11 - Numerical Methods

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MAE 6720/ASTRO 6579, Spring 2022

Numerical Methods

While we have assembled many powerful tools for the analysis of orbital systems, at the end of the day, the only way of taking into account all of the effects that might be affecting the orbit of a spacecraft or natural body is via numerical integration. Similarly, if we wish to move beyond the impulsive and patched conic models for orbital design, we must turn to numerical optimization methods. Because of the importance of these topics, we must understand, at least at some basic level, what our computational tools are doing under the hood. We will review basic techniques in numerical integration, and then consider special classes of integrators for Hamiltonian systems. Next, we will consider the field of numerical optimization, and specific techniques for trajectory design.

Numerical Integration for Initial Value Problems

$$\begin{aligned}\text{IVP: For } \dot{\mathbf{x}} = f(\mathbf{x}, t); \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n; \quad f : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n \\ \text{Find } \mathbf{x}(t); \quad t \in [t_0, \pm\infty)\end{aligned}$$

A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is **Lipschitz continuous** if $\exists$ a real, positive constant K s.t. $|f(x_1) - f(x_2)| \leq K|x_1 - x_2| \; \forall x_1, x_2$

Picard-Lindelöf (Cauchy-Lipshitz) Theorem

If f is Lipshitz continuous in $\mathbf{x}$ and continuous in t, $\exists!$ solution $\mathbf{x}(t)$ for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ for some $\epsilon > 0$.

$$\mathbf{x}(t) - \mathbf{x}(t_0) = \int_{t_0}^{t} f(\mathbf{x}(s), s)\,\mathrm{d}s \quad \Longrightarrow \quad \underbrace{\phi_{k+1}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} f(\phi_k(s), s)\,\mathrm{d}s}_{\text{Picard Iteration}}$$

Numerical Propagation

$$\phi(t_{k+1}) = \phi(t_k) + \int_{t_k}^{t_{k+1}} f(\phi(s), s)\,\mathrm{d}s = \phi(t_k) + \underbrace{(t_{k+1} - t_k)}_{\triangleq \Delta t} \frac{\partial \phi}{\partial t}(t_k) + \frac{\Delta t^2}{2} \frac{\partial^2 \phi}{\partial t^2}(t_k) + \mathcal{O}(\Delta t^3)$$

$$\boxed{\mathbf{x}_{k+1} = \begin{cases} \mathbf{x}_k + \Delta t f(\mathbf{x}_k, t_k) & \text{Forward Euler Method (explicit)} \\ \mathbf{x}_k + \Delta t f(\mathbf{x}_{k+1}, t_{k+1}) & \text{Backward Euler Method (implicit)} \end{cases}}$$

$$\text{Truncation Error: } e_{k+1} \triangleq |\mathbf{x}_{k+1} - \phi(t_{k+1})|$$

For Forward Euler, the Global Truncation Error is:

$$|e_{k+1}| \leq \frac{\Delta t M}{2K} \left(\exp\left(K(t - t_0)\right) - 1\right) \qquad M \triangleq \max_t \left|\frac{\partial^2 \phi}{\partial t^2}\right|$$

Some Definitions

A differential equation is **stiff** if certain numerical methods for solving it go unstable unless you use incredibly small step sizes.

The system $\dot{x} = kx; x(0) = 1; k \in \mathbb{C}$ has solution $x(t) = \exp(kt)$, which approaches 0 in the limit $\lim_{t\to\infty} x(t) = 0$ for $\Re(k) < 0$. Any numerical method with this behavior for fixed step size is **A-Stable**. Backward Euler is A-Stable, while Forward Euler is not.

A numerical method has **order** p when the local truncation error $e_k = \mathcal{O}(\Delta t^{p+1})$ as $\Delta t \to 0$. Euler methods are order 1, Runge-Kutta are higher.

A numerical method is **consistent** if $\lim_{\Delta t \to 0} {}^{e_k}/_{\Delta t} = 0$.

