1 - Review

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Math and Dynamics Review

The study of astrodynamics, or orbital mechanics, is essentially the study of classical mechanics (sometimes known as Newtonian mechanics). While we now know that these are only approximations, with a more accurate model available via Einstein's general relativity, for the majority of cases we rely on the laws first postulated by Isaac Newton and later expanded by Leonhard Euler. As with any study, the first step is to make sure that we have the appropriate tools and language to describe the phenomena under consideration. Since the early 20th century, thanks to the efforts of Josiah Willard Gibbs, the standard tools for studying classical mechanics are vector algebra and vector calculus, which we will review here. Remember that these handouts are not complete on their own. They are intended to accompany the recorded lectures, and to help in your note-taking and studying.

Newton's Laws of Motion

1. *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare*
   Every body preserves in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon
2. *Mutationem motus proportionalem esse vi motrici impressae; et fieri secundum lineam rectam qua vis illa imprimitur*
   The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed
3. *Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi*
   To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts

A Vector is an Element of a Vector Space

A vector space is a collection of vectors over a field of scalars...

A field ($\mathcal{F}$) is a set of scalars ($x, y, z, \ldots \in \mathcal{F}$), with two binary operators: Addition and Multiplication, obeying field axioms: $\forall x, y, z \in \mathcal{F}$

1. Both operators are **associative** $x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$
2. Both operators are **commutative** $x + y = y + x$ and $xy = yx$
3. Every field contains an **additive identity element** (0) such that: $x + 0 = x$
4. Every field contains a **multiplicative identity element** (1) such that: $1x = x$
5. Every element x has an **additive inverse** ($-x$) such that: $x + -x = 0$
6. Every element x has a **multiplicative inverse** (x^{-1}) such that: $xx^{-1} = 1$
7. Addition is **distributive** over multiplication: $x(y + z) = xy + xz$

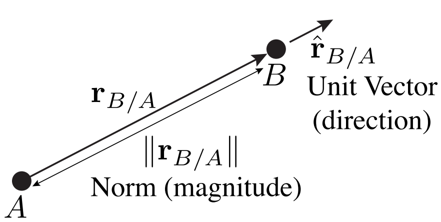
A Vector is an Element of a Vector Space

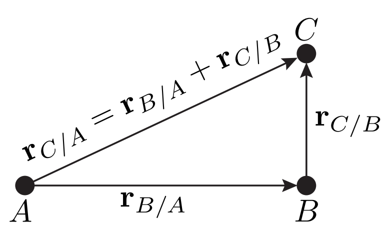
A vector space (V) is a collection of vectors ($\mathbf{a}, \mathbf{b}, \mathbf{c} \ldots \in V$) over a field of scalars ($x, y, z, \ldots \in \mathcal{F}$), with two operators: Vector Addition and Scalar Multiplication with the following properties: $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in V,\ x, y \in \mathcal{F}$

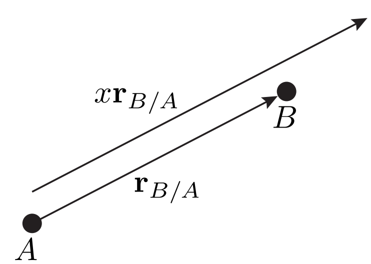
1. Commutativity of vector addition: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. Associativity of vector addition: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
3. Identity element of vector addition: $\exists \mathbf{0} \in V$ s.t. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. Inverse elements of vector addition: $\exists -\mathbf{a} \in V$ s.t. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. Compatibility of scalar multiplication: $x(y\mathbf{a}) = (xy)\mathbf{a}$
6. Distributivity of scalar multiplication over vector addition: $x(\mathbf{a} + \mathbf{b}) = x\mathbf{a} + x\mathbf{b}$
7. Distributivity of scalar multiplication over scalar addition: $(x + y)\mathbf{a} = x\mathbf{a} + y\mathbf{a}$
8. Identity element of scalar multiplication: $\exists 1 \in \mathcal{F}$ s.t. $1\mathbf{a} = \mathbf{a}$

Euclidean (Geometric) Vectors

A Euclidean vector has a magnitude and a direction. A position vector $\mathbf{r}_{B/A}$ has a magnitude of the distance between points A and B and a direction pointing from A to B.







$-\mathbf{r}_{B/A} = \mathbf{r}_{A/B}$

$\hat{\mathbf{r}}_{B/A} = \dfrac{\mathbf{r}_{B/A}}{\|\mathbf{r}_{B/A}\|}$

$\mathbf{r}_{C/B} + \mathbf{r}_{B/A} = \mathbf{r}_{C/A}$

$\Rightarrow \mathbf{r}_{C/A} \underbrace{-\mathbf{r}_{B/A}}_{\equiv +\mathbf{r}_{A/B}} = \mathbf{r}_{C/B}$

$x\mathbf{r}_{B/A} = x\|\mathbf{r}_{B/A}\|\hat{\mathbf{r}}_{B/A}$

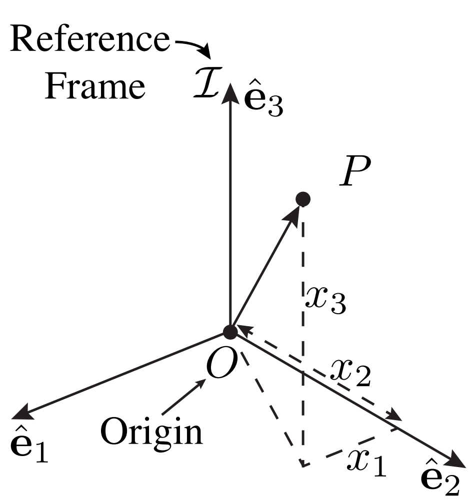
The basis of a vector space is:

1. A linearly independent set of vectors spanning the vector space
2. A subset of vectors in the space such that all vectors in the space may be written as a weighted sum of the subset
3. Non-unique

Define set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for $\mathbf{v}_i \in V$ for vector space V.

