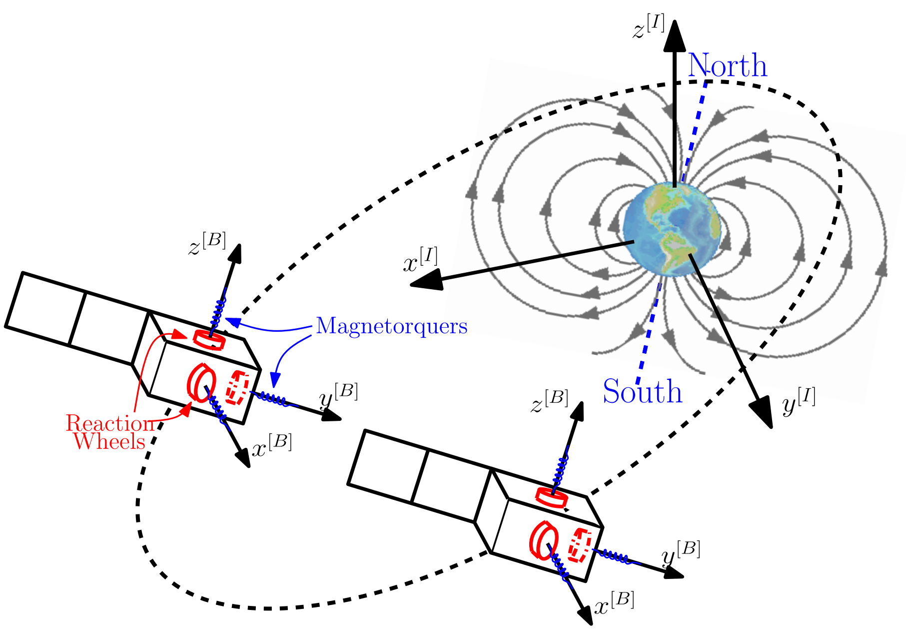
International Geomagnetic Reference Field (IGRF13)



From: http://www.geomag.bgs.ac.uk/research/modelling/IGRF.html.

See also: https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html

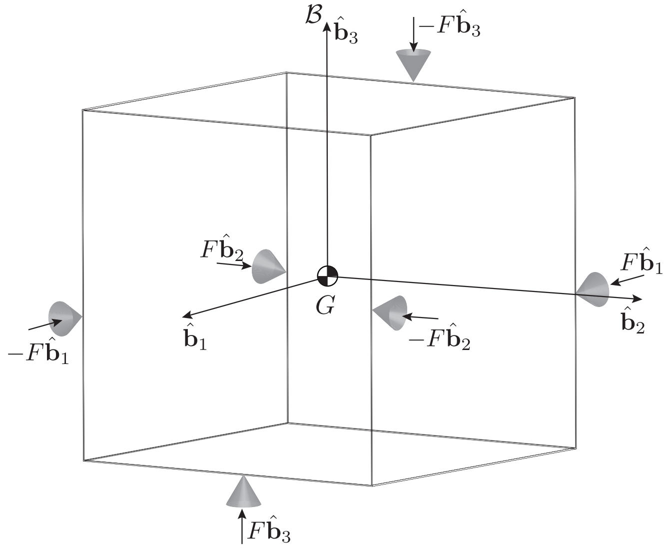
Magnetorquers and Reaction Wheels



Trégouët et al. (2015) Fig. 1

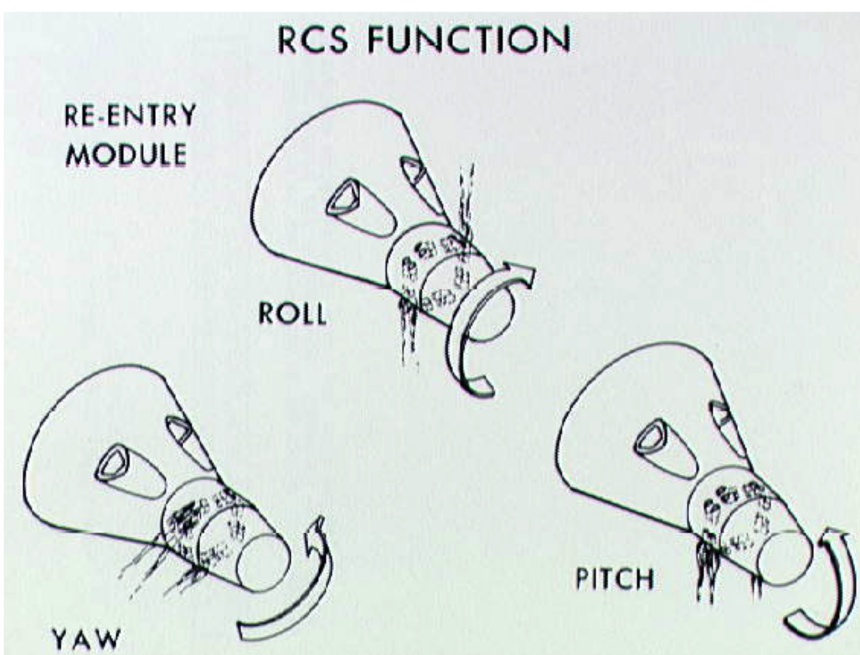
Magnetorquers are often used to do momentum offloading from reaction wheels. However, this is typically only feasible for low and mid-Earth orbits.

Reaction Control Systems



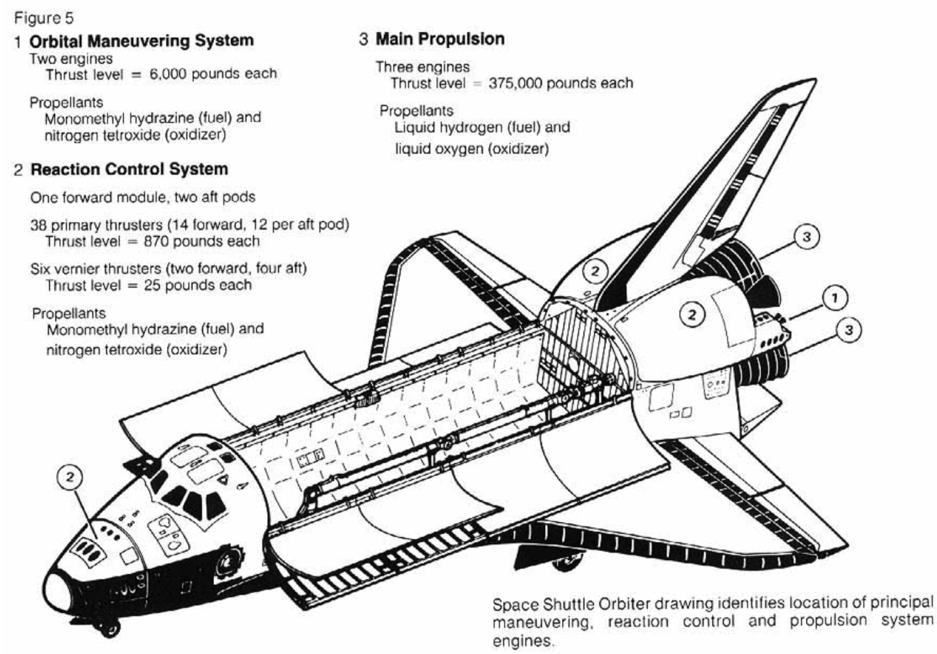
RCS works on the basis of force couples: firing thrusters, in pairs, in opposite directions, will produce a pure torque. In this example, the thrusters are all equidistant from the COM in each direction, so firing any pair would produce a pure rotation about a single axis, with no accompanying translation.

Gemini RCS



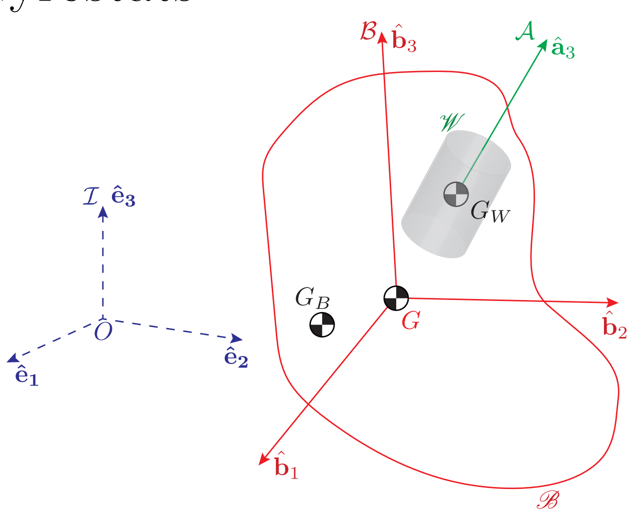
Credit: Gemini/NASA

Space Shuttle Propulsion and Attitude Control



From http://history.nasa.gov/rogersrep/v1p46.jpg

Gyrostats



Carrier/Platform ($\mathscr{B}$) + Rotors/Gyros/Wheels ($\mathscr{W}_i$) = Gyrostat

$$m = m_B + m_W$$

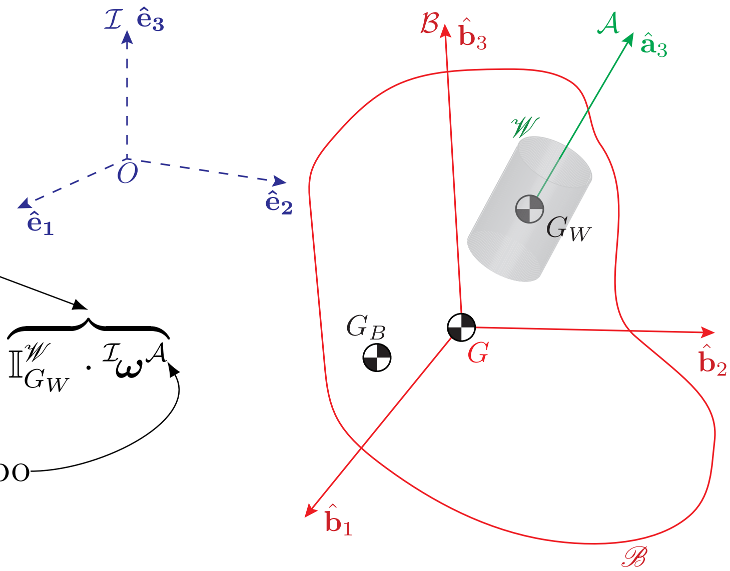
$$\mathbf{r}_{G/O} = \frac{m_B}{m}\mathbf{r}_{G_B/O} + \frac{m_W}{m}\mathbf{r}_{G_W/O}$$

$$m_B\mathbf{r}_{G_B/G} = -m_W\mathbf{r}_{G_W/G}$$

$$\begin{aligned}\mathbb{I}_G = \mathbb{I}_G^{\mathscr{B}} + \mathbb{I}_G^{\mathscr{W}} = \; & \mathbb{I}_{G_B}^{\mathscr{B}} + m_B\left[(\mathbf{r}_{G/G_B}\cdot\mathbf{r}_{G/G_B})\mathbb{U} - (\mathbf{r}_{G/G_B}\otimes\mathbf{r}_{G/G_B})\right] + \\ & \mathbb{I}_{G_W}^{\mathscr{W}} + m_W\left[(\mathbf{r}_{G/G_W}\cdot\mathbf{r}_{G/G_W})\mathbb{U} - (\mathbf{r}_{G/G_W}\otimes\mathbf{r}_{G/G_W})\right]\end{aligned}$$

NB: Here, $\mathbb{I}_G$ is the gyrostat **total** MOI about the net center of mass. $\mathbb{I}_G^{\mathscr{B}}, \mathbb{I}_G^{\mathscr{W}}$ are the MOIs of the carrier and wheel about the net COM, while $\mathbb{I}_{G_B}^{\mathscr{B}}, \mathbb{I}_{G_W}^{\mathscr{W}}$ are their MOIs about their own COMs.

Gyrostat Kinematics



$$
\begin{aligned}
{}^{\mathcal{I}}\mathbf{h}_G^{\text{tot}} &= {}^{\mathcal{I}}\mathbf{h}_G^{\mathscr{B}} + {}^{\mathcal{I}}\mathbf{h}_G^{\mathscr{W}} \\
&= {}^{\mathcal{I}}\mathbf{h}_G^{\mathscr{B}} + {}^{\mathcal{I}}\mathbf{h}_{G_W/G}^{\mathscr{W}} + {}^{\mathcal{I}}\mathbf{h}_{G_W}^{\mathscr{W}} \\
&= \overbrace{\mathbb{I}_G^{\mathscr{B}} \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}} + \overbrace{m_W \mathbf{r}_{G_W/G} \times {}^{\mathcal{I}}\mathbf{v}_{G_W/G}} + \overbrace{\mathbb{I}_{G_W}^{\mathscr{W}} \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{A}}}
\end{aligned}
$$

NB: This is **not** a typo

$$
\boxed{{}^{\mathcal{I}}\mathbf{h}_G^{\text{tot}} = \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + \mathbb{I}_{G_W}^{\mathscr{W}} \cdot {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{A}}}
$$

Gyrostat Dynamics

$$
\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left(\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + I_s \omega_s \hat{\mathbf{a}}_3\right) = \mathbf{M}_G^{(\text{ext})}
$$

$$
\mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right) + \underbrace{I_s \omega_s \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\mathbf{a}}_3\right)}_{\text{Zero for } \hat{\mathbf{a}}_3 \text{ fixed in } \mathscr{B}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + I_s \omega_s {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{a}}_3 = \mathbf{M}_G^{(\text{ext})}
$$

Torque on the wheel about $\hat{\mathbf{a}}_3$ due to the carrier:

$$
h_3 \triangleq {}^{\mathcal{I}}\mathbf{h}_{G_W}^{\mathscr{W}} \cdot \hat{\mathbf{a}}_3 = I_s \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \cdot \hat{\mathbf{a}}_3\right) + I_s \omega_s
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} h_3 = \mathbf{M}_{\mathscr{W}/\mathscr{B}} \cdot \hat{\mathbf{a}}_3
$$

Kelvin's Gyrostat

$$\text{If } I_s\omega_s \text{ is constant: } I_s \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \cdot \hat{\mathbf{a}}_3 = \mathbf{M}_{\mathscr{W}/\mathscr{B}} \cdot \hat{\mathbf{a}}_3$$

$$\mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + I_s\omega_s {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{a}}_3 = \mathbf{M}_G$$

Corresponds to constant h_3!

