10 - Non-Spherical Gravity Fields and Non-Gravitational Forces

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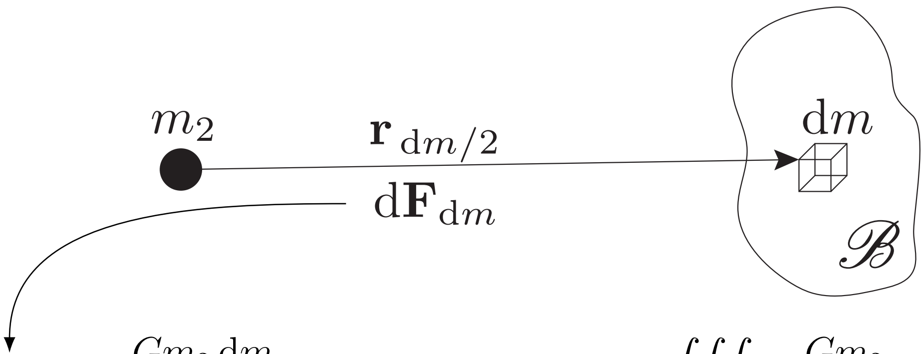
Non-Spherical Gravity Fields and Non-Gravitational Forces

Armed with our perturbation analysis tools, courtesy of Gauss and Lagrange, we are ready to analyze the effects of perturbing forces on orbits. We have already studied one important perturbation source - nth body perturbers. To this we add the gravitational effects of non-spherical central bodies. Moving beyond gravity, we consider atmospheric drag, which is the most important source of perturbations in low Earth orbits, and briefly describe additional important perturbations.

Orbiting About Extended Bodies (forces)



$$\mathrm{d}\mathbf{F}_{\mathrm{d}m} = \frac{G m_2 \,\mathrm{d}m}{\|\mathbf{r}_{\mathrm{d}m/2}\|^3}\mathbf{r}_{\mathrm{d}m/2} \qquad \Rightarrow \qquad \mathbf{F} = \iiint_{\mathscr{B}} \frac{G m_2}{\|\mathbf{r}_{\mathrm{d}m/2}\|^3}\mathbf{r}_{\mathrm{d}m/2}\,\mathrm{d}m$$

$$\mathbf{F} = m_2 \nabla U \qquad\qquad U = G\iiint_{\mathscr{B}} \frac{\mathrm{d}m}{\|\mathbf{r}_{\,\mathrm{d}m/2}\|}$$

Orbiting About Extended Bodies (coordinates)

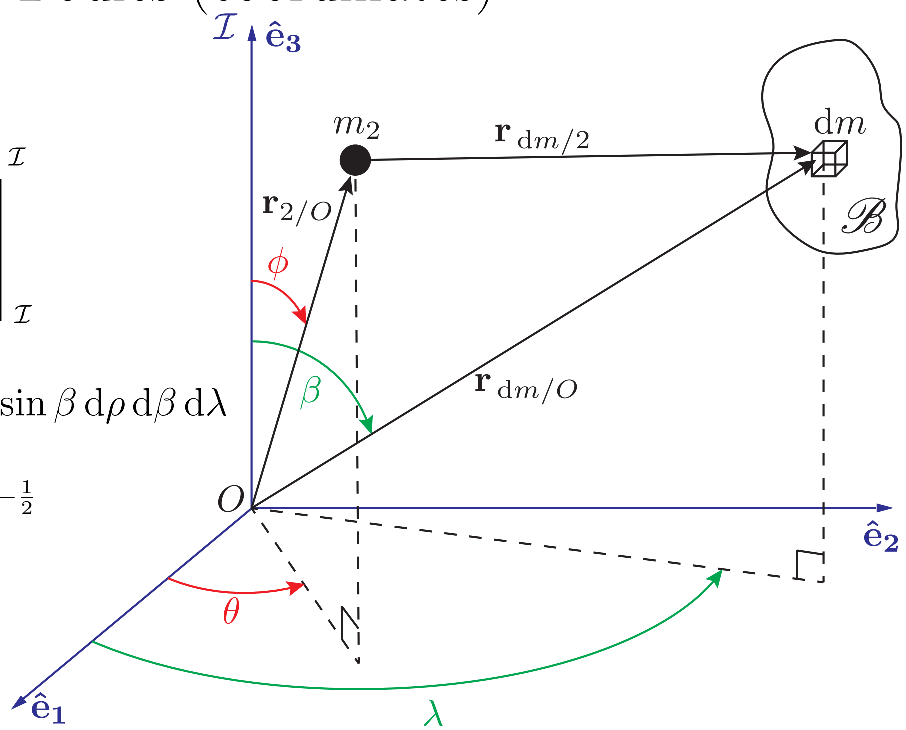
$$\left[\mathbf{r}_{2/O}\right]_{\mathcal{I}} = r\begin{bmatrix}\cos\theta\sin\phi\\ \sin\theta\sin\phi\\ \cos\phi\end{bmatrix}_{\mathcal{I}}$$

$$\left[\mathbf{r}_{\,\mathrm{d}m/O}\right]_{\mathcal{I}} = \rho\begin{bmatrix}\cos\lambda\sin\beta\\ \sin\lambda\sin\beta\\ \cos\beta\end{bmatrix}_{\mathcal{I}}$$

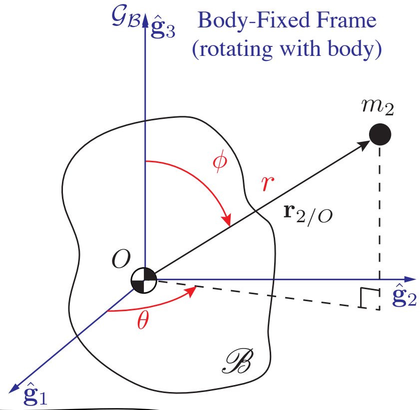
$$U(r,\theta,\phi) = G\iiint_{\mathscr{B}} \underbrace{\frac{\Gamma(\rho,\lambda,\beta)}{\|\mathbf{r}_{\,\mathrm{d}m/2}\|}}\rho^2\sin\beta\,\mathrm{d}\rho\,\mathrm{d}\beta\,\mathrm{d}\lambda$$

$$\frac{1}{r}\left[1 - \frac{2\rho}{r}\cos\left(\beta-\phi\right) + \left(\frac{\rho}{r}\right)^2\right]^{-\frac{1}{2}}$$

$$= \frac{1}{r}\sum_{\ell=0}^{\infty} P_\ell\left(\cos(\beta-\phi)\right)\left(\frac{\rho}{r}\right)^{\ell}$$



Laplace's Equation in Spherical Coordinates



$$\nabla^2 U = 0$$

$$U(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$0 = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right)} + \underbrace{\frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial U}{\partial \phi}\right)} + \underbrace{\frac{1}{r^2 \sin^2\phi}\frac{\partial^2 U}{\partial \theta^2}}$$

$$0 = \overbrace{\left(\frac{r^2 \sin^2\phi}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{2r\sin^2\phi}{R}\frac{\mathrm{d}R}{\mathrm{d}r}\right)} + \overbrace{\left(\frac{\cos\phi\sin\phi}{\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\phi} + \frac{\sin^2\phi}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}\right)} + \underbrace{\overbrace{\left(\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}\right)}}_{\triangleq -m^2}$$

$$\Theta(\theta) = A_m e^{im\theta} = S_m \sin(m\theta) + C_m \cos(m\theta) \qquad m = -\infty \ldots \infty$$