- S is **linearly independent** if $\sum a_i \mathbf{v}_i = 0 \Leftrightarrow a_i \equiv 0 \; \forall i, a_i \in \mathcal{F}$
- S **spans** V if $\exists\, a_i \in \mathcal{F}$ such that $\mathbf{b} = \sum a_i \mathbf{v}_i \; \forall \mathbf{b} \in V$

Reference Frames (Bases) and Vector Components



$$\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$$

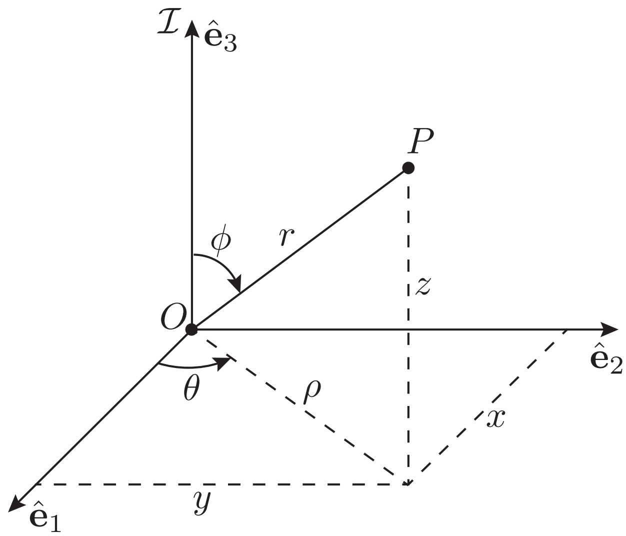
$$\mathbf{r}_{P/O} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3$$

x_i are **Cartesian** coordinates

$$\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathcal{I}}$$

Coordinate Systems

A single reference frame can have an infinite number of coordinate systems



Polar/Cylindrical Coordinates
θ - Azimuthal Angle

$$\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}} = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{bmatrix}_{\mathcal{I}}$$

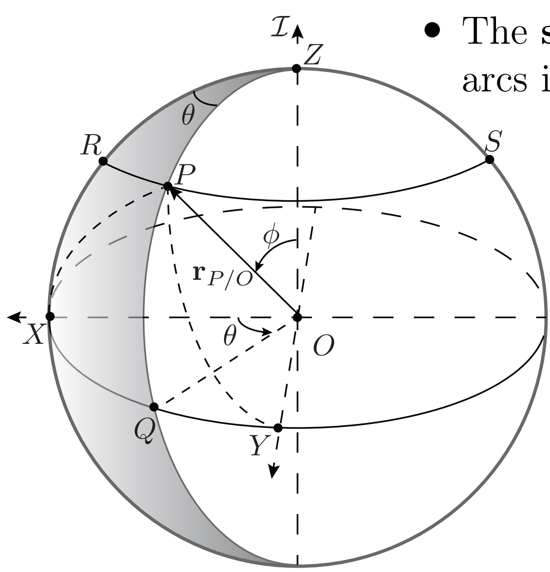
Spherical Coordinates
ϕ - Polar (Zenith) Angle

$$\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}} = r\begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}_{\mathcal{I}}$$

NB: θ and ϕ definitions are frequently reversed. Spherical coordinates are sometimes defined with an elevation angle (the complement to the zenith)

Spherical Trigonometry

- A plane passing through a sphere's center intersects the sphere in a **great circle**, which has **poles** perpendicular to the plane.



Adapted from Green (1985)

- The **spherical angle** between intersecting great circle arcs is the angle between their planes:

$$\text{Spherical Angle } XZQ \equiv \angle XOQ \equiv \theta$$
$$\equiv \text{Great Circle Arc } \widehat{XQ}$$

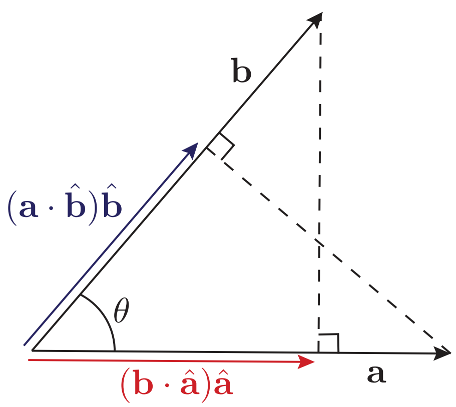
- A plane that does not pass through the sphere's center intersects the sphere in a **small** circle.

$$\left[\hat{\mathbf{r}}_{P/O}\right]_{\mathcal{I}} = \begin{bmatrix} \cos\widehat{XP} \\ \sin\widehat{YP} \\ \cos\widehat{ZP} \end{bmatrix}_{\mathcal{I}}$$