Multiple Wheels

$$ {}^{\mathcal{I}}\mathbf{h}_G^{\text{tot}} = \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + \sum_{i=1}^{N} I_{s_i} \underbrace{\omega_{s_i}}_{\text{Controlled by wheels}} \underbrace{\hat{\mathbf{a}}_{3_i}}_{\text{Controlled by CMGs}} $$

Dynamic Balance (the setup)

We can use ACS components to cancel the effects of non-zero products of inertia to achieve equilibrium orientation conditions.

- Consider a gyrostat with N wheels, and define a carrier-fixed frame $\mathcal{B} = \left(G, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\right)$ for total center of mass G

Body-fixed frame of wheel $\mathcal{W}_i$

- Total angular momentum contribution of the wheels: $\mathbf{h} \triangleq \sum_{i=1}^{N} \mathbb{I}_{G_{W_i}}^{\mathcal{W}_i} \cdot {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{A}_i}$

Center of mass of wheel $\mathcal{W}_i$

- **Total** spacecraft moment of inertia matrix: $[\mathbb{I}_G]_{\mathcal{B}} = \begin{bmatrix} I_s & I_{12} & I_{13} \\ I_{12} & I_t & 0 \\ I_{13} & 0 & I_t \end{bmatrix}_{\mathcal{B}}$
- Given: ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \omega_s \hat{\mathbf{b}}_1$, Want: $\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\mathbf{h}_G^{\mathrm{tot}} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = 0$

Dynamic Balance (the solution)

$$\text{Want: } \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\mathbf{h}_G^{\mathrm{tot}} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = 0$$

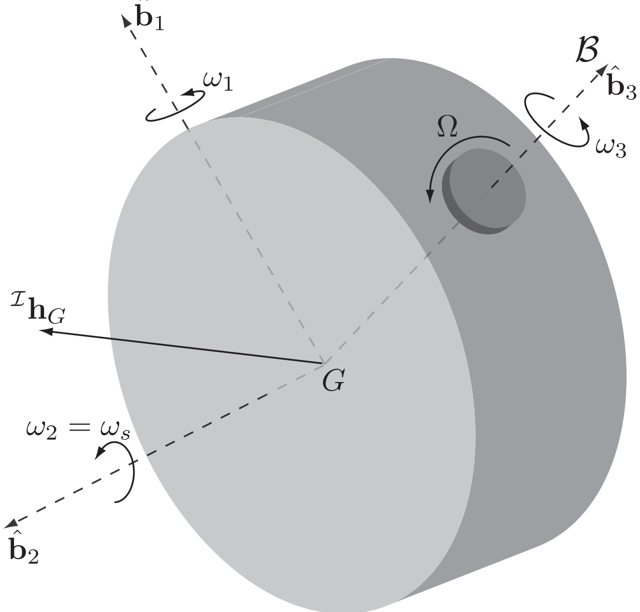
$$ {}^{\mathcal{I}}\mathbf{h}_G^{\mathrm{tot}} = \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + \mathbf{h} \quad \Longrightarrow \quad \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\mathbf{h}_G^{\mathrm{tot}} = \mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{h}$$

In equilibrium condition: ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{h} = 0$

$$[\mathbf{h}]_{\mathcal{B}} \triangleq \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}_{\mathcal{B}} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_s \\ 0 & \omega_s & 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} I_s & I_{12} & I_{13} \\ I_{12} & I_t & 0 \\ I_{13} & 0 & I_t \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} \omega_s \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_s \\ 0 & \omega_s & 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}_{\mathcal{B}} = 0$$

$$\boxed{\begin{bmatrix} 0 \\ -I_{13}\omega_s^2 \\ I_{12}\omega_s^2 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} 0 \\ -h_3\omega_s \\ h_2\omega_s \end{bmatrix}_{\mathcal{B}} = 0 \quad \Longrightarrow \quad \begin{array}{l} h_1 = \text{Arbitrary Constant} \\ h_2 = -I_{12}\omega_s \\ h_3 = -I_{13}\omega_s \end{array}}$$

Nutation Dampers



$$\left[\mathbb{I}_G^{\mathrm{tot}}\right]_{\mathcal{B}} = \begin{bmatrix} I_T & 0 & 0 \\ 0 & I_S & 0 \\ 0 & 0 & I_T \end{bmatrix}_{\mathcal{B}}$$

$$\left[\mathbb{I}_{G_W}^{\mathrm{wheel}}\right]_{\mathcal{B}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_W \end{bmatrix}_{\mathcal{B}}$$

Recall:

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{h}_G^{\mathrm{tot}} &= \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + \mathbb{I}_{G_W}^{\mathscr{W}} \cdot {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{A}} \\ &= \mathbb{I}_G^{\mathrm{tot}} \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + I_W \Omega \hat{\mathbf{b}}_3 \end{aligned}$$

Frame attached to wheel

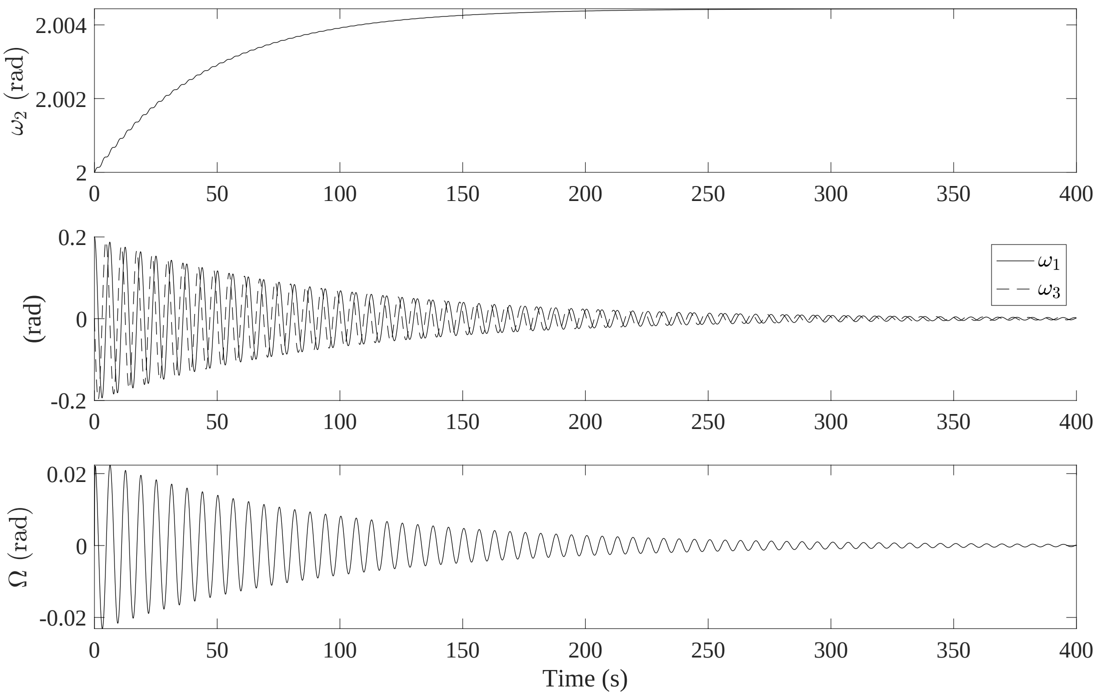
Nutation dampers are mechanical devices capable of dissipating energy used to damp out disturbances the perturb a spinning spacecraft off of its symmetry axis. The version shown here consists of a small wheel mounted orthogonally to the spin axis, which has a friction torque of magnitude $-D\Omega$ for a spin rate Ω.

Nutation Damper Equations of Motion

$$\begin{aligned} \dot{\omega}_1 &= \omega_2 \left(\left(\frac{I_S}{I_T} - 1 \right) \omega_3 - \Omega \frac{I_W}{I_T} \right) \\ \dot{\omega}_2 &= \Omega \frac{I_W}{I_S} \omega_1 \\ \dot{\omega}_3 &= \left[\Omega \frac{D}{I_T} - \omega_1 \omega_2 \left(\frac{I_S}{I_T} - 1 \right) \right] \left(1 - \frac{I_W}{I_T} \right)^{-1} \\ \dot{\Omega} &= \left[-\Omega \frac{D}{I_T} \frac{I_T}{I_W} + \omega_1 \omega_2 \left(\frac{I_S}{I_T} - 1 \right) \right] \left(1 - \frac{I_W}{I_T} \right)^{-1} \end{aligned}$$

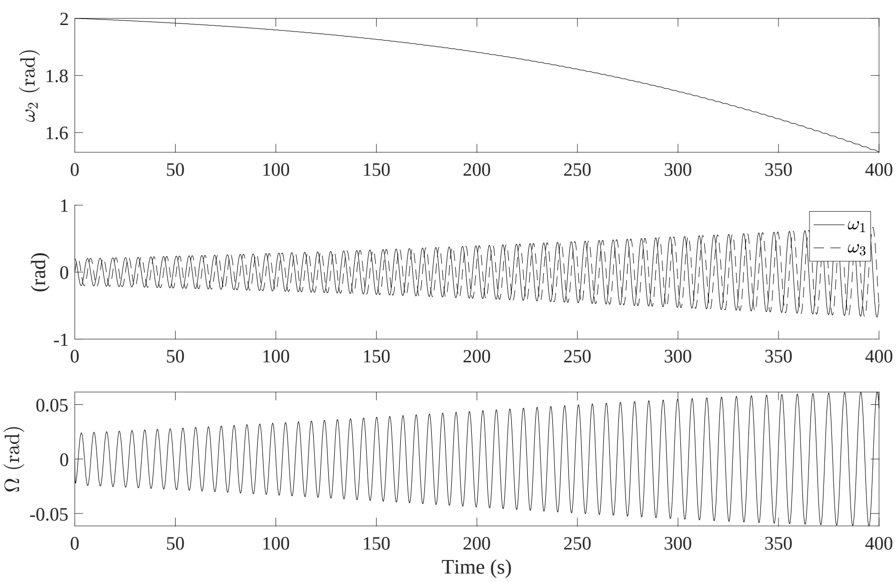
Note that all of the equations are functions of three ratios: $\dfrac{I_S}{I_T}$, $\dfrac{I_W}{I_T}$, and $\dfrac{D}{I_T}$.

Nutation Dampers in Action



An initial disturbance torque (about $\hat{\mathbf{b}}_1$) causes the nutation damper to spin up. It then spins down again due to the viscous damping. Note that the spin rate about $\hat{\mathbf{b}}_2$ does not return to its original value—the added angular momentum results in a slightly increased steady state spin about the symmetry axis.

Nutation Dampers in Action ($I_s < I_t$)



A nutation damper on a minor axis spinner has the exact opposite effect—any disturbance gets magnified and the damper spin rate continuously increases instead of damping out.

Recall Torque Free Motion

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0 \end{aligned}$$

$$[\mathbb{I}_G]_{\mathcal{B}} = \begin{bmatrix} I_t & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & I_s \end{bmatrix}_{\mathcal{B}} \qquad \begin{aligned} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} &= \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3 \\ \omega_3 \equiv \omega_s &\gg \omega_1, \omega_2 \end{aligned}$$

$$\boxed{\begin{aligned} \ddot{\omega}_1 + \omega_n^2\omega_1 &= 0 \\ \ddot{\omega}_2 + \omega_n^2\omega_2 &= 0 \end{aligned} \qquad \omega_n^2 \triangleq \frac{(I_3 - I_2)(I_3 - I_1)\omega_3^2}{I_1 I_2} = \left(\frac{I_s - I_t}{I_t}\omega_s\right)^2}$$

ω_n is called the **nutation frequency**

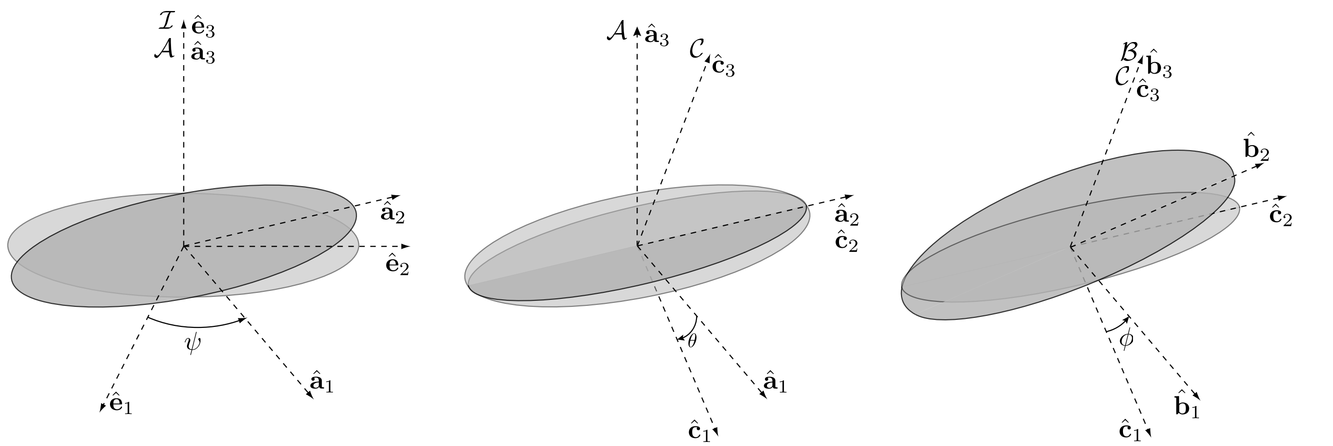
Nutation Angle and Coning

$$\text{Assuming} \quad \begin{aligned} \omega_1(t=0) &= \omega_0 \\ \omega_2(t=0) &= 0 \end{aligned} \implies \left.\begin{aligned} \dot{\omega}_1 &= -\omega_n\omega_2 \\ \dot{\omega}_2 &= \omega_n\omega_1 \end{aligned}\right\} \begin{aligned} \omega_1 &= \omega_0\cos(\omega_n t) \\ \omega_2 &= \omega_0\sin(\omega_n t) \end{aligned}$$

$$\left[{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}} = [\mathbb{I}_G]_{\mathcal{B}}\left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} = \begin{bmatrix} I_t\omega_0\cos(\omega_n t) \\ I_t\omega_0\sin(\omega_n t) \\ I_s\omega_s \end{bmatrix}_{\mathcal{B}}$$

$$\boxed{\cos\theta \triangleq \frac{{}^{\mathcal{I}}\mathbf{h}_G}{\|{}^{\mathcal{I}}\mathbf{h}_G\|} \cdot \hat{\mathbf{b}}_3 = \frac{I_s\omega_s}{\sqrt{I_t^2\omega_0^2 + I_s^2\omega_s^2}}}$$