Laplace's Equation in Spherical Coordinates (continued)

$$0 = \underbrace{\left(\frac{r^2}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{2r}{R}\frac{\mathrm{d}R}{\mathrm{d}r}\right)}_{\triangleq \ell(\ell+1)} + \frac{1}{\sin^2\phi}\left(\frac{\cos\phi\sin\phi}{\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\phi} + \frac{\sin^2\phi}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} - m^2\right)$$

$$\Rightarrow R_\ell(r) = A_\ell r^\ell + B_\ell r^{-\ell-1}$$

$$0 = \ell(\ell+1) - \frac{m^2}{\sin^2\phi} + \frac{\cos\phi}{\sin\phi}\frac{1}{\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\phi} + \frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}$$

Associated Legendre Differential Equation

$$x \triangleq \cos\phi \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]\Phi = 0$$

Azimuthally Symmetric Bodies

$$m = 0 \Rightarrow \Theta = 1 \quad \Longrightarrow \quad \overbrace{\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right] + \ell(\ell+1)\Phi = 0}_{\Phi_\ell(\phi) = P_\ell(\cos\phi)}}^{\text{Legendre's Differential Equation}}$$

$$U(r,\phi) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell r^{-\ell-1}\right] P_\ell(\cos\phi) \quad \Longrightarrow \quad A_\ell = 0 \;\forall \ell$$

Body Total Mass — Body Equatorial Radius

$$\boxed{U(r,\phi) = \frac{G\, m_{\mathscr{B}}}{r}\left[1 - \sum_{k=2}^{\infty} J_k \left(\frac{R_{\mathscr{B}}}{r}\right)^k P_k(\cos\phi)\right]}$$

Non-dimensional coefficients named after Harold Jeffreys. $J_1 = 0$ due to symmetry.

J Values for Solar System Bodies

$(\times 10^{-6})$	**Earth**	**Mars**	**Moon**	**Venus**	**Mercury**
J_2	1082.6	1955.5	203.23	4.4044	22.5
J_3	-2.5327	31.450	8.4759	-2.1082	4.49
J_4	-1.6196	-15.377	-9.5919	-2.1474	6.5

$(\times 10^{-6})$	**Jupiter**	**Saturn**	**Uranus**	**Neptune**
J_2	14696.572	16290.573	3341.29	3408.43
J_3	-0.042	0.059	—	—
J_4	-586.609	-935.314	-30.44	-33.40

From: Lemoine et al. 1998 (Earth), Lemoine et al. 2001 (Mars), Konopliv et al. 2001 (Moon), Konopliv et al. 1999 (Venus), Iess et al. 2018 (Jupiter), Iess et al. 2019 (Saturn), Smith et al. 2012 (Mercury)

Arbitrary Bodies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]\Phi=0$$

Solved by the associated Legendre functions:

$$P_\ell^m(x)=\frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}\left(x^2-1\right)^\ell=(-1)^m\,(1-x^2)^{m/2}\,\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(P_\ell(x)\right)$$

$$P_\ell^{-m}(x)=(-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)$$

Combining Φ and Θ terms yields the spherical harmonics:

$$Y_\ell^m(\theta,\phi)=\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\phi)\,e^{im\theta}$$

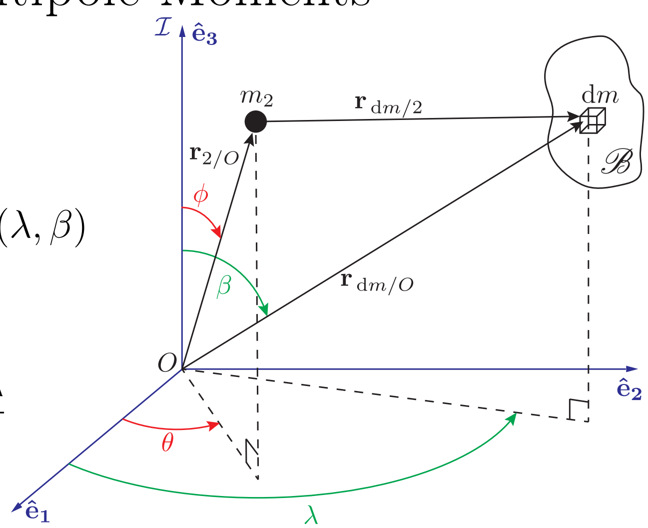
$$U(r,\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}B_{\ell m}r^{-(\ell+1)}Y_\ell^m(\theta,\phi)$$

Legendre Addition Theorem and Multipole Moments

$$P_\ell\left(\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}\right)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y_\ell^m(\hat{\mathbf{x}})\overline{Y}_\ell^m(\hat{\mathbf{y}})$$

$$\|\mathbf{r}_{\,\mathrm{d}m/2}\|^{-1}=4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{\rho^\ell}{r^{\ell+1}}Y_\ell^m(\theta,\phi)\overline{Y}_\ell^m(\lambda,\beta)$$

$$U(r,\theta,\phi)=G\iiint_{\mathscr{B}}\frac{\Gamma(\rho,\lambda,\beta)\rho^2\sin(\beta)\,\mathrm{d}\rho\,\mathrm{d}\beta\,\mathrm{d}\lambda}{\|\mathbf{r}_{\,\mathrm{d}m/2}\|}$$



Multipole moments

$$q_\ell^m\triangleq\iiint_{\mathscr{B}}Y_\ell^m(\theta,\phi)\overline{Y}_\ell^m(\lambda,\beta)\rho^{\ell+2}\Gamma(\rho,\lambda,\beta)\sin\beta\,\mathrm{d}\rho\,\mathrm{d}\beta\,\mathrm{d}\lambda$$

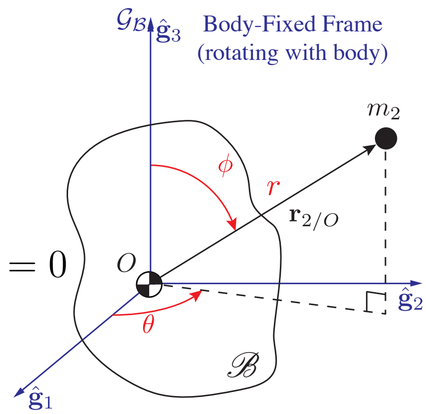
Back to Arbitrary Bodies

$$\left[\mathbf{r}_{\mathrm{d}m/O}\right]_{\mathcal{I}} = \left[\xi \quad \eta \quad \zeta\right]_{\mathcal{I}}^{T}$$

Monopole moment: $q_1^0 = \sqrt{\frac{3}{4\pi}} \int_{\mathscr{B}} \zeta \Gamma(\mathbf{r}_{\mathrm{d}m/O}) \mathrm{d}^3\mathbf{r}_{\mathrm{d}m/O} = 0$

Dipole moment: $q_1^1 = -\sqrt{\frac{3}{8\pi}} \int_{\mathscr{B}} (\xi - i\eta)\, \Gamma(\mathbf{r}_{\mathrm{d}m/O}) \mathrm{d}^3\mathbf{r}_{\mathrm{d}m/O} = 0$

$$U(r,\theta,\phi) = \frac{Gm_{\mathscr{B}}}{r} + G\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} q_\ell^m r^{-(\ell+1)} Y_\ell^m(\theta,\phi)$$



$$U(r,\theta,\phi) = \frac{Gm_{\mathscr{B}}}{r}\left[1 + \sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} P_\ell^m(\cos\phi) \times \left(C_\ell^m \cos(m\theta) + S_\ell^m \sin(m\theta)\right)\right]$$