Vector Products

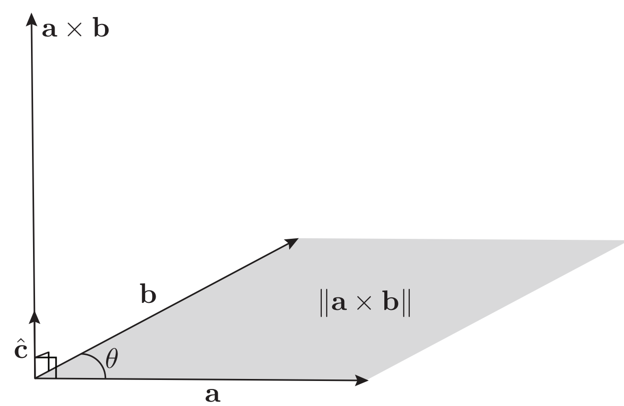
(Scalar) Dot Product

- $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $x\mathbf{a} \cdot y\mathbf{b} = xy(\mathbf{a} \cdot \mathbf{b})$



(Vector) Cross Product

- $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta \hat{\mathbf{c}}$
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $y\mathbf{a} \times \mathbf{b} = y\,(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times y\mathbf{b}$

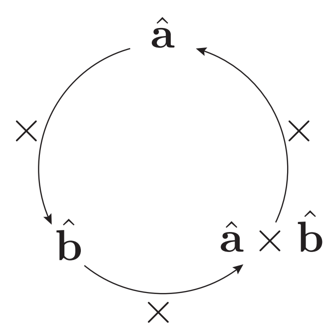


$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|\|\mathbf{a}\| \cos(0) = \|\mathbf{a}\|^2$$

More Vector Products

If vector $\mathbf{a}$ is perpendicular to vector $\mathbf{b}$ ($\mathbf{a} \perp \mathbf{b}$):

1. $\mathbf{a} \cdot \mathbf{b} = 0$
2. $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{a}} \times \hat{\mathbf{b}}$ is a reference frame



- Scalar Triple Product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
- Vector Triple Product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$

Tensors and the Outer Product

A rank-2 tensor (dyadic) is defined as the outer product of two vectors:
$\mathbb{T} \triangleq \mathbf{a} \otimes \mathbf{b}$

$$\mathcal{I} = (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \begin{cases} \mathbf{a} = \sum_i a_i \hat{\mathbf{e}}_i \Rightarrow a_i = \mathbf{a} \cdot \hat{\mathbf{e}}_i \qquad \mathbf{b} = \sum_i b_i \hat{\mathbf{e}}_i \Rightarrow b_i = \mathbf{b} \cdot \hat{\mathbf{e}}_i \\ \mathbb{T} = \sum_i \sum_j T_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \Rightarrow T_{ij} = \hat{\mathbf{e}}_i \cdot \mathbb{T} \cdot \hat{\mathbf{e}}_j = a_i b_j \end{cases}$$

(Tensor) Outer Product

- $(\mathbf{a} + \mathbf{b}) \otimes \mathbf{c} = \mathbf{a} \otimes \mathbf{c} + \mathbf{b} \otimes \mathbf{c}$
- $\mathbf{c} \otimes (\mathbf{a} + \mathbf{b}) = \mathbf{c} \otimes \mathbf{a} + \mathbf{c} \otimes \mathbf{b}$
- $x(\mathbf{a} \otimes \mathbf{b}) = (x\mathbf{a}) \otimes \mathbf{b} = \mathbf{a} \otimes (x\mathbf{b})$
- $(\mathbf{a} \otimes \mathbf{b}) \otimes \mathbf{c} = \mathbf{a} \otimes (\mathbf{b} \otimes \mathbf{c})$

All Vector and Tensor Operations Can Be Written as Matrix Multiplications

$$[\mathbf{a}]_\mathcal{I} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_\mathcal{I} \qquad [\mathbf{b}]_\mathcal{I} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_\mathcal{I} \qquad [\mathbb{T}]_\mathcal{I} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_\mathcal{I} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}_\mathcal{I}$$

$$[\mathbf{a} \cdot \mathbf{b}]_\mathcal{I} = [\mathbf{a}]_\mathcal{I}^T [\mathbf{b}]_\mathcal{I} \qquad [\mathbf{a} \cdot \mathbb{T}]_\mathcal{I} = \left([\mathbf{a}]_\mathcal{I}^T [\mathbb{T}]_\mathcal{I}\right)^T \qquad [\mathbb{T}]_\mathcal{I} = [\mathbf{a} \otimes \mathbf{b}]_\mathcal{I} = [\mathbf{a}]_\mathcal{I} [\mathbf{b}]_\mathcal{I}^T$$

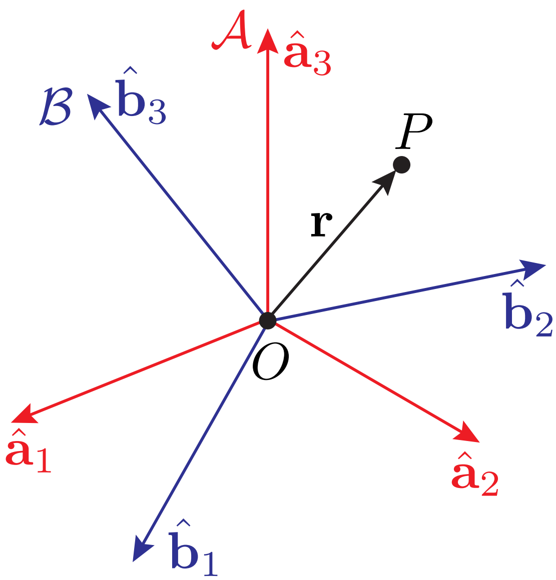
$$[\mathbf{a} \times \mathbf{b}]_\mathcal{I} = [\mathbf{a}\times]_\mathcal{I} [\mathbf{b}]_\mathcal{I} \qquad [\mathbb{T} \cdot \mathbf{a}]_\mathcal{I} = [\mathbb{T}]_\mathcal{I} [\mathbf{a}]_\mathcal{I}$$