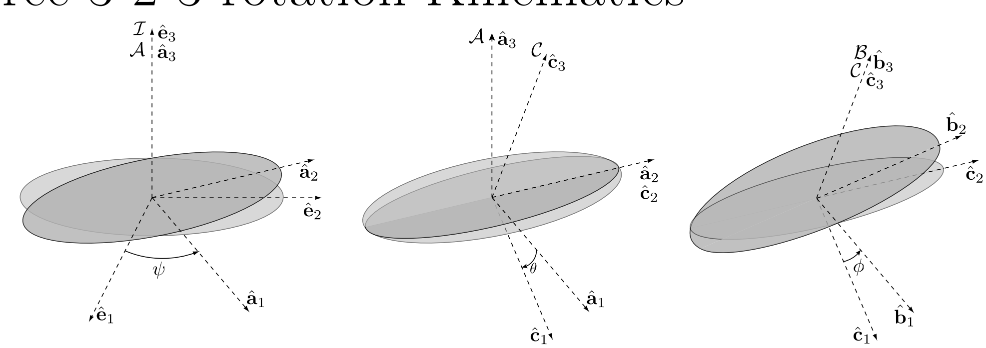
Recall the Body-2 3-2-3 $(\psi, \theta, \phi)_{\mathcal{B}}^{\mathcal{I}}$ rotation



$$^{\mathcal{B}}C^{\mathcal{I}} = {}^{\mathcal{B}}C^{\mathcal{C}\,\mathcal{C}}C^{\mathcal{A}\,\mathcal{A}}C^{\mathcal{I}} =$$

$$\begin{bmatrix} -\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\cos(\theta) & \sin(\phi)\cos(\psi) + \sin(\psi)\cos(\phi)\cos(\theta) & -\sin(\theta)\cos(\phi) \\ -\sin(\phi)\cos(\psi)\cos(\theta) - \sin(\psi)\cos(\phi) & -\sin(\phi)\sin(\psi)\cos(\theta) + \cos(\phi)\cos(\psi) & \sin(\phi)\sin(\theta) \\ \sin(\theta)\cos(\psi) & \sin(\psi)\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Torque Free 3-2-3 rotation Kinematics



$$\begin{aligned} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} &= \dot{\psi}\hat{\mathbf{a}}_3 + \dot{\theta}\hat{\mathbf{c}}_2 + \dot{\phi}\hat{\mathbf{b}}_3 \\ &= \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3 \end{aligned}$$

$$\begin{aligned} \omega_1 &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi &&= \omega_0\cos(\omega_n t) \\ \omega_2 &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi &&= \omega_0\sin(\omega_n t) \\ \omega_3 &= \dot{\phi} + \dot{\psi}\cos\theta &&= \omega_s \end{aligned}$$

$$\omega_1^2 + \omega_2^2 = \dot{\psi}^2\sin^2\theta = \omega_0^2 \Longrightarrow \boxed{\dot{\psi} = \frac{\omega_0}{\sin\theta}}$$

Torque-Free 3-2-3 rotation Kinematics Continued

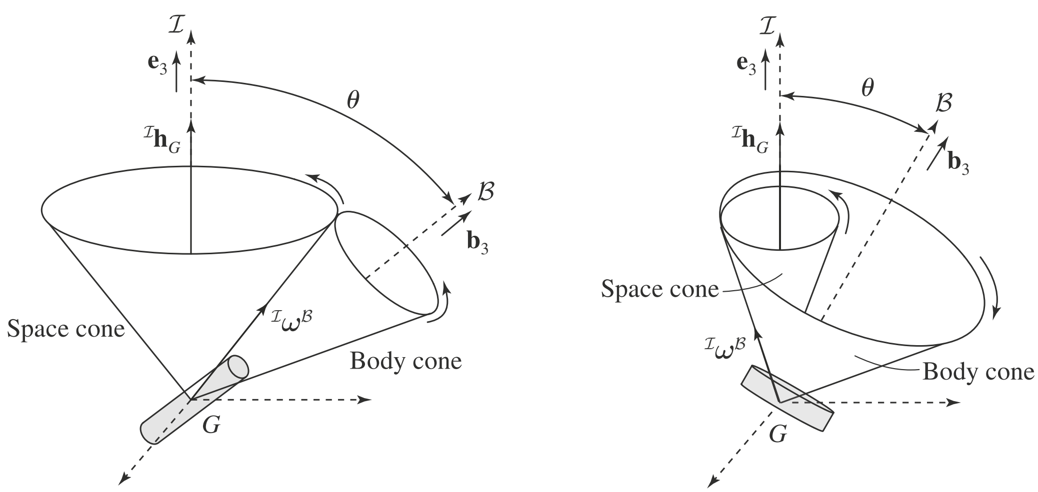
$$\left[{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}} = [h_G \hat{\mathbf{e}}_3]_{\mathcal{B}} = h_G \begin{bmatrix} -\sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \underbrace{\cos\theta}_{(1)} \end{bmatrix}_{\mathcal{B}}$$

$$= [\mathbb{I}_G]_{\mathcal{B}} \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} = \begin{bmatrix} -I_t\dot{\psi}\sin\theta\cos\phi \\ I_t\dot{\psi}\sin\theta\sin\phi \\ \underbrace{I_s\left(\dot{\phi}+\dot{\psi}\cos\theta\right)}_{(2)} \end{bmatrix}_{\mathcal{B}}$$

$$\boxed{\dot{\psi} = \frac{h_G}{I_t}}$$

$$\left.\begin{array}{c} (1) \equiv (2) \implies h_G\cos\theta = I_s\left(\dot{\phi}+\dot{\psi}\cos\theta\right) \\ \omega_s = \omega_3 = \dot{\phi}+\dot{\psi}\cos\theta \end{array}\right\} \boxed{\dot{\phi} = \frac{I_t - I_s}{I_t}\omega_s}$$

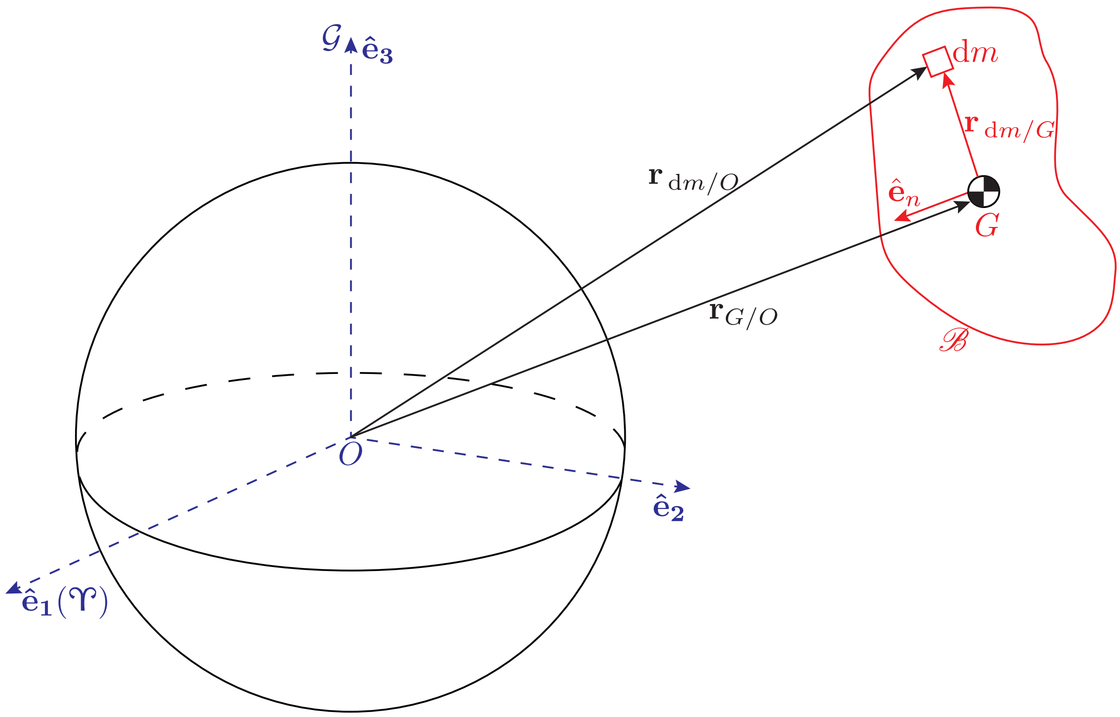
Body and Space Cones



$I_t > I_s$ $\quad\quad$ $I_t < I_s$

Kasdin & Paley (2009) Fig. 11.21

An Arbitrary Rigid Body in Orbit



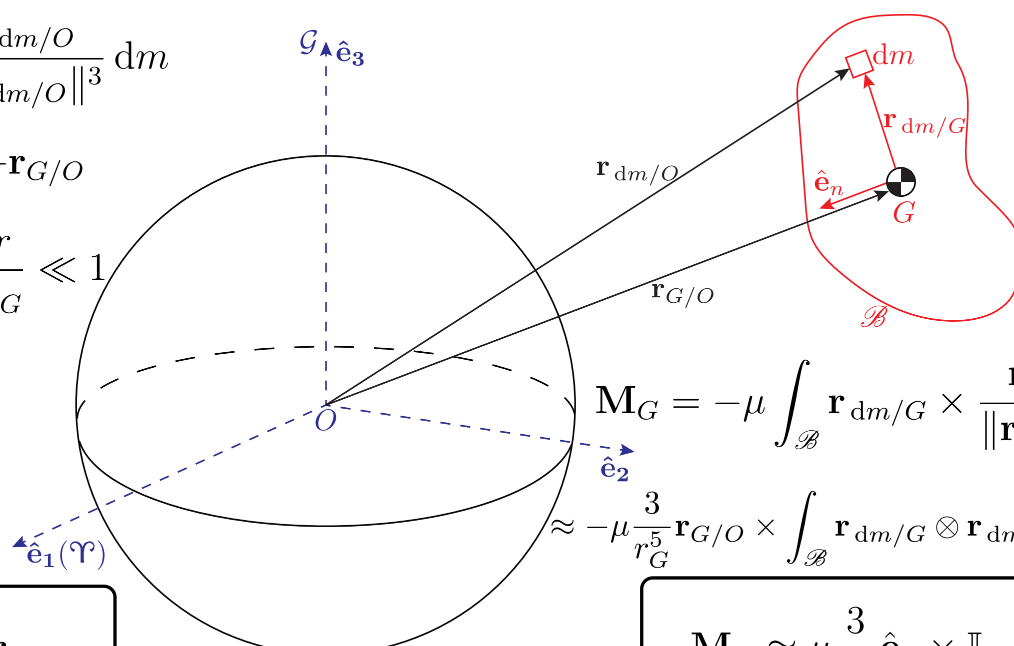
First Order Gravitational Effects

$$\mathbf{F}_G = -\mu \int_{\mathscr{B}} \frac{\mathbf{r}_{\,\mathrm{d}m/O}}{\|\mathbf{r}_{\,\mathrm{d}m/O}\|^3}\,\mathrm{d}m$$

$$\mathbf{r}_{\,\mathrm{d}m/O} = \mathbf{r}_{\,\mathrm{d}m/G} + \mathbf{r}_{G/O}$$

$$\left.\begin{aligned} r &\triangleq \|\mathbf{r}_{\,\mathrm{d}m/G}\| \\ r_G &\triangleq \|\mathbf{r}_{G/O}\| \end{aligned}\right\} \frac{r}{r_G} \ll 1$$

$$m_s \triangleq \int_{\mathscr{B}} \mathrm{d}m$$



$$\mathbf{M}_G = -\mu \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/G} \times \frac{\mathbf{r}_{\,\mathrm{d}m/O}}{\|\mathbf{r}_{\,\mathrm{d}m/O}\|^3}\,\mathrm{d}m$$

$$\approx -\mu \frac{3}{r_G^5} \mathbf{r}_{G/O} \times \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/G} \otimes \mathbf{r}_{\,\mathrm{d}m/G}\,\mathrm{d}m \cdot \mathbf{r}_{G/O}$$