Runge-Kutta

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \sum_{i=1}^{s} b_i g_i \begin{cases} g_1 &= f(\mathbf{x}_k, t_k) \\ g_2 &= f(\mathbf{x}_k + \Delta t(a_{21} g_1), t_k + c_2 \Delta t) \\ g_3 &= f(\mathbf{x}_k + \Delta t(a_{31} g_1 + a_{32} g_2), t_k + c_3 \Delta t) \\ & \dots \\ g_i &= f\left(\mathbf{x}_k + \Delta t \sum_{j=1}^{s} a_{ij} g_j, t_k + c_i \Delta t\right) \end{cases}$$

Butcher Tableau:
$$\begin{array}{c|ccccc} 0 & 1 \\ c_2 & a_{21} \\ c_3 & a_{31} & a_{32} \\ \vdots & \vdots & \vdots & \ddots \\ c_s & a_{s1} & a_{s2} & \dots & a_{s,s-1} \\ \hline & b_1 & b_2 & \dots & b_{s-1} & b_s \end{array} \implies \begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^T \end{array}$$

Consistent when
$$\sum_{j=1}^{\infty} a_{ij} = c_i \quad i = 2 \dots s.$$

Richardson Extrapolation and Burlisch-Stoer

$$e \triangleq a_0 \Delta t^{m_0} + a_1 \Delta t^{m_1} + a_2 \Delta t^{m_2} + \dots \qquad \Delta t^{m_i} > \Delta t^{m_{i+1}} \quad \forall i$$
$$e = \phi - \mathbf{x}(\Delta t) \quad \Longrightarrow \quad \phi = \mathbf{x}(\Delta t) + a_0 \Delta t^{m_0} + \mathcal{O}(\Delta t^{m_1})$$

$$\phi = \frac{h^{m_0}\mathbf{x}\left(\frac{\Delta t}{h}\right) - \mathbf{x}(\Delta t)}{h^{m_0} - 1} + \mathcal{O}(\Delta t^{m_1}) \quad \Longrightarrow \quad \mathbf{x}_{k+1}(\Delta t) = \frac{h^{m_k}\mathbf{x}_k\left(\frac{\Delta t}{h}\right) - \mathbf{x}_k(\Delta t)}{h^{m_k} - 1}$$

$$\left.\begin{aligned}
&\mathbf{x}_0 \triangleq \mathbf{x}(t); \qquad \Delta t \triangleq hn \\
&\mathbf{x}_1 = \mathbf{x}_0 + hf(\mathbf{x}_0, t) \\
&\dots \\
&\mathbf{x}_j = \mathbf{x}_{j-2} + 2hf(\mathbf{x}_{j-1}, t + (j-1)h) \qquad j > 1 \\
&\dots \\
&\mathbf{x}(t + \Delta t) \approx \tfrac{1}{2}\left(\mathbf{x}_n + \mathbf{x}_{n-1} + hf(\mathbf{x}_n, t + \Delta t)\right)
\end{aligned}\right\} e = |\mathbf{x}(t+\Delta t) - \phi(t+\Delta t)| = \sum_{i=1}^{\infty} a_i h^{2i}$$

Liouville's Theorem

For a Hamiltonian system with coordinates q_i and conjugate momenta p_i, define a **phase space distribution function** ρ such that $\rho(\mathbf{q}, \mathbf{p})$ determines the probability that the system will be found in the infinitesimal phase space volume $\mathrm{d}^n\mathbf{q}\mathrm{d}^n\mathbf{p}$. The evolution of ρ is governed by:

The Liouville Equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right) = 0$$

Liouville's Theorem

The distribution function is constant along any trajectory in phase space.