$$[\mathbf{b} \times \mathbf{a}]_\mathcal{I} = [\mathbf{b}\times]_\mathcal{I} [\mathbf{a}]_\mathcal{I} \qquad [\mathbf{a} \times \mathbb{T}]_\mathcal{I} = [\mathbf{a}\times]_\mathcal{I} [\mathbb{T}]_\mathcal{I} \qquad [\mathbf{a}\times]_\mathcal{I} \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_\mathcal{I}$$

$$= -[\mathbf{a}\times]_\mathcal{I} [\mathbf{b}]_\mathcal{I} \qquad [\mathbb{T} \times \mathbf{a}]_\mathcal{I} = [\mathbb{T}]_\mathcal{I} [\mathbf{a}\times]_\mathcal{I}$$

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2 \Longrightarrow [\mathbf{a} \cdot \mathbf{a}]_\mathcal{I} = [\mathbf{a}]_\mathcal{I}^T [\mathbf{a}]_\mathcal{I} = a_1^2 + a_2^2 + a_3^2 \quad \text{so} \quad \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Multiple Reference Frames and Direction Cosine Matrices



$$[\mathbf{r}]_{\mathcal{B}} = {}^{\mathcal{B}}C^{\mathcal{A}}\,[\mathbf{r}]_{\mathcal{A}}$$

Direction Cosine Matrix (DCM)

- DCMs are Orthogonal Matrices:

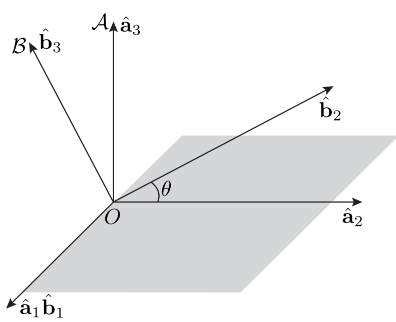
$$^{\mathcal{A}}C^{\mathcal{B}} = \left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{-1} = \left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{T}$$

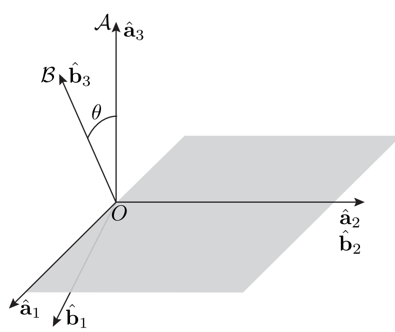
- DCMs are Composed by Multiplication:

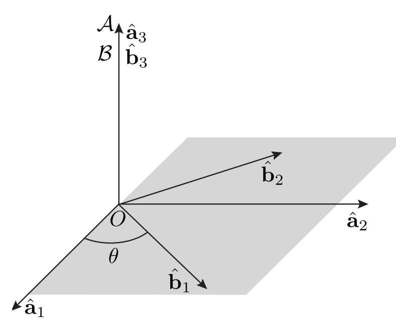
$$^{\mathcal{I}}C^{\mathcal{F}_1}\,{}^{\mathcal{F}_1}C^{\mathcal{F}_2}\,{}^{\mathcal{F}_2}C^{\mathcal{F}_3}\ldots{}^{\mathcal{F}_{\mathcal{N}-1}}C^{\mathcal{F}_\mathcal{N}} = {}^{\mathcal{I}}C^{\mathcal{F}_\mathcal{N}}$$

$$[\mathbb{T}]_{\mathcal{B}} = {}^{\mathcal{B}}C^{\mathcal{A}}\,[\mathbb{T}]_{\mathcal{A}}\,{}^{\mathcal{A}}C^{\mathcal{B}}$$

Simple Direction Cosine Matrices







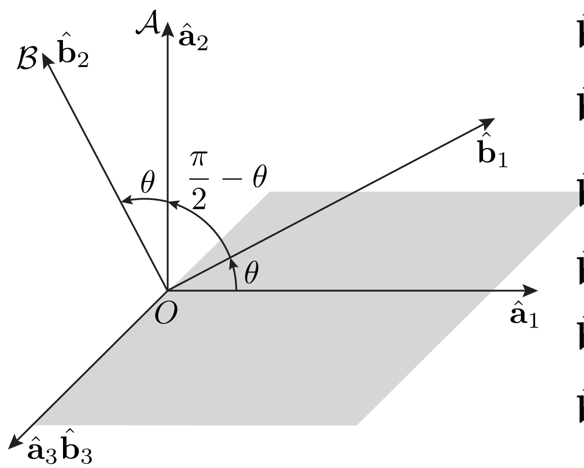
$$^{\mathcal{B}}C^{\mathcal{A}} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}}_{\triangleq\, C_1(\theta)}\quad \underbrace{\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}}_{\triangleq\, C_2(\theta)}\quad \underbrace{\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}}_{\triangleq\, C_3(\theta)}$$

Any DCM can be decomposed into three rotations about non-repeating frame axes.