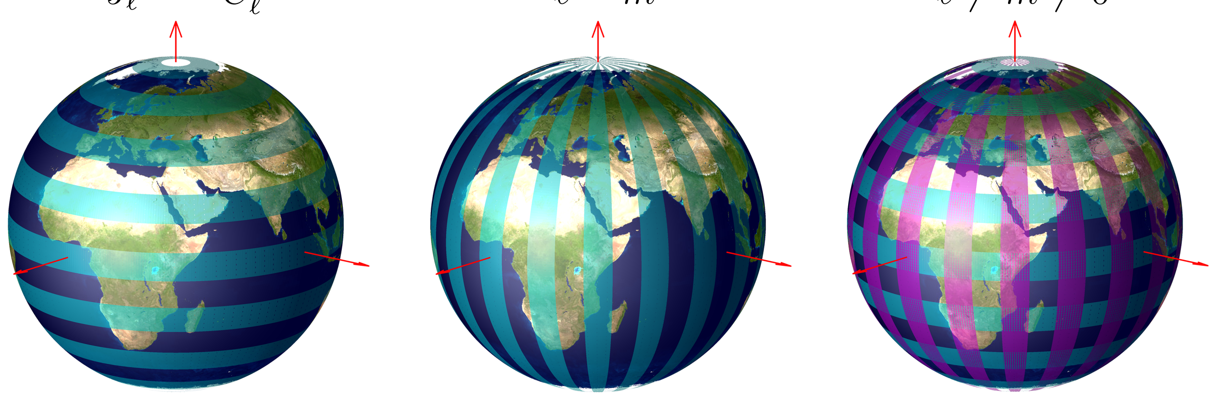
Tabulated Coefficients

Measured Body Shape

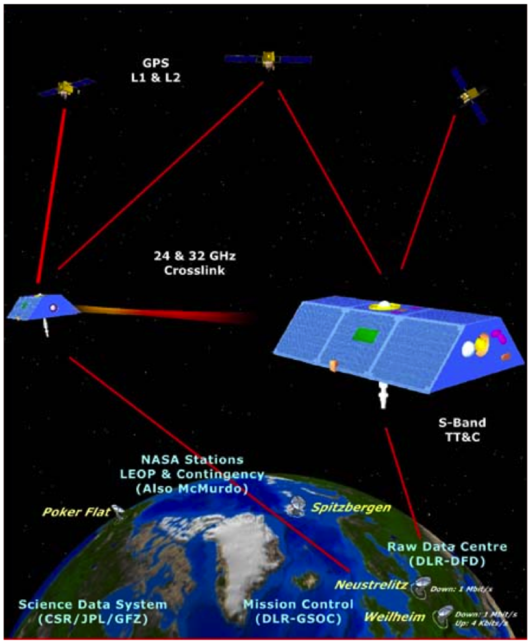
$$U(r,\theta,\phi) = \frac{\mu}{r}\left[1 - \sum_{\ell=2}^{\infty}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} J_\ell P_\ell(\cos\phi) + \sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} P_\ell^m(\cos\phi) \times (C_\ell^m \cos(m\theta) + S_\ell^m \sin(m\theta))\right]$$

ℓ = degree, m = order

Zonal Harmonics	**Sectoral Harmonics**	**Tesseral Harmonics**
Bands of Latitude	Bands of Longitude	Tiles of lat/lon
$J_\ell = -C_\ell^0$	$\ell = m$	$\ell \neq m \neq 0$



State of the Art: GRACE



- Launched in 2002 with original 5 year mission (decommissioned in 2017)
- Followup (GRACE-FO) launched in 2018
- Provides monthly gravity anomaly mapping (degree 60-90)
- Static geopotential maps available from:
  - International Centre for Global Earth Models (ICGEM)
  - National Geospatial-Intelligence Agency (NGA)
- Current Standard is Earth Gravitational Model 2008 (EGM2008)

  http://earth-info.nga.mil/GandG/update/index.php?action=home
- See also: http://icgem.gfz-potsdam.de/tom_longtime

Normalizations – Be Careful!

Description of Files Related to Using the EGM2008 Global Gravitational Model to Compute Geoid Undulations with Respect to WGS 84

(1) EGM2008_to2190_TideFree.gz

This file contains the fully-normalized, unit-less, spherical harmonic coefficients of the Earth's gravitational potential $\{\overline{C}_{nm}, \overline{S}_{nm}\}$ and their associated (calibrated) error standard deviations $\{sigma\overline{C}_{nm}, sigma\overline{S}_{nm}\}$, as implied by the EGM2008 model. The $\{\overline{C}_{nm}, \overline{S}_{nm}\}$ coefficients are consistent with the expression:

$$V(r,\theta,\lambda) = \frac{GM}{r}\left[1 + \sum_{n=2}^{N\max}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}\left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\right)\overline{P}_{nm}(\cos\theta)\right] \quad (1)$$

From: README_WGS84_2.pdf

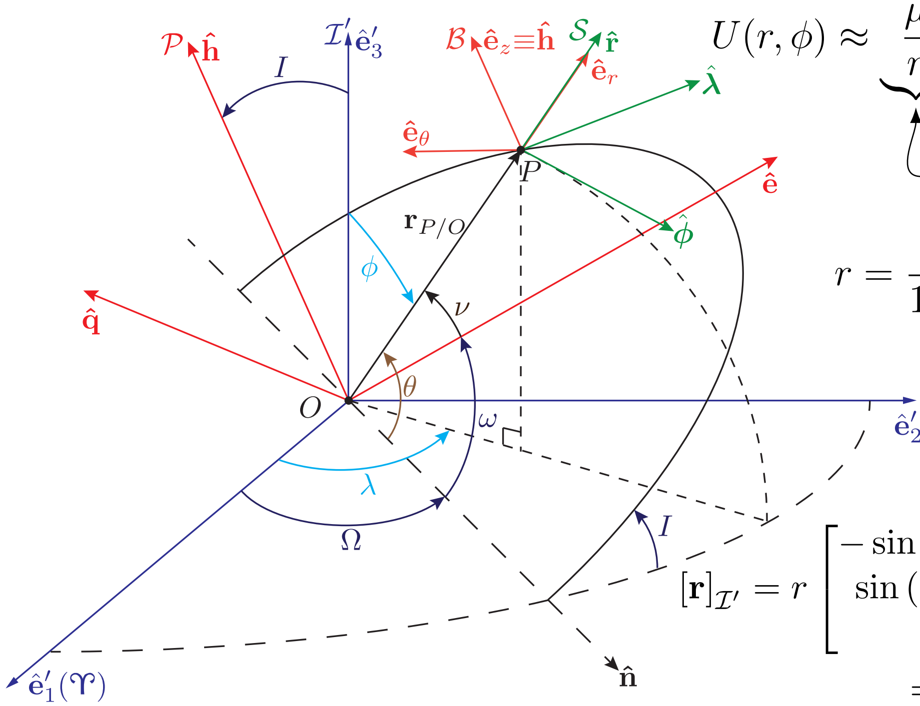
$$C_\ell^m = \left[\frac{(\ell - m)!(2\ell + 1)(2 - \delta_{0m})}{(l+m)!}\right]^{\frac{1}{2}} \overline{C}_\ell^m$$

For example:

$$\overline{C}_2^0 = -4.841651437908150 \times 10^{-4}$$

$$C_2^0 = \left[\frac{(2-0)!(2\times 2+1)(2-\delta_{00})}{(2+0)!}\right]^{\frac{1}{2}} \overline{C}_2^0 = -0.001082626173852 = -J_2$$

Oblateness (J_2) Perturbation Setup



$$U(r,\phi) \approx \underbrace{\frac{\mu}{r}}_{} \underbrace{- \frac{\mu}{r} J_2 \left(\frac{R_{\mathscr{B}}}{r}\right)^2 \left(\frac{3\cos^2\phi - 1}{2}\right)}_{\text{Perturbing Potential}}$$