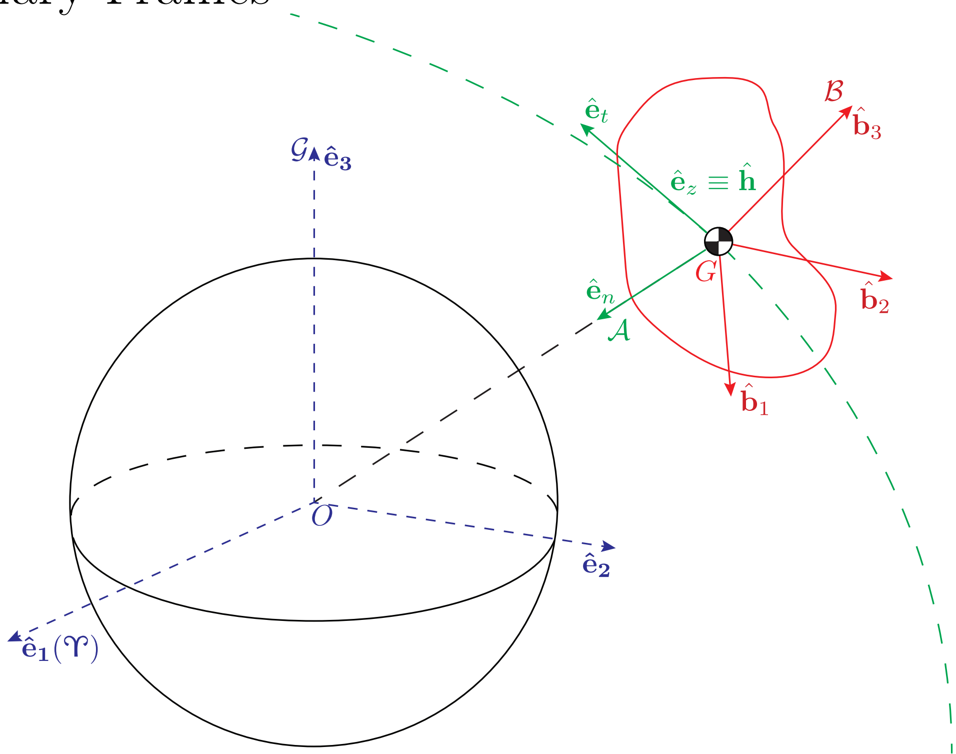
$$\boxed{\mathbf{F}_G \approx -\mu \frac{m_s}{r_G^3} \mathbf{r}_{G/O}}$$

$$\boxed{\mathbf{M}_G \approx \mu \frac{3}{r_G^3} \hat{\mathbf{e}}_n \times \mathbb{I}_G \cdot \hat{\mathbf{e}}_n}$$

Some Helpful Auxiliary Frames

$$\mathcal{A} = (G, \hat{\mathbf{e}}_t, \hat{\mathbf{e}}_n, \hat{\mathbf{e}}_z)$$

$$\mathcal{B} = (G, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$$



$$\hat{\mathbf{e}}_n \equiv -\hat{\mathbf{r}}_{G/O}$$

$$[\mathbb{I}_G]_{\mathcal{B}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}}$$

Torque on Spacecraft Due to Planet

$$\mathbf{M}_G \approx \mu \frac{3}{r_G^3} \hat{\mathbf{e}}_n \times \mathbb{I}_G \cdot \hat{\mathbf{e}}_n$$

$$[\mathbf{M}_G]_{\mathcal{B}} = \frac{3\mu}{r_G^3} [\hat{\mathbf{e}}_n \times]_{\mathcal{B}} [\mathbb{I}_G]_{\mathcal{B}} \underbrace{[\hat{\mathbf{e}}_n]_{\mathcal{B}}}_{{}^{\mathcal{B}}C^{\mathcal{A}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{A}}}$$

$$\boxed{[\mathbf{M}_G]_{\mathcal{B}} = \frac{3\mu}{r_G^3} \begin{bmatrix} {}^{\mathcal{B}}C^{\mathcal{A}}_{22} {}^{\mathcal{B}}C^{\mathcal{A}}_{32} (-I_2 + I_3) \\ {}^{\mathcal{B}}C^{\mathcal{A}}_{12} {}^{\mathcal{B}}C^{\mathcal{A}}_{32} (I_1 - I_3) \\ {}^{\mathcal{B}}C^{\mathcal{A}}_{12} {}^{\mathcal{B}}C^{\mathcal{A}}_{22} (-I_1 + I_2) \end{bmatrix}_{\mathcal{B}}}$$

Back to Euler's Equations

$$[\mathbb{I}_G]_{\mathcal{B}_p}\left[{}^{\mathcal{B}}\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}_p} + \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}_p}[\mathbb{I}_G]_{\mathcal{B}_p}\left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}_p} = [\mathbf{M}_G]_{\mathcal{B}_p} \quad \Longrightarrow$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} \dot\omega_1 \\ \dot\omega_2 \\ \dot\omega_3 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_{\mathcal{B}} = 3\frac{\mu}{r_G^3}\begin{bmatrix} {}^{\mathcal{B}}C^{\mathcal{A}}_{22}{}^{\mathcal{B}}C^{\mathcal{A}}_{32}\left(-I_2+I_3\right) \\ {}^{\mathcal{B}}C^{\mathcal{A}}_{12}{}^{\mathcal{B}}C^{\mathcal{A}}_{32}\left(I_1-I_3\right) \\ {}^{\mathcal{B}}C^{\mathcal{A}}_{12}{}^{\mathcal{B}}C^{\mathcal{A}}_{22}\left(-I_1+I_2\right) \end{bmatrix}_{\mathcal{B}}$$

$$\boxed{\begin{aligned} I_1\dot\omega_1 &= (I_2 - I_3)\left(\omega_2\omega_3 - \frac{3\mu}{r_G^3}{}^{\mathcal{B}}C^{\mathcal{A}}_{22}{}^{\mathcal{B}}C^{\mathcal{A}}_{32}\right) \\ I_2\dot\omega_2 &= (I_3 - I_1)\left(\omega_1\omega_3 - \frac{3\mu}{r_G^3}{}^{\mathcal{B}}C^{\mathcal{A}}_{12}{}^{\mathcal{B}}C^{\mathcal{A}}_{32}\right) \\ I_3\dot\omega_3 &= (I_1 - I_2)\left(\omega_1\omega_2 - \frac{3\mu}{r_G^3}{}^{\mathcal{B}}C^{\mathcal{A}}_{12}{}^{\mathcal{B}}C^{\mathcal{A}}_{22}\right) \end{aligned}}$$

We now have equations of motion for the components of ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$, but they are functions of components of the DCM ${}^{\mathcal{B}}C^{\mathcal{A}}$, which themselves are changing in time. So, in order to numerically integrate this system, we have to augment our state with at least some subset of the elements of ${}^{\mathcal{B}}C^{\mathcal{A}}$.

Circular Orbit Case

$$\begin{aligned} I_1\dot\omega_1 &= (I_2 - I_3)\left(\omega_2\omega_3 - 3n^2{}^{\mathcal{B}}C^{\mathcal{A}}_{22}{}^{\mathcal{B}}C^{\mathcal{A}}_{32}\right) \\ I_2\dot\omega_2 &= (I_3 - I_1)\left(\omega_1\omega_3 - 3n^2{}^{\mathcal{B}}C^{\mathcal{A}}_{12}{}^{\mathcal{B}}C^{\mathcal{A}}_{32}\right) \\ I_3\dot\omega_3 &= (I_1 - I_2)\left(\omega_1\omega_2 - 3n^2{}^{\mathcal{B}}C^{\mathcal{A}}_{12}{}^{\mathcal{B}}C^{\mathcal{A}}_{22}\right) \end{aligned}$$

$${}^{\mathcal{G}}\boldsymbol{\omega}^{\mathcal{A}} = n\hat{\mathbf{h}} \Longrightarrow \left[{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_{\mathcal{B}} - {}^{\mathcal{B}}C^{\mathcal{A}}\begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}_{\mathcal{A}}$$

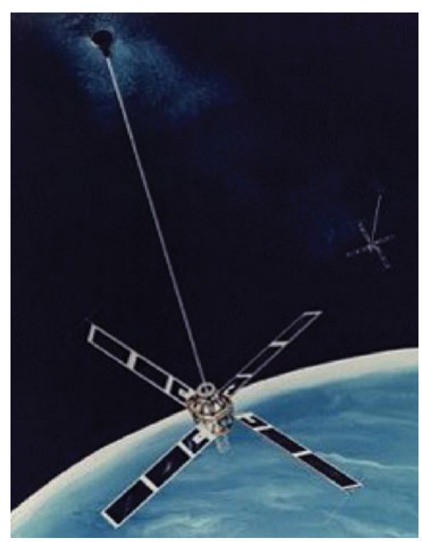
$$\begin{aligned} {}^{\mathcal{B}}\dot{C}^{\mathcal{A}} &= -\left[{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}}{}^{\mathcal{B}}C^{\mathcal{A}} \\ {}^{\mathcal{B}}\dot{C}^{\mathcal{A}}_{i2} &= \begin{bmatrix} -{}^{\mathcal{B}}C^{\mathcal{A}}_{22}\left(n{}^{\mathcal{B}}C^{\mathcal{A}}_{33} - \omega_3\right) + {}^{\mathcal{B}}C^{\mathcal{A}}_{32}\left(n{}^{\mathcal{B}}C^{\mathcal{A}}_{23} - \omega_2\right) \\ {}^{\mathcal{B}}C^{\mathcal{A}}_{12}\left(n{}^{\mathcal{B}}C^{\mathcal{A}}_{33} - \omega_3\right) - {}^{\mathcal{B}}C^{\mathcal{A}}_{32}\left(n{}^{\mathcal{B}}C^{\mathcal{A}}_{13} - \omega_1\right) \\ -{}^{\mathcal{B}}C^{\mathcal{A}}_{12}\left(n{}^{\mathcal{B}}C^{\mathcal{A}}_{23} - \omega_2\right) + {}^{\mathcal{B}}C^{\mathcal{A}}_{22}\left(n{}^{\mathcal{B}}C^{\mathcal{A}}_{13} - \omega_1\right) \end{bmatrix} \end{aligned}$$

Circular Orbit Case using Euler Parameters

- At this point, we need to keep track of two full columns of ${}^{\mathcal{B}}C^{\mathcal{A}}$—six scalar elements, which include many redundancies
- Numerical integrators will often add noise to your state which will violate the inherent constraints between DCM components, leading to much larger errors in the overall integration
- Instead, we can use Euler Parameters/Quaternions and only carry four elements with only one constraint between them
- Recall that ${}^{\mathcal{B}}C^{\mathcal{A}} = \mathbf{\Xi}^T(\mathbf{q})\mathbf{\Psi}(\mathbf{q})$
- Similarly $\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{q} = \frac{1}{2}\mathbf{q} \odot \begin{bmatrix} {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \\ 0 \end{bmatrix} = \frac{1}{2}\mathbf{\Xi}(\mathbf{q})\,{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}$
- We can replace all instances of DCM components in our attitude equations of motion with quaternion components!

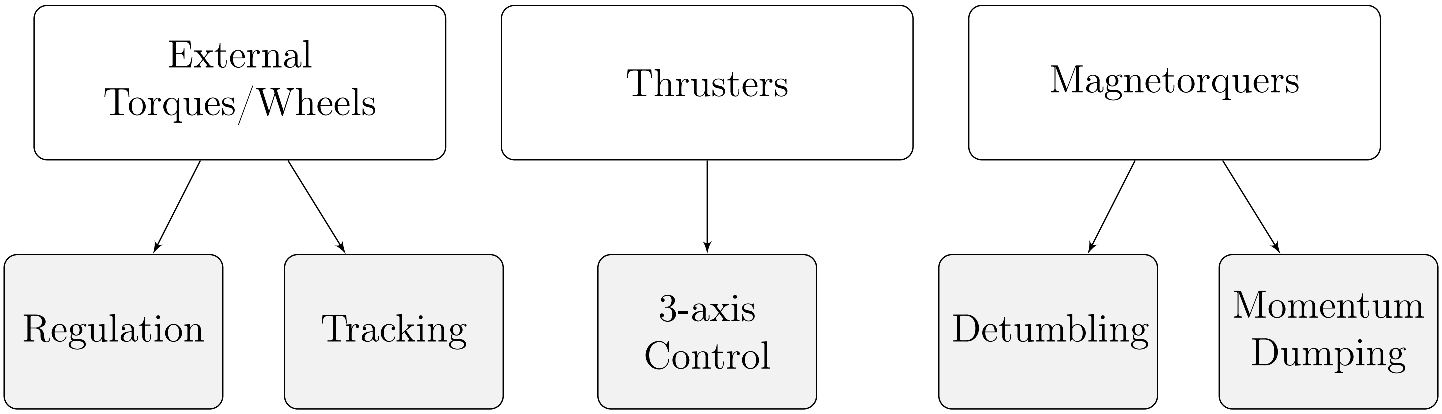
Gravity-Gradient Stabilization

UniCubeSat-GG (2012)
From: https://eoportal.org/web/eoportal/satellite-missions/content/-/article/unicubesat



Transit (1963)
From: https://www.gpsworld.com/origins-gps-part-1/

Attitude Control Maneuvers



Regulation vs. Tracking

While often synonymous in classical control, for spacecraft attitude control, regulation typically refers to maintaining a specific orientation (rejecting disturbances) and tracking typically means following a prescribed orientation trajectory. Regulation to a desired inertial orientation can often be accomplished with classical feedback control techniques, while more complex orientation control is often implemented via optimal and modern control techniques.

Regulation

$$\text{Dynamics}\quad\begin{cases}\mathbb{I}_G \cdot \dfrac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \left(\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right) = \mathbf{M}_G \\ \\ {}^{\mathcal{B}}\dot{C}^{\mathcal{I}} = -\left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}}{}^{\mathcal{B}}C^{\mathcal{I}}\end{cases}$$

$$\underbrace{\Delta C}_{\text{Error DCM}} \triangleq {}^{\mathcal{B}}C^{\mathcal{I}}\,\underbrace{C_d}_{\text{Desired DCM}}{}^{-1} \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\Delta C = {}^{\mathcal{B}}\dot{C}^{\mathcal{I}}C_d^{-1}$$

$$\text{Control Law: }\mathbf{M}_G = -\overbrace{k_p}^{\text{Positive Constants}}\underbrace{\boldsymbol{\Delta\epsilon}}_{\text{Vector part of quaternion representiation of }\Delta C} - \overbrace{k_d}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$$

Regulation Continued

$$\text{Control Law: }\mathbf{M}_G = -k_p\boldsymbol{\Delta\epsilon} - k_d{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$$

$$\Downarrow$$

$$\mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \mathbf{M}_G - {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \left(\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right)$$

$$\Rightarrow \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \mathbb{I}_G^{-1} \cdot \left[-k_p\boldsymbol{\Delta\epsilon} - k_d{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} - {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \left(\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right)\right]$$

$$\text{Need: }\boldsymbol{\Delta\epsilon} \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = 0$$

Gyrostat Control

$${}^{\mathcal{I}}\mathbf{h}_G^{\text{tot}} = \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + \underbrace{\mathbb{I}_{G_W}^{\mathscr{W}} \cdot {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{A}}}_{\triangleq\, \mathbf{h}}$$

$$\mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = -{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \underbrace{\left(\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + \mathbf{h}\right)}_{\text{Conserved}} - \underbrace{\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{h}}_{\text{Wheel Torque}}$$

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} = -{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{h} - \overbrace{\mathbf{M}^{\mathscr{W}}}^{\text{Wheel input torque}}$$

$$\mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = -{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \left(\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right) + \mathbf{M}^{\mathscr{W}}$$

$$\boxed{\mathbf{M}^{\mathscr{W}} = -k_p \boldsymbol{\Delta\epsilon} - k_d {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}}$$

Quaternion Feedback

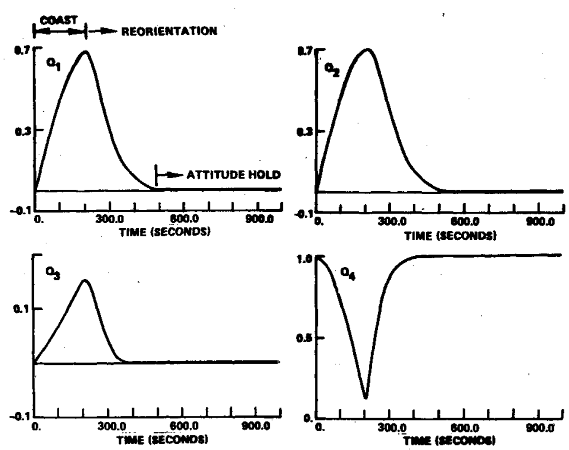


Fig. 5 Digital simulation results (quaternions).