More on DCMs

Each entry of a DCM is the cosine of the angle between each pair of unit vectors of the two frames the DCM maps between:

$$\left.\begin{aligned}\mathcal{A} &= (O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3) \\ \mathcal{B} &= (O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)\end{aligned}\right\} \left[{}^{\mathcal{A}}C^{\mathcal{B}}\right]_{ij} = \hat{\mathbf{a}}_i \cdot \hat{\mathbf{b}}_j \Longrightarrow {}^{\mathcal{B}}C^{\mathcal{A}} = \left({}^{\mathcal{A}}C^{\mathcal{B}}\right)^T \Longrightarrow \left[{}^{\mathcal{B}}C^{\mathcal{A}}\right]_{ij} = \hat{\mathbf{b}}_i \cdot \hat{\mathbf{a}}_j$$



$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_1 = \cos\theta$$

$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_2 = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_1 = \cos\left(\theta + \frac{\pi}{2} - \theta + \theta\right) = -\sin\theta$$

$$\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_2 = \cos\theta$$

$$\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_3 = 1$$

$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_3 = \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_3 = \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_1 = \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_2 = \cos\left(\frac{\pi}{2}\right) = 0$$

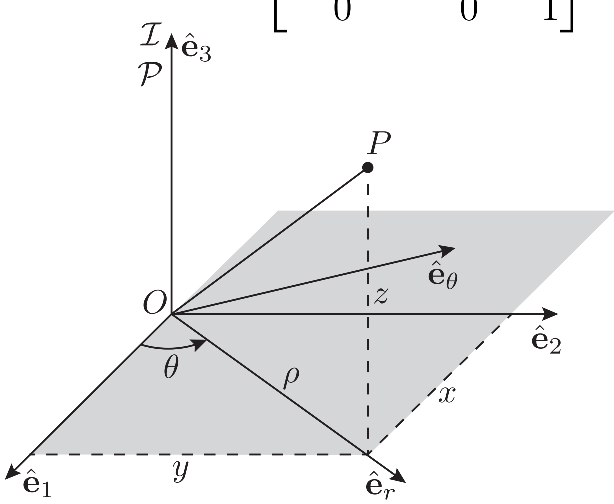
Polar/Cylindrical Reference Frames

$$\mathcal{P} = (O, \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$$

$${}^{\mathcal{P}}C^{\mathcal{I}} \equiv C_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\left[\mathbf{r}_{P/O}\right]_{\mathcal{P}}}_{\begin{bmatrix}\rho \\ 0 \\ z\end{bmatrix}_{\mathcal{P}}} = {}^{\mathcal{P}}C^{\mathcal{I}} \underbrace{\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}}}_{\begin{bmatrix}x \\ y \\ z\end{bmatrix}_{\mathcal{I}}}$$

$$\begin{bmatrix}\rho \\ 0 \\ z\end{bmatrix}_{\mathcal{P}} = {}^{\mathcal{P}}C^{\mathcal{I}} \begin{bmatrix}x \\ y \\ z\end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} x\cos(\theta) + y\sin(\theta) \\ -x\sin(\theta) + y\cos(\theta) \\ z \end{bmatrix}_{\mathcal{P}}$$



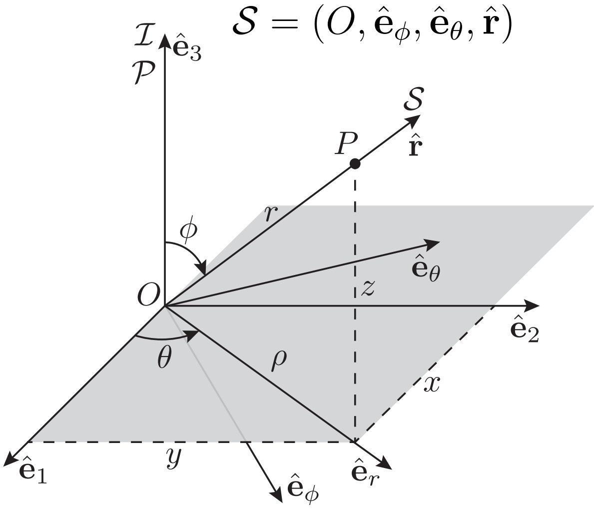
$$\begin{bmatrix}x \\ y \\ z\end{bmatrix}_{\mathcal{I}} = \underbrace{{}^{\mathcal{I}}C^{\mathcal{P}}}_{\left({}^{\mathcal{P}}C^{\mathcal{I}}\right)^T} \begin{bmatrix}\rho \\ 0 \\ z\end{bmatrix}_{\mathcal{P}} = \begin{bmatrix}\rho\cos(\theta) \\ \rho\sin(\theta) \\ z\end{bmatrix}_{\mathcal{I}}$$

Useful for tracking an object moving in-plane whose position is most easily described in polar coordinates. $\hat{\mathbf{e}}_r$ will always point at the object.