Central Body Potential

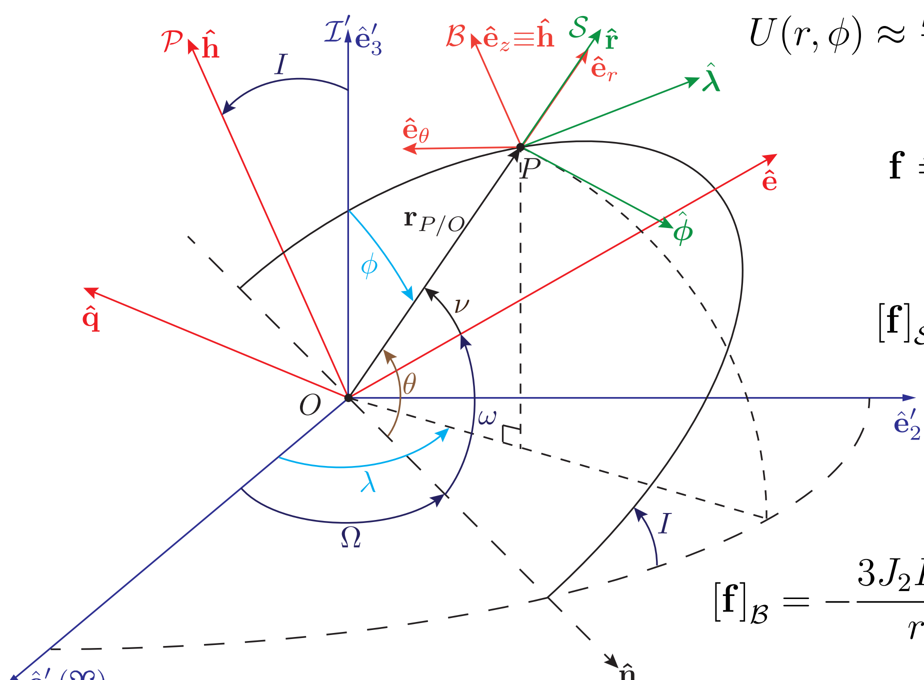
$$r = \frac{a(1-e^2)}{1 + e\cos\nu} \qquad \theta \equiv \omega + \nu$$

$$[\mathbf{r}]_{\mathcal{I}'} = r \begin{bmatrix} \sin(\phi)\cos(\lambda) \\ \sin(\lambda)\sin(\phi) \\ \cos(\phi) \end{bmatrix}_{\mathcal{I}'}$$

$$[\mathbf{r}]_{\mathcal{I}'} = r \begin{bmatrix} -\sin(\Omega)\sin(\theta)\cos(I) + \cos(\Omega)\cos(\theta) \\ \sin(\Omega)\cos(\theta) + \sin(\theta)\cos(I)\cos(\Omega) \\ \sin(I)\sin(\theta) \end{bmatrix}_{\mathcal{I}'}$$

$$\implies \cos\phi = \sin I \sin\theta$$

Oblateness (J_2) Perturbation Analysis (Gauss)



$$U(r,\phi) \approx \frac{\mu}{r} \underbrace{- \frac{\mu}{r} J_2 \left(\frac{R_{\mathscr{B}}}{r}\right)^2 \left(\frac{3\cos^2\phi - 1}{2}\right)}$$

$$\mathbf{f} \triangleq -\boldsymbol{\nabla}\left(\frac{\mu J_2 R_{\mathscr{B}}^2}{r^3}\left(\frac{3\cos^2\phi - 1}{2}\right)\right)$$

$$[\mathbf{f}]_{\mathcal{S}} = \frac{3 J_2 R_{\mathscr{B}}^2 \mu}{r^4} \begin{bmatrix} \frac{1}{2}\left(3\cos^2(\phi) - 1\right) \\ \sin(\phi)\cos(\phi) \\ 0 \end{bmatrix}_{\mathcal{S}}$$

$$\mathcal{S} = \left(P, \hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\lambda}}\right)$$

$$[\mathbf{f}]_{\mathcal{B}} = -\frac{3 J_2 R_{\mathscr{B}}^2 \mu}{r^4} \begin{bmatrix} \frac{1}{2}\left(1 - 3\sin^2(I)\sin^2(\theta)\right) \\ -\sin^2(I)\sin(\theta)\cos(\theta) \\ \sin(I)\cos(I)\sin(\theta) \end{bmatrix}_{\mathcal{B}}$$

Nodal Regression

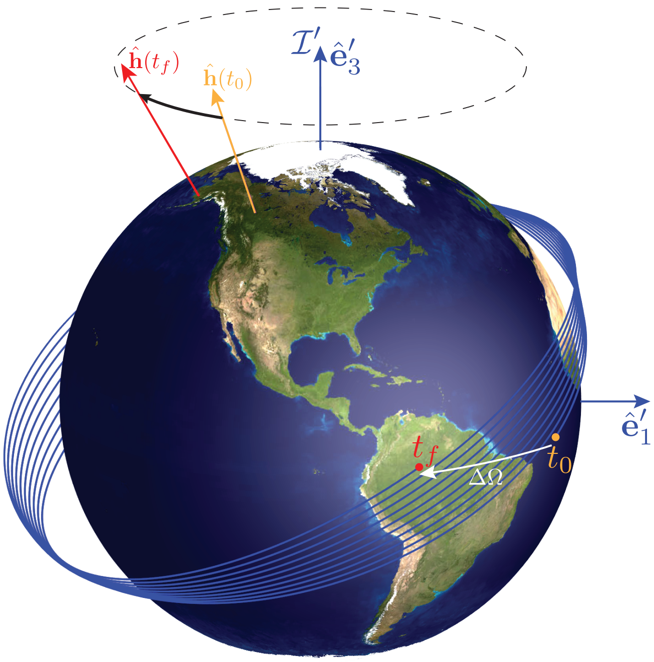


Figure based on Vallado (2013)

$$\dot{\Omega} = \frac{f_h r \sin(\theta)}{h \sin(I)} = -\frac{3\mu J_2 R_{\mathscr{B}}^2}{r^3 h} \cos(I) \sin^2(\theta)$$

$$\frac{h}{r^2} = \dot{\theta} + \dot{\Omega}\cos I$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\theta} = \frac{r^2}{h}\dot{\Omega}\left[1 - \frac{r^2}{h}\dot{\Omega}\cos(I)\right]^{-1}$$

$$\approx -3J_2\left(\frac{R_{\mathscr{B}}}{\ell}\right)^2 \cos(I)\sin^2(\theta)\left(1 + e\cos(\theta - \omega)\right)$$

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\theta}\right)_{\mathrm{av}} \triangleq \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\theta}\right)\mathrm{d}\theta$$

$$\boxed{\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\theta}\right)_{\mathrm{av}} = -\frac{3}{2}J_2\left(\frac{R_{\mathscr{B}}}{\ell}\right)^2\cos(I)}$$

Nodal Regression (Lagrange's Equations)

$$\left.\begin{aligned} \dot{\Omega} &= \frac{1}{\sqrt{a\mu\left(1-e^2\right)}\sin(I)}\frac{\partial U^{(1)}}{\partial I} \\ U^{(1)} &= -\frac{J_2\mu R_{\mathscr{B}}^2}{r^3}\left(\frac{3\cos^2\phi - 1}{2}\right) \\ \cos\phi &= \sin I \sin\theta \end{aligned}\right\} \dot{\Omega} = -\frac{3\mu J_2 R_{\mathscr{B}}^2}{r^3 \underbrace{\sqrt{a\mu\left(1-e^2\right)}}_{\equiv h}}\cos(I)\sin^2(\theta)$$