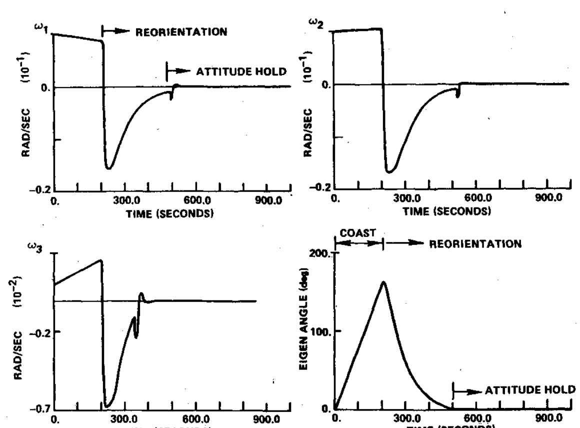


Fig. 6 Digital simulation results (body rates and eigenangle).

From: Wie & Barba (1985)

Detumbling with Magnetorquers

Magnetic Control Torque

Geomagnetic Field Vector

$$\text{Recall: } \mathbf{M} = -\mathbf{B} \times \mathbf{N}$$

Magnetic Dipole Moment due to Magnetorquer

$$\text{Control Law: } \mathbf{N} = \frac{k}{\|\mathbf{B}\|} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \frac{\mathbf{B}}{\|\mathbf{B}\|} \text{ for positive gain } k$$

$$\mathbf{M} = \frac{k}{\|\mathbf{B}\|}\left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \frac{\mathbf{B}}{\|\mathbf{B}\|}\right) = k\left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{B}}\right) \times \hat{\mathbf{B}} \;\Rightarrow\; \text{Control torque is } \perp \hat{\mathbf{B}}$$

For magnetorquer i (of N) producing dipole $m_i\hat{\mathbf{n}}_i$, can write a bang-bang detumbling control:

$$m_i = -\max(m_i)\,\mathrm{sgn}\left(\hat{\mathbf{n}}_i \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{B}\right)$$

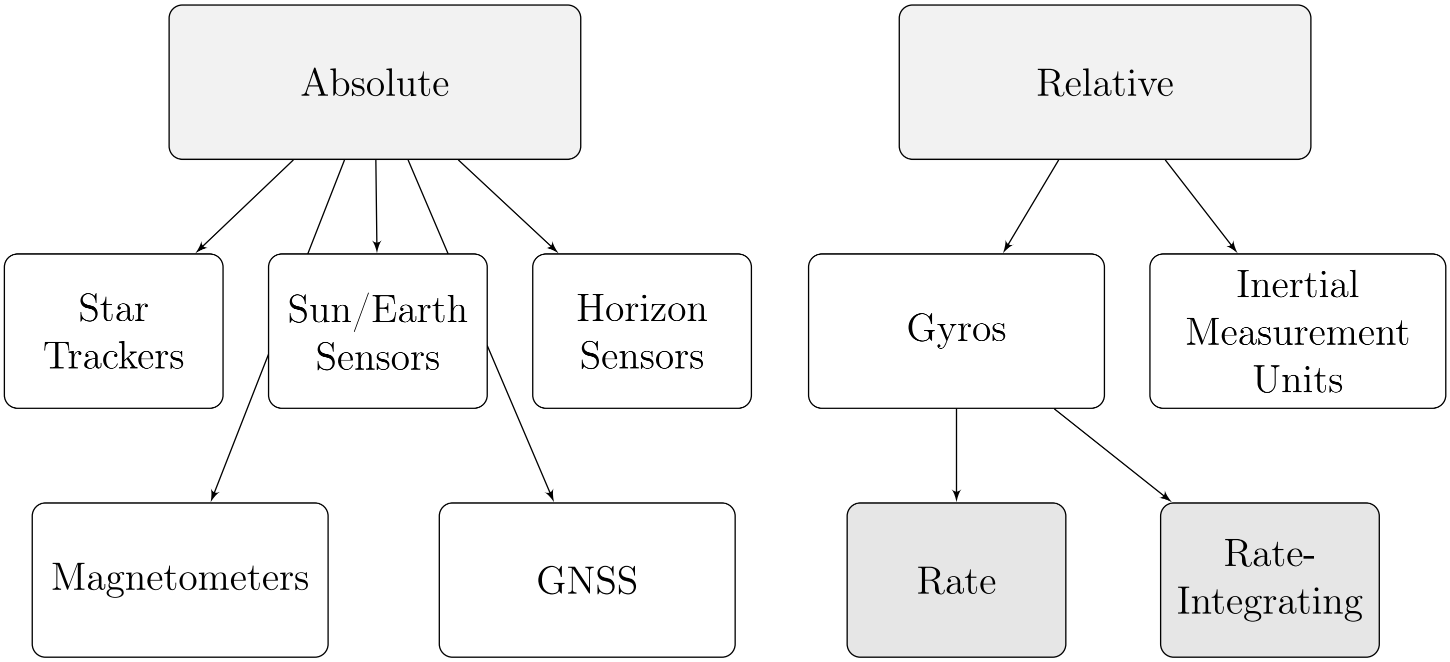
Momentum Dumping

Wheel angular momentum

$$\text{Control Law: } \mathbf{N} = \frac{k}{\|\mathbf{B}\|}\mathbf{h} \times \frac{\mathbf{B}}{\|\mathbf{B}\|} \text{ for positive gain } k$$

$$\mathbf{M} = \frac{k}{\|\mathbf{B}\|}\left(\mathbf{h} \times \frac{\mathbf{B}}{\|\mathbf{B}\|}\right) = k\left(\mathbf{h} \times \hat{\mathbf{B}}\right) \times \hat{\mathbf{B}} \;\Rightarrow\; \text{Control torque is } \perp \hat{\mathbf{B}}$$

Attitude Sensing



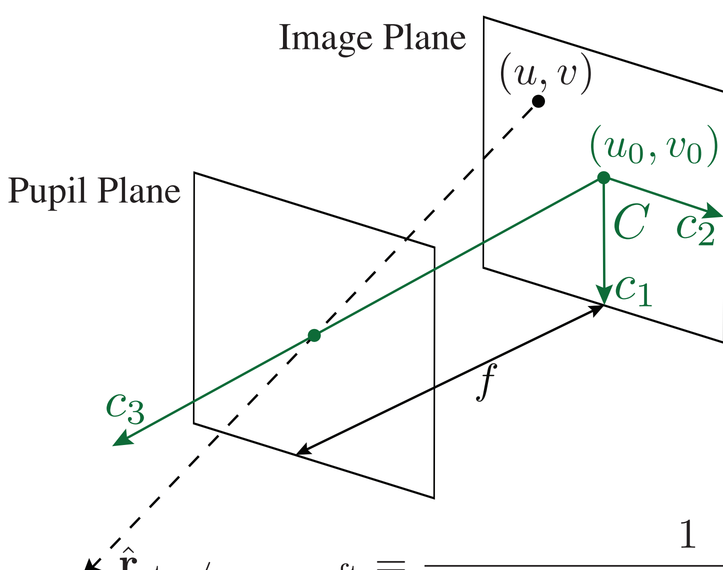
Attitude Sensors

Sensor	**Raw Accuracy**
Magnetometer	$\pm 0.5° - 5°$
Sun Sensor (coarse)	$\pm 1° - 5°$
Sun Sensor (spinning slit)	$\pm 0.1° - 0.5°$
Sun Sensor (fine)	$\pm 0.01° - 0.05°$
Star Tracker	$\pm 0.1 - 5$ arcsec/star
Gyro	$\pm 0.001 - 1°$/hour
MEMS Gyro	$\pm 0.01 - 1°$/hour
Ring Laser Gyro	$\pm 0.001 - 0.1°$/hour
Fiber Optic Gyro	± 0.01 arcsec/hour $- 0.1°$/hour

Attitude Control Scenarios

Required Pointing Accuracy	**ACS Approach**	**ADCS Hardware**
$> 5°$	Gravity Gradient Stabilization Possible	No sensing needs; boom and associated hardware
$1 - 5°$	Spin Stabilization Possible	Sun/Earth/Horizon sensors; magnetorquer/magnetometer possible
$0.1 - 1°$	3-Axis Control likely needed	Star Trackers/Gyros; Reaction wheels/RCS/Magnetorquers
$< 0.1°$	3-Axis Control Required	Precise Star Trackers/Gyros; Reaction wheels/RCS

Image Geometry



$$\hat{\mathbf{r}}_{\text{star/spacecraft}} = \frac{1}{\sqrt{f^2 + (u - u_0)^2 + (v - v_0)^2}} \begin{bmatrix} u - u_0 \\ v - v_0 \\ f \end{bmatrix}_C$$

Star Tracker Software

- Star trackers rely heavily on post-processing to identify and match stars to catalog data
- Many different algorithms and implementations exist (most proprietary)
- More recently, many university projects have been open sourcing their work. For example: http://openstartracker.org/

Tri-Axial Attitude Determination (TRIAD)

- Given: $[\hat{\mathbf{r}}_1]_\mathcal{B}, [\hat{\mathbf{r}}_2]_\mathcal{B}, [\hat{\mathbf{r}}_1]_\mathcal{I}, [\hat{\mathbf{r}}_2]_\mathcal{I}$, Find: ${}^\mathcal{I}C^\mathcal{B}$
- Typically assume that one measurement ($[\hat{\mathbf{r}}_1]_\mathcal{B}$) is more precise than the other
- For 3 mutually orthogonal unit vectors $\hat{\mathbf{v}}_1 \perp \hat{\mathbf{v}}_2 \perp \hat{\mathbf{v}}_3$:

$$\begin{bmatrix}\hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3\end{bmatrix}_\mathcal{B} = {}^\mathcal{B}C^\mathcal{I} \begin{bmatrix}\hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3\end{bmatrix}_\mathcal{I}$$

$$\boxed{{}^\mathcal{B}C^\mathcal{I} = \begin{bmatrix}\hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3\end{bmatrix}_\mathcal{B} \underbrace{\left(\begin{bmatrix}\hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3\end{bmatrix}_\mathcal{I}\right)^{-1}}_{\left(\begin{bmatrix}\hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3\end{bmatrix}_\mathcal{I}\right)^T}}$$

- For both frames, the mutually orthogonal unit vectors are formed as:

$$\begin{aligned}
\hat{\mathbf{v}}_1 &= \hat{\mathbf{r}}_1 \\
\hat{\mathbf{v}}_2 &= (\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2)\left(\|\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2\|\right)^{-1} \\
\hat{\mathbf{v}}_3 &= (\hat{\mathbf{r}}_1 \times (\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2))\left(\|\hat{\mathbf{r}}_1 \times (\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2)\|\right)^{-1}
\end{aligned}$$

Wahba's Problem

- Find the orthogonal matrix A with determinant 1 that minimizes the cost function:

$$J(A) = \frac{1}{2}\sum_{i=1}^{N} a_i \|\mathbf{b}_i - A\mathbf{r}_i\|^2 \quad \text{where } a_i \geq 0$$

- For the attitude determination problem, $A = {}^{\mathcal{B}}C^{\mathcal{I}}$, $\mathbf{b}_i = [\mathbf{r}_i]_{\mathcal{B}}$, and $\mathbf{r}_i = [\mathbf{r}_i]_{\mathcal{I}}$:

$$\begin{aligned} J({}^{\mathcal{B}}C^{\mathcal{I}}) &= \frac{1}{2}\sum_{i=1}^{N} a_i \left\| [\mathbf{r}_i]_{\mathcal{B}} - {}^{\mathcal{B}}C^{\mathcal{I}}\, [\mathbf{r}_i]_{\mathcal{I}} \right\|^2 \\ &= \underbrace{\sum_{i=1}^{N} a_i}_{\triangleq \lambda_0} - \mathrm{Tr}\left({}^{\mathcal{B}}C^{\mathcal{I}} B^T\right) \quad \text{where} \quad \underbrace{B \triangleq \sum_{i=1}^{N} a_i\, [\mathbf{r}_i]_{\mathcal{B}}\, [\mathbf{r}_i]_{\mathcal{I}}^T}_{\text{Attitude Profile Matrix}} \end{aligned}$$

Whaba's problem is a generalization of TRIAD, allowing for measurement of arbitrary numbers of unit vectors, and the incorporation of knowledge of the precision of different measurements.