Spherical Reference Frames

$$ {}^{\mathcal{S}}C^{\mathcal{I}} = C_2(\phi)C_3(\theta) = \begin{bmatrix} \cos(\phi)\cos(\theta) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \\ \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta) & \cos(\phi) \end{bmatrix} $$

$$ \mathcal{S} = (O, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta, \hat{\mathbf{r}}) $$



$$ \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{I}}}_{\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}}} = \underbrace{{}^{\mathcal{I}}C^{\mathcal{S}}}_{\left({}^{\mathcal{S}}C^{\mathcal{I}}\right)^T} \underbrace{\begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}_{\mathcal{S}}}_{\left[\mathbf{r}_{P/O}\right]_{\mathcal{S}}} = r \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}_{\mathcal{I}} $$

Useful for tracking an object moving in 3D whose position is most easily described in spherical coordinates. $\hat{\mathbf{r}}$ will always point at the object.

Vector Derivatives in Time

- A vector $\mathbf{r}_{P/O} = a_1\hat{\mathbf{a}}_1 + a_2\hat{\mathbf{a}}_2 + a_3\hat{\mathbf{a}}_3$ is differentiable in time at a time t_1 with respect to frame $\mathcal{A} = (O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)$ if $a_1(t), a_2(t), a_3(t)$ are differentiable at $t = t_1$. Then:

$$ {}^{\mathcal{A}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}_{P/O}\bigg|_{t=t_1} = \frac{\mathrm{d}a_1}{\mathrm{d}t}\bigg|_{t=t_1}\hat{\mathbf{a}}_1 + \frac{\mathrm{d}a_2}{\mathrm{d}t}\bigg|_{t=t_1}\hat{\mathbf{a}}_2 + \frac{\mathrm{d}a_3}{\mathrm{d}t}\bigg|_{t=t_1}\hat{\mathbf{a}}_3 $$

- The unit vectors defining a frame **always** have zero time derivatives with respect to that frame (but not necessarily to other frames)

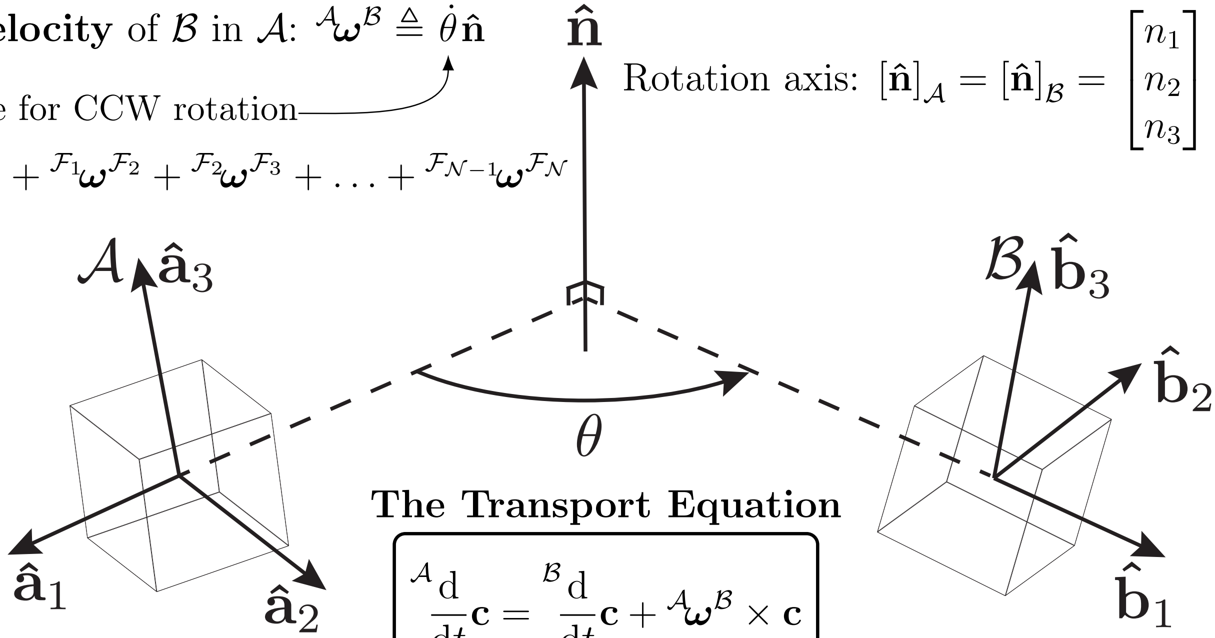
Vector Differentiation Across Reference Frames

Angular Velocity of $\mathcal{B}$ in $\mathcal{A}$: ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \triangleq \dot{\theta}\,\hat{\mathbf{n}}$

Positive for CCW rotation

Rotation axis: $[\hat{\mathbf{n}}]_{\mathcal{A}} = [\hat{\mathbf{n}}]_{\mathcal{B}} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{F}_{\mathcal{N}}} = {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{F}_1} + {}^{\mathcal{F}_1}\boldsymbol{\omega}^{\mathcal{F}_2} + {}^{\mathcal{F}_2}\boldsymbol{\omega}^{\mathcal{F}_3} + \ldots + {}^{\mathcal{F}_{\mathcal{N}-1}}\boldsymbol{\omega}^{\mathcal{F}_{\mathcal{N}}}$$



The Transport Equation

$$\frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\mathbf{c} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{c} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{c}$$

NB: Counter-clockwise is defined by looking *down* along the axis of rotation.