Apsidal Rotation

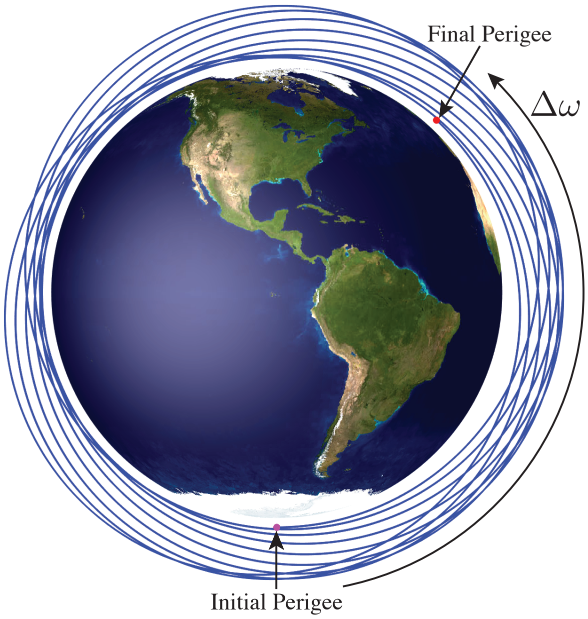


Figure based on Vallado (2013)

$$\dot{\omega} = \frac{1}{\sqrt{a\mu}}\left(\frac{\sqrt{1-e^2}}{e}\frac{\partial U^{(1)}}{\partial e} - \frac{\cot(I)}{\sqrt{1-e^2}}\frac{\partial U^{(1)}}{\partial I}\right)$$

$$\underbrace{U^{(1)} = -\frac{J_2 \mu R_{\mathscr{B}}^2}{r^3}\left(\frac{3\cos^2\phi - 1}{2}\right)}$$

$$\dot{\omega} = \frac{3\mu J_2 R_{\mathscr{B}}^2}{r^3 h}\cos^2(I)\sin^2(\theta)$$

$$\boxed{\left(\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\right)_{\mathrm{av}} = \frac{3}{2}J_2\left(\frac{R_{\mathscr{B}}}{\ell}\right)^2\left(2 - \frac{5}{2}\sin^2(I)\right)}$$

Averaging

$$U^{(1)} = -\frac{J_2 \mu R_{\mathscr{B}}^2}{r^3}\left(\frac{3\cos^2\phi - 1}{2}\right) = -\frac{J_2 \mu R_{\mathscr{B}}^2}{2r^3}\left(3\sin^2 I \sin^2\theta - 1\right)$$

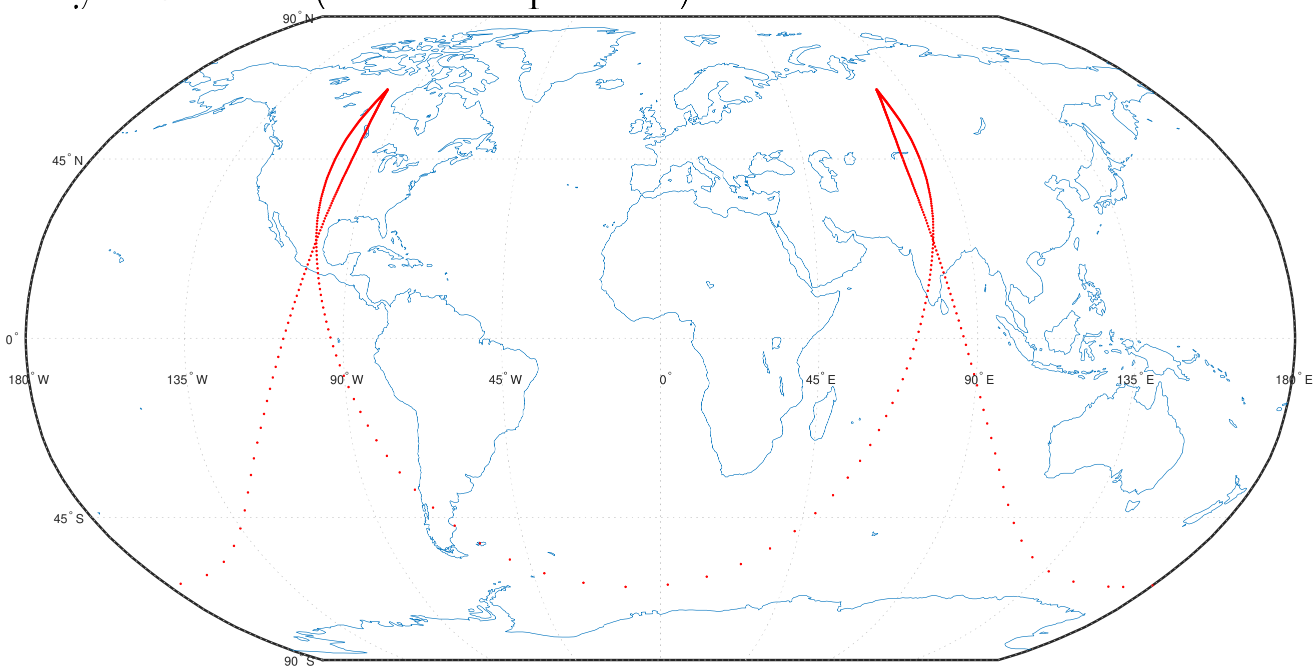
$$U^{(1)}_{\mathrm{sec}} = \frac{1}{2\pi}\int_0^{2\pi} U^{(1)}\,\mathrm{d}M$$

$$\overline{r^{-3}} = \frac{1}{2\pi}\int_0^{2\pi} r^{-3}\,\mathrm{d}M = \frac{1}{2\pi}\int_0^{2\pi} r^{-3}\frac{\mathrm{d}M}{\mathrm{d}t}\underbrace{\frac{\mathrm{d}t}{\mathrm{d}\nu}}_{r^2/h}\mathrm{d}\nu = a^{-3}\left(1-e^2\right)^{-3/2}$$

$$\overline{r^{-3}\sin^2\theta} = \frac{1}{2}a^{-3}\left(1-e^2\right)^{-3/2}$$

$$\boxed{U^{(1)}_{\mathrm{sec}} = -\frac{J_2 \mu R_{\mathscr{B}}^2}{2a^3\left(1-e^2\right)^{3/2}}\left(\frac{3}{2}\sin^2 I - 1\right)}$$

Molniya Orbits (12 hour period)



Points are spaced equally in time, so that the spacecraft spends most of its time (at apogee) over North American and Asia.

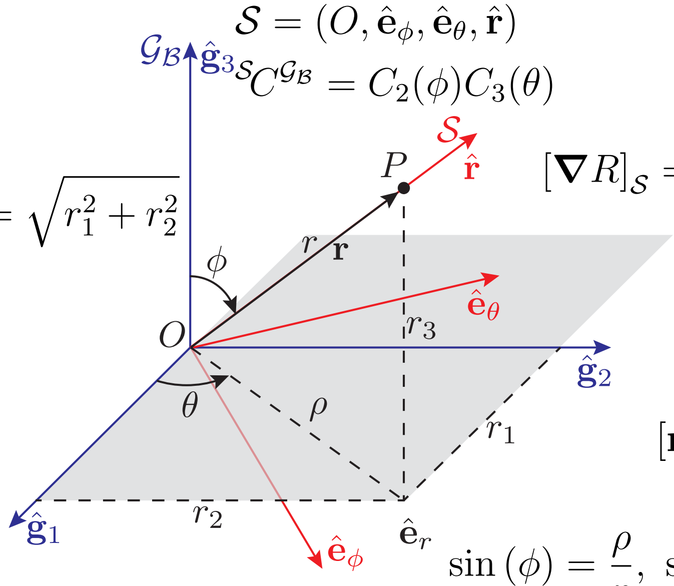
Central Body Shape Perturbing Forces

Perturbing Acceleration $\mathbf{f} = \boldsymbol{\nabla} R = \boldsymbol{\nabla}\left(\frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} P_\ell^m(\cos\phi)\left[C_\ell^m\cos(m\theta) + S_\ell^m\sin(m\theta)\right]\right)$

$$\mathcal{S} = (O, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta, \hat{\mathbf{r}})$$

$$^{\mathcal{S}}C^{\mathcal{G}_\mathcal{B}} = C_2(\phi)C_3(\theta)$$

$$\rho = \sqrt{r_1^2 + r_2^2}$$



$$[\boldsymbol{\nabla} R]_\mathcal{S} = \begin{bmatrix} \frac{1}{r}\frac{\partial R}{\partial \phi} \\ \frac{1}{r\sin\phi}\frac{\partial R}{\partial \theta} \\ \frac{\partial R}{\partial r}\end{bmatrix}_\mathcal{S} \qquad [\boldsymbol{\nabla} R]_{\mathcal{G}_\mathcal{B}} = {}^{\mathcal{G}_\mathcal{B}}C^{\mathcal{S}}\,[\boldsymbol{\nabla} R]_\mathcal{S}$$

$$[\mathbf{r}]_{\mathcal{G}_\mathcal{B}} \triangleq \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}_{\mathcal{G}_\mathcal{B}} = {}^{\mathcal{G}_\mathcal{B}}C^{\mathcal{S}} \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}_\mathcal{S} \implies$$

$$\sin(\phi) = \frac{\rho}{r},\ \sin(\theta) = \frac{r_2}{\rho},\ \cos(\phi) = \frac{r_3}{r},\ \cos(\theta) = \frac{r_1}{\rho}$$