Singular Value Decomposition and Wahba's Problem

$$B = U \underbrace{\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}}_{\triangleq \Sigma} V^T = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |U| \end{bmatrix}}_{\triangleq U_+} \underbrace{\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3|U||V| \end{bmatrix}}_{\triangleq \Sigma'} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |V| \end{bmatrix}^T}_{\triangleq V_+^T}$$

Define a rotation matrix:
$W \triangleq U_+^T {}^{\mathcal{B}}C^{\mathcal{I}} V_+ = I\cos\theta - \sin\theta\, [\hat{\mathbf{n}}_\times]_{\mathcal{I}} + (1-\cos\theta)\, [\hat{\mathbf{n}}]_{\mathcal{I}}\, [\hat{\mathbf{n}}]_{\mathcal{I}}^T$

$$\mathrm{Tr}\left({}^{\mathcal{B}}C^{\mathcal{I}} B^T\right) = \mathrm{Tr}(W\Sigma') = [\hat{\mathbf{n}}]_{\mathcal{I}}^T\, \Sigma'\, [\hat{\mathbf{n}}]_{\mathcal{I}} + \cos\theta \left(\mathrm{Tr}(\Sigma') - [\hat{\mathbf{n}}]_{\mathcal{I}}^T\, \Sigma'\, [\hat{\mathbf{n}}]_{\mathcal{I}}\right)$$

This is maximized for $\theta = 0$ ($W = I$) and therefore:

$$\boxed{{}^{\mathcal{B}}C^{\mathcal{I}} \approx U_+ V_+^T = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |U||V| \end{bmatrix} V^T}$$

The Three-Body Problem and N-Body Problem

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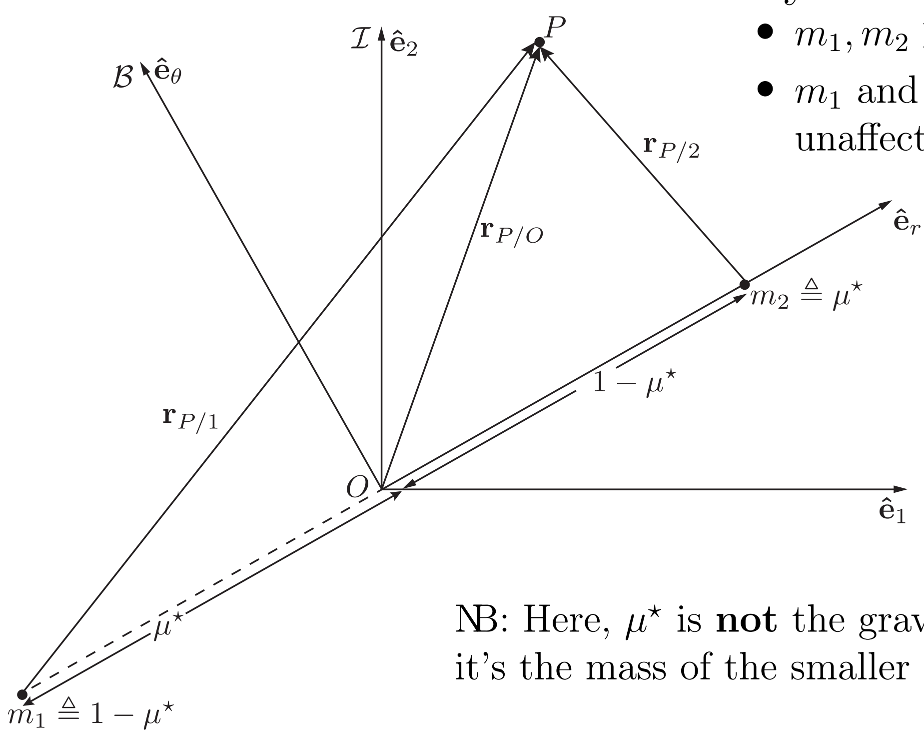
Cornell University

MAE 4060/5065, Fall 2021

The Three-Body Problem and N-Body Problem

As soon as any additional body is added to the two-body problem, we lose the ability to write down a fully analytical solution. We have previously dealt with this by treating additional bodies as perturbers to a two-body system, slowly modifying the Keplerian elements of the two-body orbit. There are cases, however (e.g., when operating near the boundaries of spheres of influence) where this model ceases being useful, as the two-body elements evolve rapidly and significantly over the course of a single orbit. Here, we must explicitly deal with (at least) three co-orbiting bodies, which drives us towards numerical integration in order to accurately predict how an orbit evolves. However, in the case of three bodies, if we make certain additional assumptions, we can still find a conserved quantity that allows us to predict some of the system's behavior (although not the exact trajectories). Study of the three-body problem is incredibly important in modern astrodynamics, as three-body orbital design allows us to create incredibly fuel-efficient trajectories in cases where our spacecraft are in close proximity to multiple bodies (for example, the moon systems of Jupiter and Saturn). Three-body analysis also opens the possibility of creating stable, periodic orbits about empty points in space! In the case of an N-body system, we still have conservation of energy and momentum (assuming no forces other than gravity), but to track the exact trajectory of any one particle in the system, we are forced to rely on numerical integration.

The Circular Restricted Three-Body Problem (CR3BP)



- $m_1, m_2 \gg m_p$
- m_1 and m_2 in mutual circular orbits, unaffected by m_p

$$\mathbf{r}_{1/O} = -\mu^\star \hat{\mathbf{e}_\mathbf{r}}$$
$$\mathbf{r}_{2/O} = (1-\mu^\star)\hat{\mathbf{e}_\mathbf{r}}$$

NB: Here, $\mu^\star$ is **not** the gravitational parameter—it's the mass of the smaller major body

Canonical units for the CR3BP are defined such that $G = 1$
1 MU $= m_1 + m_2$, 1 DU $= \|\mathbf{r}_{1/2}\|$, and 2π TU $= T_{p,1,2}$ (the orbital period of m_1 and m_2).

CR3BP Dynamics

$$\mathbf{F}_P = -\frac{Gm_1 m_P}{\|\mathbf{r}_{P/1}\|^3}\mathbf{r}_{P/1} - \frac{Gm_2 m_P}{\|\mathbf{r}_{P/2}\|^3}\mathbf{r}_{P/2} \qquad {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = n\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3$$

$$^{\mathcal{B}}\mathbf{a}_{P/O} + 2\hat{\mathbf{e}}_3 \times {}^{\mathcal{B}}\mathbf{v}_{P/O} + \hat{\mathbf{e}}_3 \times \left(\hat{\mathbf{e}}_3 \times \mathbf{r}_{P/O}\right) = -G\left(\frac{m_1}{\|\mathbf{r}_{P/1}\|^3}\mathbf{r}_{P/1} + \frac{m_2}{\|\mathbf{r}_{P/2}\|^3}\mathbf{r}_{P/2}\right)$$

$$\left[\mathbf{r}_{P/O}\right]_{\mathcal{B}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}}$$

$$\mathbf{F}_P = -\nabla V \quad \text{where} \quad V = -\left(\frac{1-\mu^\star}{r_1} + \frac{\mu^\star}{r_2}\right)$$

$$r_1 \triangleq \|\mathbf{r}_{P/1}\| = \sqrt{(x+\mu^\star)^2 + y^2 + z^2}$$
$$r_2 \triangleq \|\mathbf{r}_{P/2}\| = \sqrt{(x-(1-\mu^\star))^2 + y^2 + z^2}$$

$$\boxed{\begin{aligned} \ddot{x} - 2\dot{y} - x &= -\frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} - y &= -\frac{\partial V}{\partial y} \\ \ddot{z} &= -\frac{\partial V}{\partial z} \end{aligned}}$$

Remember that, just as in the Clohessy-Wiltshire equations, x, y, z are rotating frame coordinates.

A New Potential

$$U \triangleq -\frac{1}{2}(x^2+y^2) - \left(\frac{1-\mu^\star}{r_1} + \frac{\mu^\star}{r_2}\right)$$

$$\left.\begin{aligned} \ddot{x} - 2\dot{y} - x &= -\frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} - y &= -\frac{\partial V}{\partial y} \\ \ddot{z} &= -\frac{\partial V}{\partial z} \end{aligned}\right\} \Longrightarrow \boxed{\begin{aligned} \ddot{x} &= -\frac{\partial U}{\partial x} + 2\dot{y} \\ \ddot{y} &= -\frac{\partial U}{\partial y} - 2\dot{x} \\ \ddot{z} &= -\frac{\partial U}{\partial z} \end{aligned}}$$

NB: U is not a true potential, as it incorporates both elements related to gravitational forces, as well as the fictitious forces that arise whenever we do dynamics in rotating frame. It is incredibly useful, however, for simplifying the CR3BP equations.

The Jacobi Constant

$$\frac{1}{2}\left({}^{\mathcal{B}}\mathbf{v}_{P/O} \cdot {}^{\mathcal{B}}\mathbf{v}_{P/O}\right) + U(x,y,z) = C \triangleq \text{Jacobi Constant}$$

$${}^{\mathcal{B}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{P/O} - \hat{\mathbf{e}}_3 \times \mathbf{r}_{P/O}$$

$$\underbrace{\frac{1}{2}\left({}^{\mathcal{I}}\mathbf{v}_{P/O} \cdot {}^{\mathcal{I}}\mathbf{v}_{P/O}\right) - \left(\frac{1-\mu^\star}{r_1} + \frac{\mu^\star}{r_2}\right)}_{\text{KE+PE}} \quad - \quad \underbrace{{}^{\mathcal{I}}\mathbf{v}_{P/O} \cdot \left(\hat{\mathbf{e}}_3 \times \mathbf{r}_{P/O}\right)}_{\hat{\mathbf{e}}_3 \cdot \left(\mathbf{r}_{P/O} \times {}^{\mathcal{I}}\mathbf{v}_{P/O}\right) = \hat{\mathbf{e}}_3 \cdot {}^{\mathcal{I}}\mathbf{h}_{P/O}} \quad = C$$

$$\boxed{\mathcal{E} - h\cos(I) = C}$$

Set by initial conditions

Total Energy of Mass P

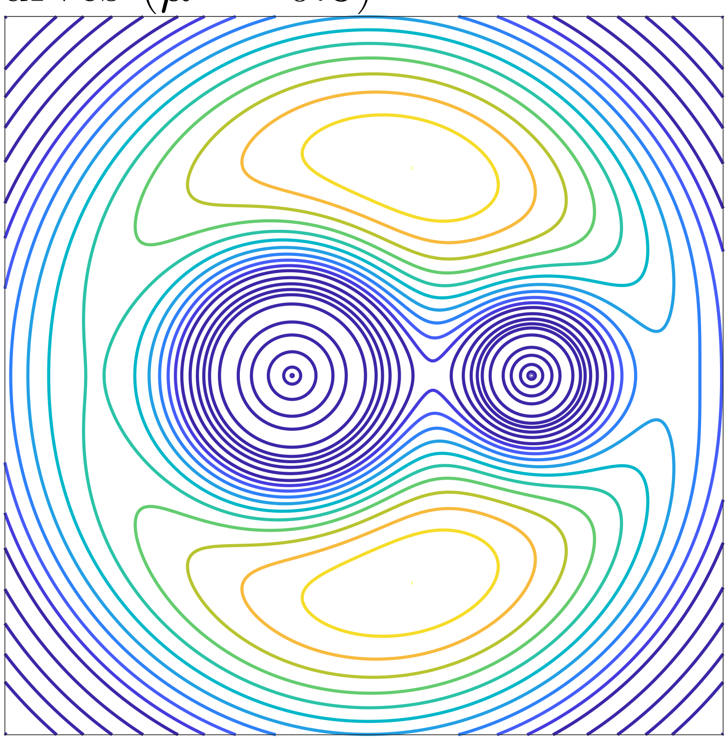
Angular Momentum of Mass P

Angle between orbit of mass P and $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ plane

The Jacobi Constant

C is the only conserved quantity in the CR3BP, but its definition is not unique. In many texts, you will find it defined as -2 times the version we define here. By the governing equation ($\frac{1}{2}\left\|{}^{\mathcal{B}}\mathbf{v}_{P/O}\right\|^2 + U(x,y,z) = C$), C represents an upper bound on U (since the first term is positive, $C \geq U$, with the equality holding in cases of zero rotating-frame velocities). This means that contours of U are zero velocity curves in the rotating frame for given values of C—-that is, for a given value of C, the particle in a 3-body system can only be located within the corresponding region of smaller or equal U. A particle with a given C value cannot cross a larger contour of U—in the rotating frame, when the particle approaches such a contour line, it appears to turn around.