Ruth (1983) - A Canonical Integration Technique

$$\text{If: } H = T(p) + V(q) \quad \Longrightarrow \quad \begin{aligned} q(t+\Delta t) &= q + \Delta t \left(\frac{\partial T}{\partial p}\right)\bigg|_{p=p(t)} \\ p(t+\Delta t) &= p - \Delta t \left(\frac{\partial V}{\partial q}\right)\bigg|_{q=q(t+\Delta t)} \end{aligned}$$

Symplectic Mapping

$$S : \begin{bmatrix} q \\ p \end{bmatrix}_t \rightarrow \begin{bmatrix} q \\ p \end{bmatrix}_{t+\Delta t}$$

$$\text{For higher order mappings:} \quad \begin{aligned} q_0 &= q(t) \qquad & q_{i+1} &= q_i + c_i \Delta t \left(\frac{\partial T}{\partial p}\right)\bigg|_{p=p_i} \\ p_0 &= p(t) \qquad & p_{i+1} &= p - d_i \Delta t \left(\frac{\partial V}{\partial q}\right)\bigg|_{q=q_{i+1}} \end{aligned}$$

Neri (1987) - Lie Algebras and Canonical Integration

Poisson Brackets

$$\{f, g\} \triangleq \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \qquad\qquad D_G(\cdot) \triangleq \{\cdot, G\}$$

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} = \{q_i, H\} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} = \{p_i, H\} \end{aligned}\right\} \dot{\mathbf{z}} = \{\mathbf{z}, H(\mathbf{z})\} = D_H(\mathbf{z}) \Longrightarrow \mathbf{z}(t+\Delta t) = e^{\Delta t D_H}\mathbf{z}(t)$$

$$\text{If: } H = T + V \quad \Longrightarrow \quad \mathbf{z}(t+\Delta t) = e^{\Delta t(D_T + D_V)}\mathbf{z}(t)$$

$$e^{\Delta t(D_T+D_V)} = \prod_{i=1}^{k} e^{c_i \Delta t D_T} e^{d_i \Delta t D_V} + \mathcal{O}(\Delta t^{n+1})$$

Yoshida (1990) - Construction of Higher Order Symplectic Integrators

2^{nd} Order Integrator

$$S_2(\Delta t) \triangleq e^{\frac{\Delta t}{2} D_T} e^{\Delta t D_V} e^{\frac{\Delta t}{2} D_T}$$

$$S_4(\Delta t) = S_2(x_1 \Delta t) S_2(x_0 \Delta t) S_2(x_1 \Delta t) \qquad \left.\begin{aligned} x_0 + 2x_1 &= 1 \\ x_0^3 + 2x_1^3 &= 0 \end{aligned}\right\} x_0 = -\frac{2^{\frac{1}{3}}}{2 - 2^{\frac{1}{3}}},\ x_1 = \frac{1}{2 - 2^{\frac{1}{3}}}$$

$$\text{In General: } S(\Delta t) = \prod_{i=1}^{k} e^{c_i \Delta t D_T} e^{d_i \Delta t D_V} + \mathcal{O}(\Delta t^{n+1})$$

Baker–Campbell–Hausdorff (Dynkin, 1947)

$$e^Z = e^X e^Y, \quad X, Y \text{ non-commutative}$$

$$Z = \ln\left(e^X e^Y\right) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i + s_i > 0 \\ 1 \le i \le n}} \frac{[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \cdots X^{r_n} Y^{s_n}]}{\left(\sum_{i=1}^{n} (r_i + s_i)\right) \left(\prod_{i=1}^{n} r_i! s_i!\right)}$$

$$[X^{r_1} Y^{s_1} \cdots X^{r_n} Y^{s_n}] = [\underbrace{X, [X, \cdots [X,}_{r_1} \underbrace{[Y, [Y, \cdots [Y,}_{s_1} \cdots \underbrace{[X, [X, \cdots [X,}_{r_n} \underbrace{[Y, [Y, \cdots Y}_{s_n}]] \cdots]]$$