Central Body Shape Perturbing Forces (continued)

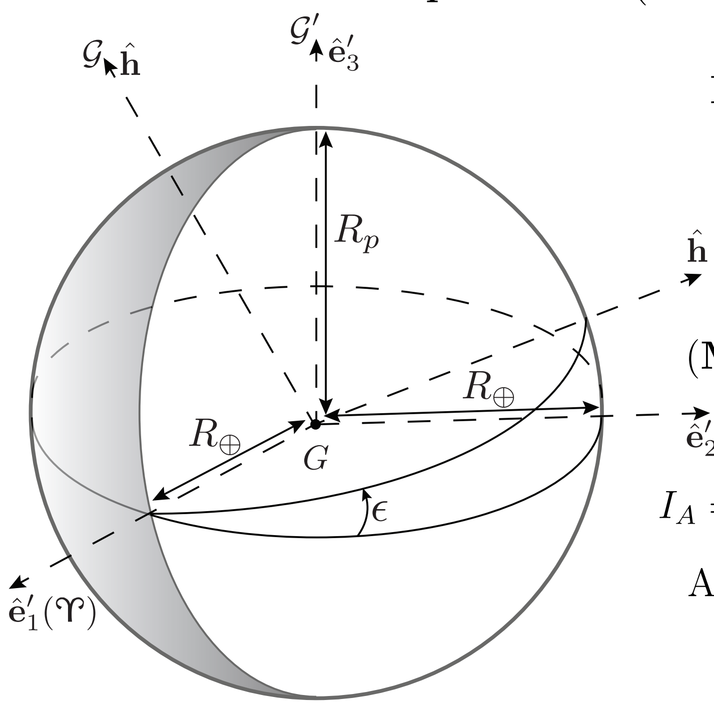
$$[\mathbf{f}]_{\mathcal{G}_\mathcal{B}} = \begin{bmatrix} \frac{r_1}{r}\left(\frac{\partial R}{\partial r} + \frac{r_3}{r\rho}\frac{\partial R}{\partial \phi}\right) - \frac{r_2}{\rho^2}\frac{\partial R}{\partial \theta} \\ \frac{r_1}{\rho^2}\frac{\partial R}{\partial \theta} + \frac{r_2}{r}\left(\frac{\partial R}{\partial r} + \frac{r_3}{r\rho}\frac{\partial R}{\partial \phi}\right) \\ \frac{r_3}{r}\frac{\partial R}{\partial r} - \frac{\rho}{r^2}\frac{\partial R}{\partial \phi} \end{bmatrix}_{\mathcal{G}_\mathcal{B}}$$

$$\frac{\partial R}{\partial r} = -\frac{\mu}{r^2}\sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell}(\ell+1)\,P_\ell^m(\cos\phi)\left[C_\ell^m\cos(m\theta) + S_\ell^m\sin(m\theta)\right]$$

$$\frac{\partial R}{\partial \theta} = \frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} mP_\ell^m(\cos\phi)\left[-C_\ell^m\sin(m\theta) + S_\ell^m\cos(m\theta)\right]$$

$$\frac{\partial R}{\partial \phi} = \frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell}\left(\ell\cos(\phi)P_\ell^m(\cos\phi) - (\ell+m)P_{\ell-1}^m(\cos\phi)\right)\left[\frac{C_\ell^m\cos(m\theta) + S_\ell^m\sin(m\theta)}{\sin(\phi)}\right]$$

Precession of the Equinoxes (Axial Precession)



$$\mathbf{F}_{\oplus/\odot} = -m_\odot\nabla U_\oplus(\mathbf{r}_{\odot/\oplus})$$

$$\mathbf{M}_{\oplus/\odot} = \mathbf{r}_{\odot/\oplus} \times \left(-m_\odot\nabla U_\oplus(\mathbf{r}_{\odot/\oplus})\right)$$

$$\mathbf{M}_{J_2} = \frac{m_\odot\mu_\oplus J_2 R_\oplus^2}{\|\mathbf{r}_{\odot/\oplus}\|^3} 3\sin\epsilon\sin l_\odot\left(\hat{\mathbf{r}}_{\odot/\oplus} \times \hat{\mathbf{e}}_3'\right)$$

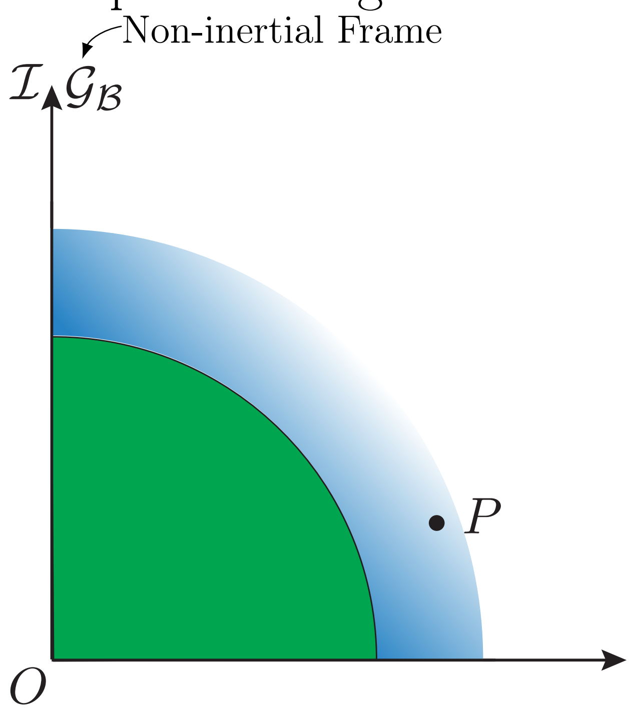
$$(\mathbf{M}_{J_2})_{\text{av}} = \frac{3}{2}\frac{Gm_\odot m_\oplus J_2 R_\oplus^2}{\|\mathbf{r}_{\odot/\oplus}\|^3}\underbrace{\sin\epsilon\left(\hat{\mathbf{h}} \times \hat{\mathbf{e}}_1'\right) \times \hat{\mathbf{e}}_3'}_{\equiv -\cos\epsilon\hat{\mathbf{h}} \times \hat{\mathbf{e}}_3'}$$

$$I_A \triangleq \hat{\mathbf{e}}_3' \cdot \mathbb{I} \cdot \hat{\mathbf{e}}_3' \qquad I_T \triangleq \hat{\mathbf{e}}_1' \cdot \mathbb{I} \cdot \hat{\mathbf{e}}_1' = \hat{\mathbf{e}}_2' \cdot \mathbb{I} \cdot \hat{\mathbf{e}}_2'$$