Newton's Second Law

Inertial Frame Derivative — Inertially Fixed Point — Mass (Assumed Constant)

$$\mathbf{F}_P = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{p}_{P/O}\right) = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left(m_P\,{}^{\mathcal{I}}\mathbf{v}_{P/O}\right) = m_P\,{}^{\mathcal{I}}\mathbf{a}_{P/O}$$

Resultant Force on P — Linear Momentum — Inertial Velocity and Acceleration

$$\mathbf{M}_{P/O} = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_{P/O}\right) = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_{P/O} \times {}^{\mathcal{I}}\mathbf{p}_{P/O}\right) = \mathbf{r}_{P/O} \times \mathbf{F}_P$$

Net Moment (Torque) about O — Angular Momentum of P about O

Euler's Laws

I. The product of the inertial acceleration of the center of mass of a rigid body and its total mass is equal to the total external force applied to the body

$$\mathbf{F}_G = m_G {}^{\mathcal{I}}\mathbf{a}_{G/O}$$

II. The rate of change of the inertial angular momentum of a rigid body about a fixed point O in the inertial frame is equal to the total external moment applied to the body about O

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\mathbf{h}_O = \mathbf{M}_O$$

Center of Mass

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \sum_{i=1}^{N} \mathbf{r}_{i/O} m_i$$

- As $N \to \infty$: $m_i \to 0$

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/O} \,\mathrm{d}m$$

- If the density is given by $\rho(\mathbf{r}_{\,\mathrm{d}m/O})$:

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/O} \rho(\mathbf{r}_{\,\mathrm{d}m/O}) \,\mathrm{d}V$$

- The center of mass corollary:

$$\sum_{i=1}^{N} m_i \mathbf{r}_{i/G} = 0 \qquad \text{or} \qquad \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/G} \rho(\mathbf{r}_{\,\mathrm{d}m/G}) \,\mathrm{d}V = 0$$

Angular Momentum of a Rigid Body

$$
{}^{\mathcal{I}}\mathbf{h}_O = \sum_{i=1}^{N} m_i \mathbf{r}_{i/O} \times {}^{\mathcal{I}}\mathbf{v}_{i/O} \quad \xrightarrow[N \to \infty]{m_i \to \mathrm{d}m} \quad {}^{\mathcal{I}}\mathbf{h}_O = \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/O} \times {}^{\mathcal{I}}\mathbf{v}_{\,\mathrm{d}m/O}\,\mathrm{d}m
$$

Internal forces between i and j

$$
\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_O\right) = \underbrace{\sum_{i=1}^{N} \mathbf{r}_{i/O} \times \mathbf{F}_i^{(\mathrm{ext})}}_{\triangleq\, \mathbf{M}_O^{(\mathrm{ext})}} + \underbrace{\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\mathbf{r}_{i/O} - \mathbf{r}_{j/O}\right) \times \mathbf{F}_{i,j}}_{\text{Equals zero for rigid bodies}}
$$

This is known as the internal moment assumption

NB: The internal moment assumption for rigid bodies is effectively the only additional postulate added by Euler's laws to Newton's laws. It is possible to define internal forces within a collection of particles that violate this assumption, therefore, we take its applicability to rigid bodies as a new law.

The Separation Principle

$$
{}^{\mathcal{I}}\mathbf{h}_O = \overbrace{\text{Angular Momentum of COM about an Inertially Fixed Point}}^{{}^{\mathcal{I}}\mathbf{h}_{G/O}} + \overbrace{\text{Angular Momentum of Body about its COM}}^{{}^{\mathcal{I}}\mathbf{h}_G}
$$

$$
{}^{\mathcal{I}}\mathbf{h}_{G/O} \triangleq m_G \mathbf{r}_{G/O} \times {}^{\mathcal{I}}\mathbf{v}_{G/O} \quad \Longrightarrow \quad \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_{G/O}\right) = \mathbf{M}_{G/O} \triangleq \mathbf{r}_{G/O} \times \mathbf{F}_G
$$

$$
{}^{\mathcal{I}}\mathbf{h}_G \triangleq \begin{cases} \displaystyle\sum_{i=1}^{N} m_i \mathbf{r}_{i/G} \times {}^{\mathcal{I}}\mathbf{v}_{i/G} & \text{Particles} \\ \displaystyle\int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/G} \times {}^{\mathcal{I}}\mathbf{v}_{\,\mathrm{d}m/G}\,\mathrm{d}m & \text{Continuous Bodies} \end{cases}
$$

$$
\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_G\right) = \mathbf{M}_G \triangleq \begin{cases} \displaystyle\sum_{i=1}^{N} \mathbf{r}_{i/G} \times \mathbf{F}_i^{(\mathrm{ext})} & \text{Contact Forces } (N = \#\text{ contacts for rigid bodies}) \\ \displaystyle\int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/G} \times \mathbf{f}_{\mathrm{d}m}\,\mathrm{d}m & \text{Field Forces} \end{cases}
$$

Moment of Inertia

$$\boxed{{}^{\mathcal{I}}\mathbf{h}_G = \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}}$$

$$\mathbb{I}_G \triangleq \begin{cases} \displaystyle\sum_{i=1}^{N} m_i \left[(\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G})\mathbb{U} - (\mathbf{r}_{i/G} \otimes \mathbf{r}_{i/G})\right] & \text{Collection of Particles} \\ \displaystyle\int_{\mathscr{B}} \left[(\mathbf{r}_{dm/G} \cdot \mathbf{r}_{dm/G})\mathbb{U} - (\mathbf{r}_{dm/G} \otimes \mathbf{r}_{dm/G})\right] \, \mathrm{d}m & \text{Rigid Body} \end{cases}$$