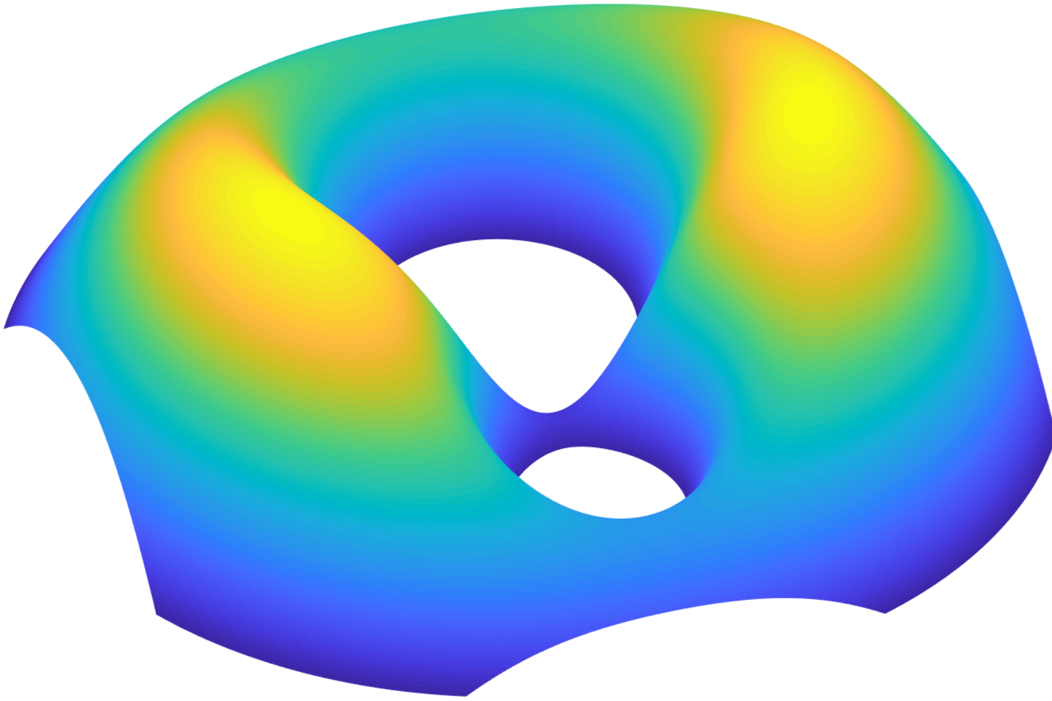
Hill Curves ($\mu^\star = 0.3$)



$$U(x,y) = U(x,-y)$$

$$U(x,y) \neq U(-x,y)$$

Hill Curves ($\mu^\star = 0.3$)



$$\lim_{x,y\to\infty} U = -\frac{x^2+y^2}{2}$$

$$\lim_{\substack{x\to-\mu^\star\\ y\to 0}} U = -\frac{1-\mu^\star}{r_1}$$

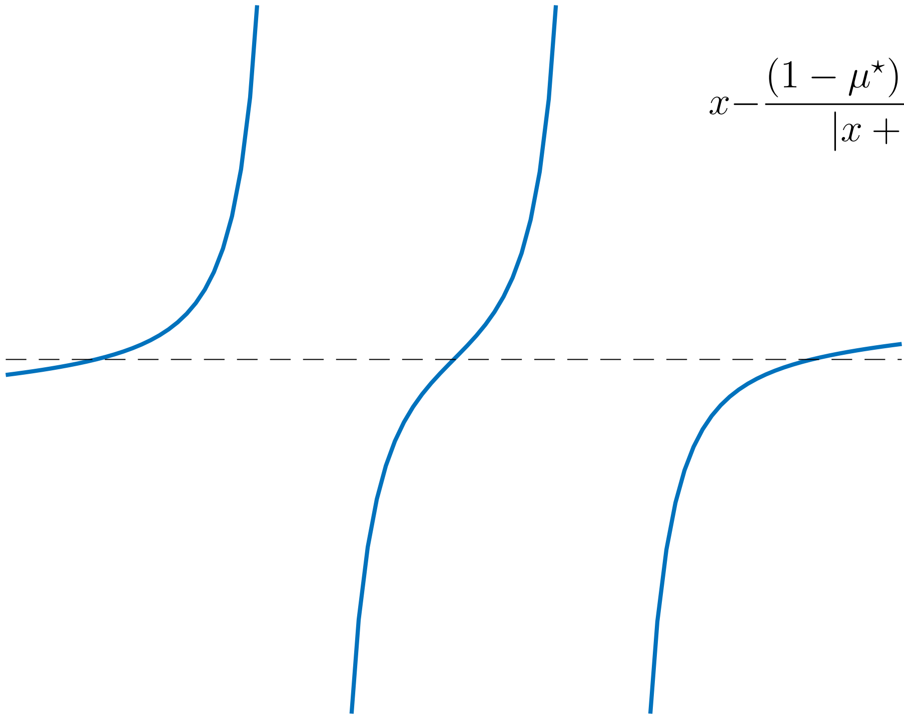
$$\lim_{\substack{x\to 1-\mu^\star\\ y\to 0}} U = -\frac{\mu^\star}{r_2}$$

CR3BP Equilibrium Points

$$\frac{\partial U}{\partial x} = -\frac{\mu^\star\left(-\mu^\star - x + 1\right)}{r_2^3} - x - \frac{\left(1-\mu^\star\right)\left(-\mu^\star - x\right)}{r_1^3} = 0$$
$$\frac{\partial U}{\partial y} = \frac{\mu^\star y}{r_2^3} - y + \frac{y\left(1-\mu^\star\right)}{r_1^3} = 0$$
$$\frac{\partial U}{\partial z} = \frac{\mu^\star z}{r_2^3} + \frac{z\left(1-\mu^\star\right)}{r_1^3} = 0$$

$\frac{\partial U}{\partial z}$ is zero for $z = 0$ so we typically focus on in-plane solutions

$y = 0$: On-Axis Equilibrium Points



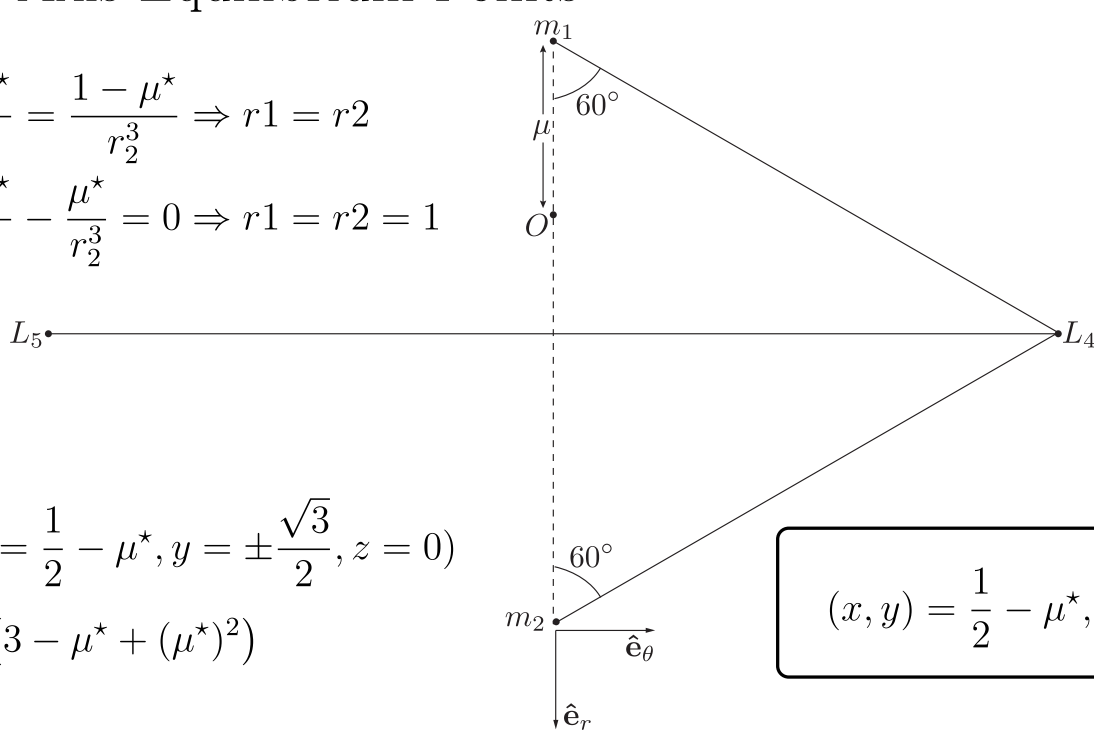
$$x - \frac{(1-\mu^\star)(x+\mu^\star)}{|x+\mu^\star|^3} - \frac{\mu^\star(x-1+\mu^\star)}{|x-1+\mu^\star|^3} = 0$$

$$\begin{aligned} x &< -\mu^\star & (L_3) \\ -\mu^\star < x &< 1-\mu^\star & (L_1) \\ x &> 1-\mu^\star & (L_2) \end{aligned}$$

$y \neq 0$: Off-Axis Equilibrium Points

$$\frac{1-\mu^\star}{r_1^3} = \frac{1-\mu^\star}{r_2^3} \Rightarrow r1 = r2$$

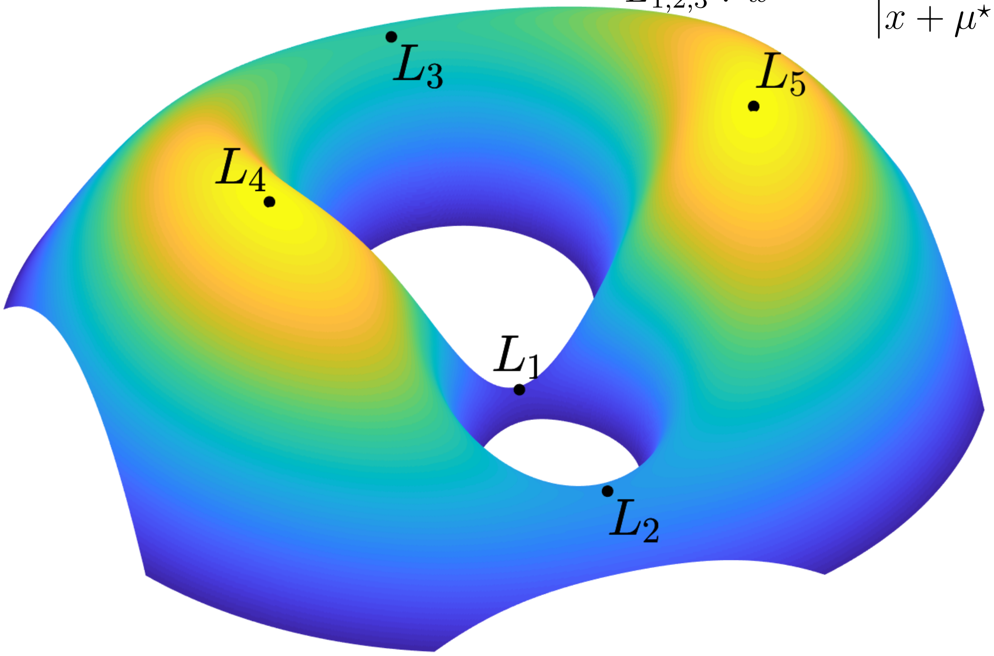
$$1 - \frac{1-\mu^\star}{r_1^3} - \frac{\mu^\star}{r_2^3} = 0 \Rightarrow r1 = r2 = 1$$



$$\begin{aligned} C_4 &= C_5 \\ &= U(x = \frac{1}{2} - \mu^\star, y = \pm\frac{\sqrt{3}}{2}, z = 0) \\ &= -\frac{1}{2}\left(3 - \mu^\star + (\mu^\star)^2\right) \end{aligned}$$

$$\boxed{(x, y) = \frac{1}{2} - \mu^\star, \pm\frac{\sqrt{3}}{2}}$$

The Lagrange Points



$$L_{1,2,3}:\ x - \frac{(1-\mu^\star)(x+\mu^\star)}{|x+\mu^\star|^3} - \frac{\mu^\star(x-1+\mu^\star)}{|x-1+\mu^\star|^3} = 0$$

$$\left[L_{4,5}\right]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} - \mu^\star \\ \\ \pm\frac{\sqrt{3}}{2} \end{bmatrix}_{\mathcal{B}}$$

Perturbation of $L_{4/5}$ Points

Consider a small displacement $\alpha\hat{\mathbf{e}}_r + \beta\hat{\mathbf{e}}_\theta$ from one of the equilibrium points L_i:

$$U \triangleq -\frac{1}{2}(x^2+y^2) - \left(\frac{1-\mu^\star}{r_1} + \frac{\mu^\star}{r_2}\right) \quad \Longleftrightarrow \quad \frac{\partial U}{\partial x} = \left.\frac{\partial U}{\partial x}\right|_{L_i} + \alpha\left.\frac{\partial^2 U}{\partial x^2}\right|_{L_i} + \beta\left.\frac{\partial^2 U}{\partial x \partial y}\right|_{L_i} + \cdots$$

$$\left.\begin{aligned} \left.\frac{\partial U}{\partial x}\right|_{L_{4/5}} &\approx -\left(\frac{3\alpha}{4} + \frac{3\sqrt{3}}{4}(1-2\mu^\star)\beta\right) \\ \left.\frac{\partial U}{\partial y}\right|_{L_{4/5}} &\approx -\left(\frac{9\beta}{4} + \frac{3\sqrt{3}}{4}(1-2\mu^\star)\alpha\right) \end{aligned}\right\} \quad \begin{aligned} \ddot{x} - 2\dot{y} &= -\frac{\partial U}{\partial x} = \frac{3\alpha}{4} + \frac{3\sqrt{3}}{4}(1-2\mu^\star)\beta \\ \ddot{y} + 2\dot{x} &= -\frac{\partial U}{\partial y} = \frac{9\beta}{4} + \frac{3\sqrt{3}}{4}(1-2\mu^\star)\alpha \end{aligned}$$

$$\left.\begin{aligned} \alpha &\triangleq Ae^{\lambda t} \\ \beta &\triangleq Be^{\lambda t} \end{aligned}\right\} \left.\begin{aligned} A\lambda^2 - 2B\lambda &= \frac{3A}{4} + \frac{3\sqrt{3}}{4}(1-2\mu^\star)B \\ B\lambda^2 + 2A\lambda &= \frac{9B}{4} + \frac{3\sqrt{3}}{4}(1-2\mu^\star)A \end{aligned}\right\} \begin{aligned} &\lambda^4 + \lambda^2 + \frac{27}{4}\mu^\star(1-\mu^\star) = 0 \Rightarrow \\ &\lambda^2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 27\mu^\star(1-\mu^\star)} \end{aligned}$$

Stability of $L_{4/5}$ Points

- If λ^2 is complex, then at least one root will have a positive real part
- For $L_{4/5}$ to be stable, we therefore required λ^2 to be strictly real:

$$\lambda^2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 27\mu^\star(1-\mu^\star)}$$

- This imposes the condition: $1 - 27\mu^\star(1-\mu^\star) \geq 0$ which requires:

$$\mu^\star \leq \frac{1}{2} - \sqrt{\frac{23}{108}} \approx 0.0385$$

- $L_{4/5}$ are stable when $m_2 \lessapprox \frac{m_1}{25}$

Perturbation of $L_{1\ldots3}$ Points

Consider a small displacement $\alpha\hat{\mathbf{e}}_r + \beta\hat{\mathbf{e}}_\theta$ from one of the equilibrium points L_i:

$$\ddot{\alpha} - 2\dot{\beta} = -\left.\frac{\partial U}{\partial x}\right|_{L_{1\ldots3}} = \alpha(1+2D)$$