Assuming axial symmetry: $J_2 \approx \dfrac{I_A - I_T}{m_\oplus R_\oplus^2}$

$$\dot{\psi} = -\frac{3}{2}\frac{Gm_\odot(I_A - I_T)}{\|\mathbf{r}_{\odot/\oplus}\|^3 I_A \omega_\oplus}\cos\epsilon$$

Atmospheric Drag



Cross-sectional Area — Relative Velocity

$$\mathbf{F}_{\text{drag}} = -\frac{1}{2}\, C_D\, A\, \rho\, v_{\text{rel}}^2\, \hat{\mathbf{v}}_{\text{rel}}$$

Drag Coefficient — Atmospheric Density

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{I}}\mathbf{v}_{P/O} - \underbrace{{}^{\mathcal{I}}\mathbf{v}_{\text{atm}/O}}_{\cancel{{}^{\mathcal{G}_\mathcal{B}}\mathbf{v}_{\text{atm}/O}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{G}_\mathcal{B}} \times \mathbf{r}_{P/O}}$$

$$\boxed{\mathbf{v}_{\text{rel}} \approx {}^{\mathcal{I}}\mathbf{v}_{P/O} - {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{G}_\mathcal{B}} \times \mathbf{r}_{P/O}}$$

$$\textbf{Ballistic Coefficient} \triangleq \frac{m}{C_D A}$$

Secular Perturbations Due to Atmospheric Drag

Planet Rotation Rate

$$Q \triangleq \left(1 - \frac{\omega_r (1-e)^{3/2}}{n\sqrt{1+e}} \cos(I)\right)$$

$$\Delta a_{\text{rev}} \approx -2\pi \frac{Q^2 A C_D}{m} a^2 \rho_p \left(I_0 + 2eI_1 + \frac{3e^2}{4}(I_0 + I_2) + \frac{e^3}{4}(3I_1 + I_3)\right) \exp\left(\frac{-ae}{H}\right)$$

Spacecraft Mass — Density at Periapsis — Atmospheric Scale Height

$$\Delta e_{\text{rev}} \approx -2\pi \frac{Q^2 A C_D}{m} a \rho_p \left(I_1 + \frac{e}{2}(I_0 + I_2) - \frac{e^2}{8}(5I_1 - I_3) + \frac{e^3}{16}(5I_0 + 4I_2 - I_4)\right) \exp\left(\frac{-ae}{H}\right)$$

Here, $I_{0\ldots4}$ are modified Bessel functions of the first kind with argument $z = \frac{ae}{H}$:

$$I_s(z) = \frac{1}{\pi}\int_0^\pi \exp(z\cos\theta)\cos(s\theta)\,\mathrm{d}\theta$$

Secular Perturbations Due to Atmospheric Drag (2)

$$\Delta I_{\text{rev}} \approx -\pi \frac{QAC_D}{2nm} \omega_r a \rho_p \sin(I) \left(I_0 - 2eI_1 + (I_2 - 2eI_1)\cos(2\omega)\right) \exp\left(\frac{-ae}{H}\right)$$

$$\Delta \Omega_{\text{rev}} \approx -\pi \frac{QAC_D}{2nm} \omega_r a \rho_p \left(I_2 - 2eI_1\right) \sin(2\omega) \exp\left(\frac{-ae}{H}\right)$$

$$\Delta \omega_{\text{rev}} \approx -\Delta \Omega_{\text{rev}} \cos(I)$$

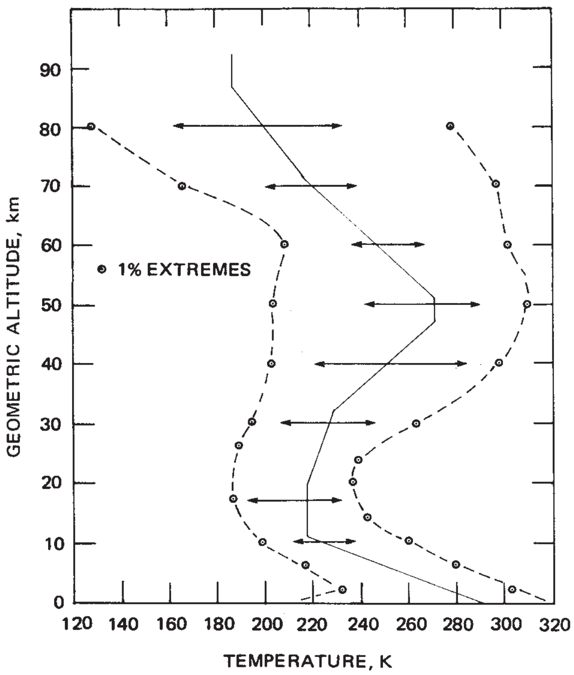
Again, here I_i are the Modified Bessel Functions of the First Kind with argument $\frac{ae}{H}$

The primary effects of drag are to circularize and shrink the orbit.

Atmosphere Variation

- The density of the Earth's upper atmosphere constantly fluctuates. Two major effects are:
  - **Incident Solar Flux** - heating from extreme ultraviolet radiation (EUV) has a near instantaneous effect
  - **Geomagnetic Interactions** - collisions with charged energetic particles cause a delayed heating effect
- Atmospheric fluctuations vary both spatially and temporally, with both random and cylcic behavior. Important cycles include:
  - **Diurnal Variations** - the atmosphere bulges in a direction lagging the direction of the sun (around 2:00 PM local time)
  - **Solar Rotation Cycle** - the same solar active regions come into view approximately every 27 days
  - **Solar Magnetic Activity Cycle** - the sun cycles in activity over a period of 11 years, as measured by the number of observed sun spots

Atmosphere Temperature Variation

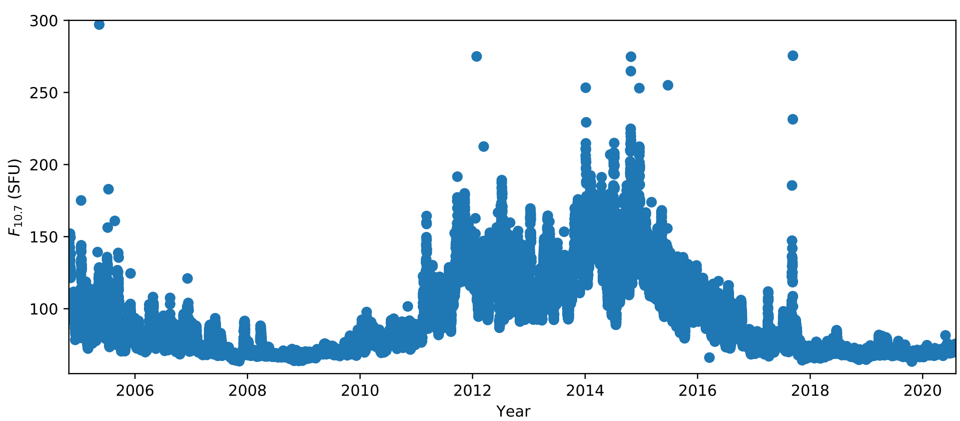


- Range of systematic variability of temperature around U.S. Standard Atmosphere, 1976.
- Arrows indicate min/max monthly measured temperatures.
- Dots are estimates of 1% min/max global temperatures.