Matrix of Inertia

$$\mathbb{I}_G = \sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}\mathbf{b}_i \otimes \mathbf{b}_j$$

$$\begin{aligned} [\mathbb{I}_G]_{\mathcal{B}} &= \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}_{\mathcal{B}} \\ &= \sum_{i=1}^{N} m_i \left(([\mathbf{r}_{i/G}]_{\mathcal{B}}^T[\mathbf{r}_{i/G}]_{\mathcal{B}})I - [\mathbf{r}_{i/G}]_{\mathcal{B}}[\mathbf{r}_{i/G}]_{\mathcal{B}}^T\right) \\ &= \int_{\mathscr{B}} \left(\|\mathbf{r}_{dm/G}\|^2 I - [\mathbf{r}_{dm/G}]_{\mathcal{B}}[\mathbf{r}_{dm/G}]_{\mathcal{B}}^T\right) \, \mathrm{d}m \\ &= \int_{\mathscr{B}} \left(\|\mathbf{r}_{dm/G}\|^2 I - [\mathbf{r}_{dm/G}]_{\mathcal{B}}[\mathbf{r}_{dm/G}]_{\mathcal{B}}^T\right) \rho(\mathbf{r}_{dm/G}) \, \mathrm{d}V \end{aligned}$$

Moments and Angular Momentum about an Arbitrary Point Q fixed to a Rigid Body

$$\mathbf{M}_Q = \mathbf{M}_G - \mathbf{r}_{Q/G} \times \sum_{i=1}^{N} \mathbf{F}_i^{(\text{ext})}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_Q\right) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_Q\right) + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_Q = \mathbf{M}_Q + \mathbf{r}_{Q/G} \times m_G {}^{\mathcal{I}}\mathbf{a}_{Q/O}$$

$$ {}^{\mathcal{I}}\mathbf{h}_Q = \mathbb{I}_Q \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \qquad \mathbb{I}_Q \triangleq \sum_{i=1}^{N} m_i \left((\mathbf{r}_{i/Q} \cdot \mathbf{r}_{i/Q})\mathbb{U} - (\mathbf{r}_{i/Q} \otimes \mathbf{r}_{i/Q}) \right) \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$$

The Parallel Axis Theorem

$$\mathbb{I}_Q = \mathbb{I}_G + m_G \left[(\mathbf{r}_{Q/G} \cdot \mathbf{r}_{Q/G})\mathbb{U} - (\mathbf{r}_{Q/G} \otimes \mathbf{r}_{Q/G}) \right]$$

Work and Energy

- A force ($\mathbf{F}_P$) does work (W) on a particle P when it displaces the particle along a trajectory (γ_P): $W_P^{\mathbf{F}_P}(\mathbf{r}_{P/O}; \gamma_P) \triangleq \int_{\gamma_P} \mathbf{F}_P \cdot {}^{\mathcal{I}}\mathrm{d}\mathbf{r}_{P/O}$

  Path Integral over trajectory
- The **Kinetic Energy** of particle P is defined as: $T_{P/O} \triangleq \frac{1}{2} m_p ({}^{\mathcal{I}}\mathbf{v}_{P/O} \cdot {}^{\mathcal{I}}\mathbf{v}_{P/O})$
- The change in kinetic energy from time t_1 to time t_2 is equal to the total work done on the particle during that time
- The work done by **Conservative Forces** depends only on the endpoints of the trajectory. $\oint \mathbf{F}_P \cdot {}^{\mathcal{I}}\mathrm{d}\mathbf{r}_{P/O} = 0$ means that $\mathbf{F}_P$ is conservative.

  Closed Path Integral
- Conservative Forces can always be written as the gradient of a scalar **Potential** (U): $\mathbf{F}_P^{(\text{cons})} = -\boldsymbol{\nabla} U_{P/O}^{(\mathbf{F}_P)}$ so

  $U_{P/O}^{(\mathbf{F}_P)}(t_2) = U_{P/O}^{(\mathbf{F}_P)}(t_1) - W_P^{(\mathbf{F}_P)}(t_1, t_2)$

Total Work and Energy

- Total Energy: $E_{P/O}(t) \triangleq T_{P/O}(t) + U_{P/O}(t)$
- Total Work: $W_P^{\text{tot}}(\mathbf{r}_{P/O}; \gamma_P) = \underbrace{W_P^{\text{c}}(t_1, t_2)}_{\substack{\text{Work due to conservative forces} \\ = \text{negative change in potential energy}}} + \underbrace{W_P^{\text{nc}}(\mathbf{r}_{P/O}; \gamma_P)}_{\substack{\text{Work due to non-conservative forces} \\ = \text{change in total energy}}}$
- Conservation of Energy: no non-conservative forces $\equiv$ constant total energy

$$E_{P/O}(t_2) = E_{P/O}(t_1) + W_P^{(\text{nc})}(\mathbf{r}_{P/O}; \gamma_P)$$

Hamilton's Principle and the Euler-Lagrange Equations

- Define the Lagrangian as KE - PE: $L \triangleq T - V$
- Define the action of a system as: $I \triangleq \int_{t_1}^{t_2} L \, \mathrm{d}t$
- Hamilton's Principle: the motion of a system is a stationary point of the action:

$$\boxed{\delta I = 0}$$

- Hamilton's Principle and Newton's 2nd law lead directly to the Euler-Lagrange Equations:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad \forall i}$$