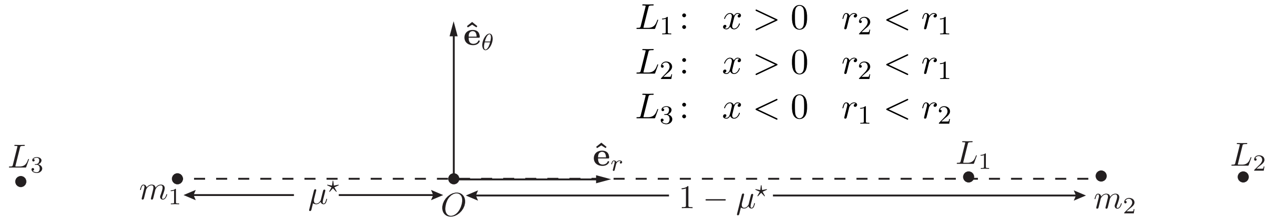
$$\ddot{\beta} + 2\dot{\alpha} = -\left.\frac{\partial U}{\partial y}\right|_{L_{1\ldots3}} = \beta(1-D)$$

$$D \triangleq \frac{1-\mu^\star}{r_1^3} + \frac{\mu^\star}{r_2^3}$$

$$\left.\begin{aligned} \alpha &\triangleq Ae^{\lambda t} \\ \beta &\triangleq Be^{\lambda t} \end{aligned}\right\} \quad \begin{aligned} \lambda^4 + (2-D)\lambda^2 + (1+2D)(1-D) = 0 \Longrightarrow \\ \lambda^2 = \left(\frac{D}{2} - 1\right) \pm \frac{1}{2}\sqrt{D(9D-8)} \end{aligned}$$

$$x - \frac{(1-\mu^\star)(x+\mu^\star)}{|x+\mu^\star|^3} - \frac{\mu^\star(x-1+\mu^\star)}{|x-1+\mu^\star|^3} = 0 \Longrightarrow 1 - D = \frac{\mu^\star(1-\mu^\star)}{x}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)$$

Stability of $L_{1 \ldots 3}$ Points



- We again require λ^2 to be strictly real and negative for stability
- D is positive by definition and we require $9D > 8$ and $D < 1$
- However, none of the three co-linear Lagrange points allows for $D < 1$
- $L_{1 \ldots 3}$ are inherently unstable

The Tisserand Criterion

$$\frac{1}{2}\left({}^{\mathcal{B}}\mathbf{v}_{P/O} \cdot \underbrace{{}^{\mathcal{B}}\mathbf{v}_{P/O}}\right) - \frac{x^2 + y^2}{2} - \left(\frac{1-\mu^\star}{r_1} + \frac{\mu^\star}{r_2}\right) = C$$

$${}^{\mathcal{B}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{P/O} - \hat{\mathbf{e}}_3 \times \mathbf{r}_{P/O}$$

$$C = \frac{1}{2}\left({}^{\mathcal{I}}\mathbf{v}_{P/O} \cdot {}^{\mathcal{I}}\mathbf{v}_{P/O}\right) - \hat{\mathbf{e}}_3 \cdot {}^{\mathcal{I}}\mathbf{h}_P - \left(\frac{1-\mu^\star}{r_1} + \frac{\mu^\star}{r_2}\right)$$

$$\frac{1}{2}\left({}^{\mathcal{I}}\mathbf{v}_{P/1} \cdot {}^{\mathcal{I}}\mathbf{v}_{P/1}\right) = \frac{1}{r_1} - \frac{1}{2a} \qquad \hat{\mathbf{e}}_3 \cdot {}^{\mathcal{I}}\mathbf{h}_P = \sqrt{a(1-e^2)}\cos(I)$$

$$\frac{1}{a} + 2\sqrt{a(1-e^2)}\cos(I) + \underbrace{2\mu^\star\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}_{\text{small}} = -2C$$

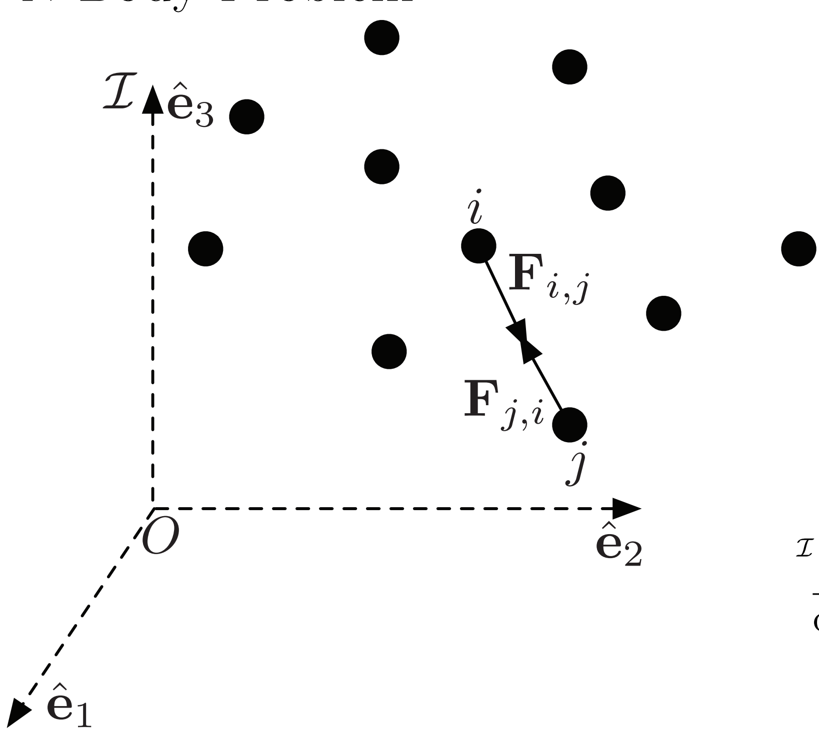
$$\boxed{T \triangleq \frac{1}{a} + 2\sqrt{a(1-e^2)}\cos(I) \approx -2C}$$

The Tisserand Criterion and Trajectory Design

Tisserand's Criterion can be used as a trajectory design tool:

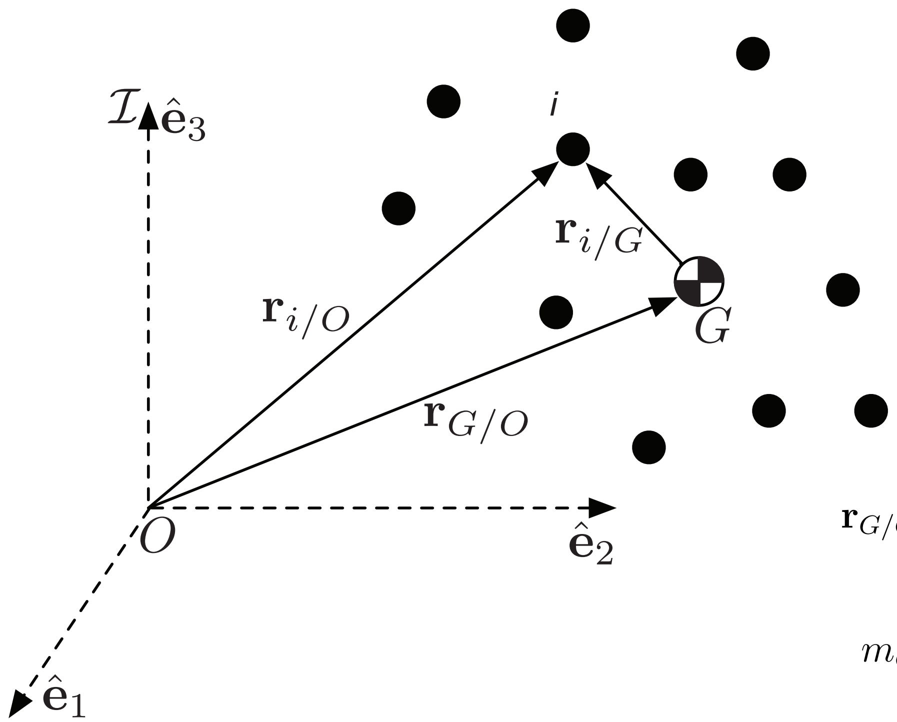
- The conditions of the approximation (small $\mu^\star$ and $r_2^{-1} - r_1^{-1}$) apply when planning deep-space flybys.
- Can therefore match a, e, I pre- and post- flyby allowing for rapid iteration on flyby trajectories
- The Tisserand criterion was explicitly used in initial TESS orbit design when modeling lunar flybys. See Gangestad et al. (2013) and Dichmann et al. (2016) for details

The N-Body Problem



$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{i/O} = -G\sum_{\substack{j=1\\j\neq i}}^{N}\frac{m_j}{\|\mathbf{r}_{i/j}\|^3}\mathbf{r}_{i/j}$$

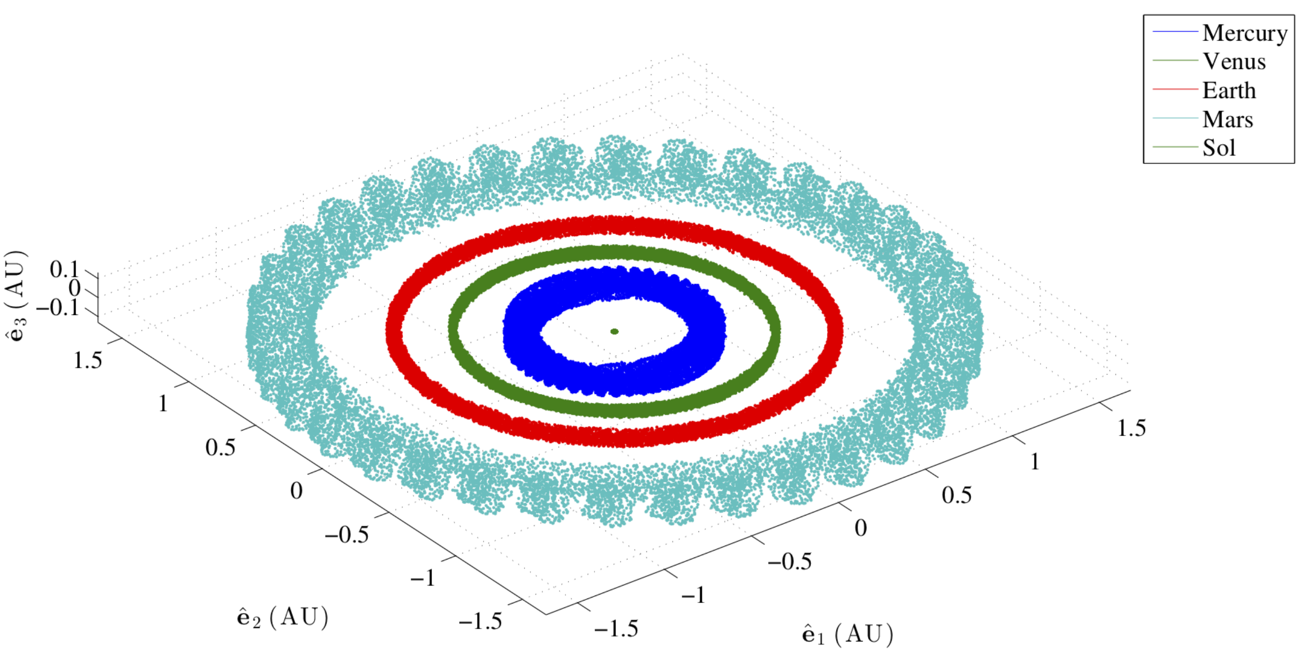
Center of Mass



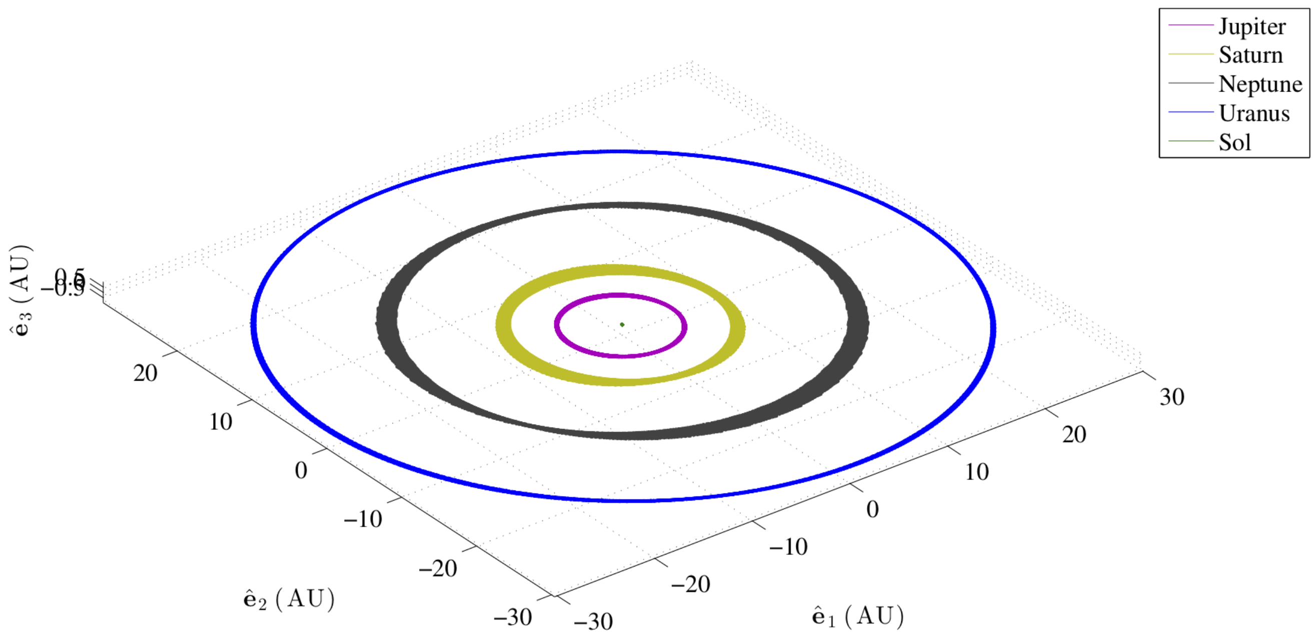
$$\mathbf{r}_{G/O} \triangleq \frac{1}{m_G} \sum_{i=1}^{N} m_i \mathbf{r}_{i/O}$$

$$m_G \triangleq \sum_{i=1}^{N} m_i$$

Inner Solar System over 100,000 Years



Outer Solar System over 100,000 Years



Earth Orbital Elements over 100,000 Years