From: U.S. Standard Atmosphere, 1976.
https://ntrs.nasa.gov/search.jsp?R=19770009539

Measuring Solar Activity

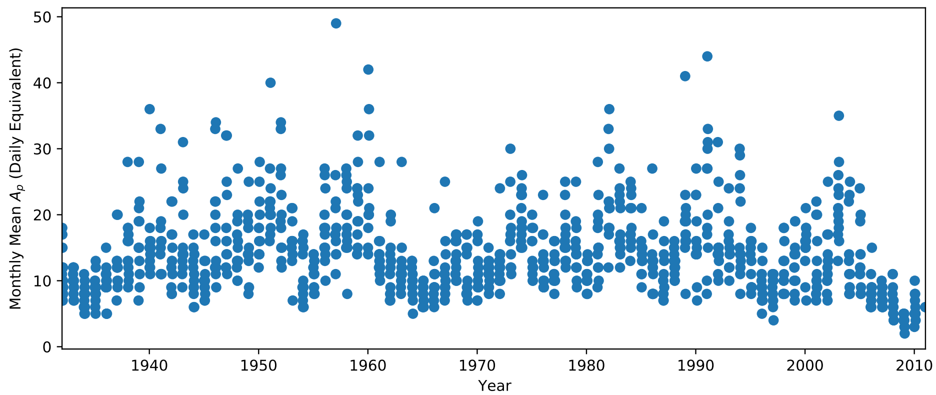
- The atmosphere absorbs all UV radiation, so we cannot directly measure EUV flux from the ground but both EUV and radiation with a wavelength of 10.7 cm (2800 MHz) originate in the same layers of the sun.
- $F_{10.7}$ is used as a proxy for EUV, and has been measured since 1940
- Define one **Solar Flux Unit** (SFU) as 1×10^{-22} watt m^{-2} Hz^{-1}



Data from: https://www.spaceweather.gc.ca/solarflux/sx-en.php

Measuring Geomagnetic Activity

- The Earth's magnetic field also varies temporally and spatially and is typically fit with a spherical harmonic model (same as the geopotential)
- Define a geomagnetic planetary index K_p to measure worldwide geomagnetic activity. Can also use the daily planetary amplitude A_p. Both are in units of gamma $= 10^{-9}$ Tesla $= 10^{-9}$ kg s m^{-1}.



Data from: https://www.ngdc.noaa.gov/stp/GEOMAG/kp_ap.html

The Exponential Atmosphere

Assuming a static atmosphere where density decays exponentially with altitude:

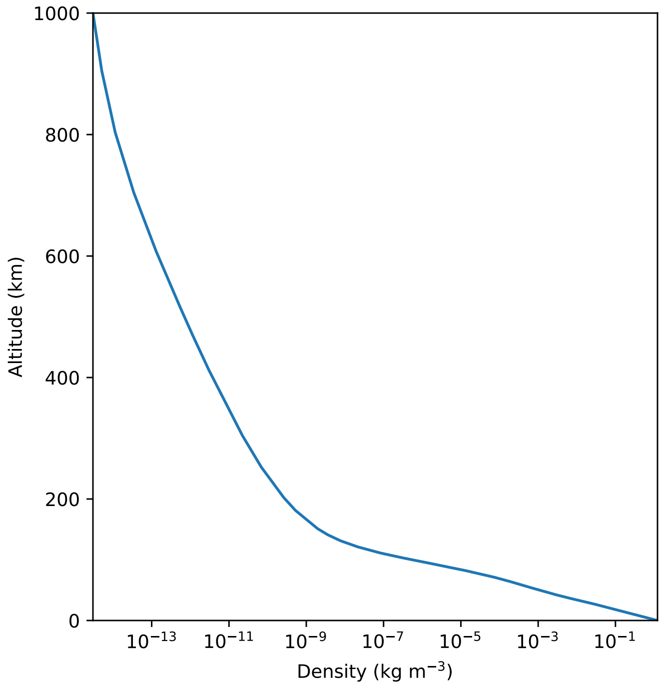
$$\rho = \rho_0 \exp\left(-\frac{h - h_0}{H}\right)$$

- ρ_0, h_0 are reference density and reference altitude (tabulated)
- h is the altitude above the ellipsoid
- H is the **Scale Height** - the fractional change in density with height. H is the increase in altitude required for ρ to drop to $^1/_e$ of its initial value:

$$H = \frac{R^\star T}{M g_0}$$

- $R^\star$ is the ideal gas constant: 8.31446261815324 J K^{-1} mol^{-1}
- T and M are the temperature and mean molecular weight of the atmosphere
- g_0 is the gravitational acceleration at the surface (9.80665 m s^{-2} on Earth)

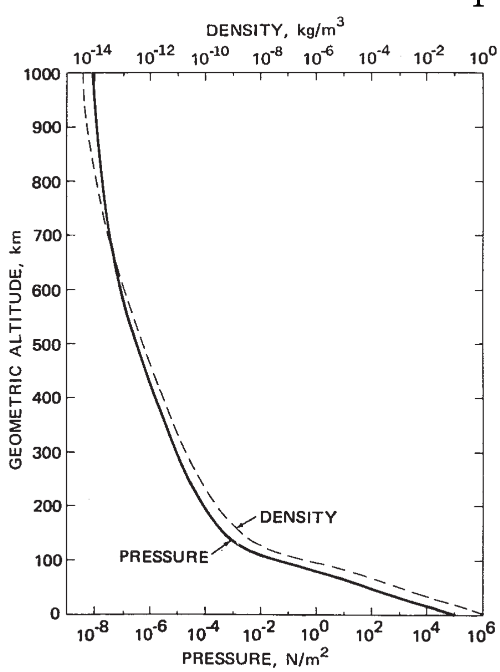
The Exponential Atmosphere (2)

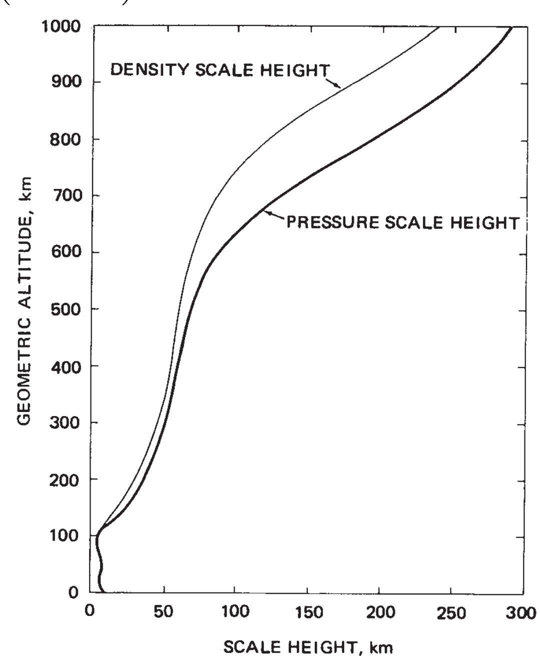


h (km)	h_0 (km)	ρ_0 (kg m^{-3})	H (km)
0-25	0	1.225	7.249
25-30	25	3.899e-2	6.349
30-40	30	1.774e-2	6.682
40-50	40	3.972e-3	7.554
50-60	50	1.057e-3	8.382
60-70	60	3.206e-4	7.714
70-80	70	8.770e-5	6.549
80-90	80	1.905e-5	5.799
90-100	90	3.396e-6	5.382
100-110	100	5.297e-7	5.877
110-120	110	9.661e-8	7.263
120-130	120	2.438e-8	9.473
130-140	130	8.484e-9	12.636
140-150	140	3.845e-9	16.149
150-180	150	2.070e-9	22.523
180-200	180	5.464e-10	29.740
200-250	200	2.789e-10	37.105
250-300	250	7.248e-11	45.546
300-350	300	2.418e-11	53.628
350-400	350	9.518e-12	53.298
400-450	400	3.725e-12	58.515
450-500	450	1.585e-12	60.828
500-600	500	6.967e-13	63.822
600-700	600	1.454e-13	71.835
700-800	700	3.614e-14	88.667
800-900	800	1.170e-14	124.64
900-1000	900	5.245e-15	181.05
1000-	1000	3.019e-15	268.00

Data from Wertz (1978)

The U.S. Standard Atmosphere (1976)





From: https://ntrs.nasa.gov/search.jsp?R=19770009539

Other Important Effects

- Radiation Pressure
- Poynting-Robertson Drag
- Yarkovsky-O'Keefe-Radzievskii-Paddack (YORP)