$$\text{Commutator:} \quad [X, Y] \triangleq XY - YX$$

Separable Hamiltonians

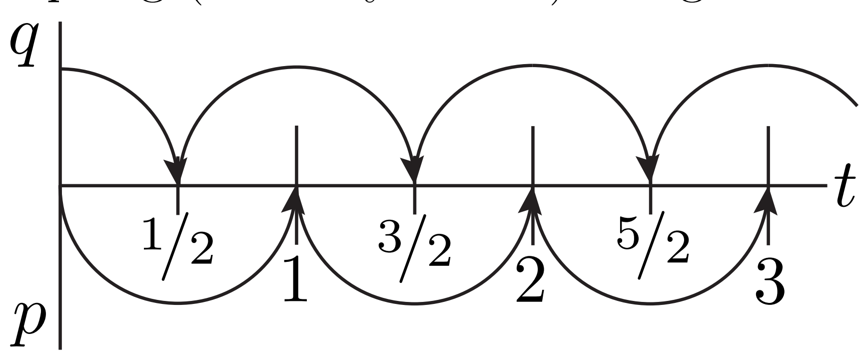
$$e^{aD} = \sum_{n=0}^{\infty} \frac{(aD)^n}{n!}$$

$$\text{If: } H = T(\mathbf{p}) + V(\mathbf{q}) \Rightarrow \quad \begin{array}{l} D_T^2 \mathbf{z} = \left\{\{\mathbf{z}, T\}, T\right\} = \{\dot{q}, T\} = 0 \\ D_V^2 \mathbf{z} = \left\{\{\mathbf{z}, V\}, V\right\} = \{-\dot{p}, V\} = 0 \end{array} \left.\vphantom{\begin{array}{l} a \\ b \end{array}}\right\} \begin{array}{l} e^{\Delta t c_i D_T} = 1 + c_i D_T \Delta t \\ e^{\Delta t d_i D_V} = 1 + d_i D_V \Delta t \end{array}$$

$$\text{Drift} \qquad e^{\Delta t c_i D_T} : \begin{bmatrix} q \\ p \end{bmatrix} \rightarrow \begin{bmatrix} q + \Delta t c_i \left.\dfrac{\partial T}{\partial p}\right|_p \\ p \end{bmatrix}$$

$$\text{Kick} \qquad e^{\Delta t d_i D_V} : \begin{bmatrix} q \\ p \end{bmatrix} \rightarrow \begin{bmatrix} q \\ p - \Delta t d_i \left.\dfrac{\partial V}{\partial p}\right|_q \end{bmatrix}$$

Leapfrog (Velocity Verlet) Integration



$$q_{k+1/2} = q_k + \frac{\Delta t}{2} \left.\frac{\partial H}{\partial p}\right|_{p_k}$$

$$p_{k+1} = p_k - \Delta t \left.\frac{\partial H}{\partial p}\right|_{q_{k+1/2}}$$

$$q_{k+3/2} = q_{k+1/2} + \Delta t \left.\frac{\partial H}{\partial p}\right|_{p_{k+1}}$$

$$\text{In general:} \quad \ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x}) \quad \text{where} \quad \mathcal{E} = \frac{1}{2}(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}) + V(\mathbf{x})$$

$$\left.\begin{array}{ll} \mathbf{x}_{i+1} & = \mathbf{x}_i + \dot{\mathbf{x}}_{i+1/2} \Delta t \\ \dot{\mathbf{x}}_{i+3/2} & = \dot{\mathbf{x}}_{i+1/2} + \underbrace{\ddot{\mathbf{x}}_{i+1}}_{\mathbf{F}(\mathbf{x}_{i+1})} \Delta t \end{array}\right\} \begin{array}{l} \mathbf{x}_{i+1} = \mathbf{x}_i + \dot{\mathbf{x}}_i \Delta t + \dfrac{\mathbf{F}(\mathbf{x}_i)}{2} \Delta t^2 \\ \dot{\mathbf{x}}_{i+1} = \dot{\mathbf{x}}_i + \dfrac{1}{2} \left(\mathbf{F}(\mathbf{x}_i) + \mathbf{F}(\mathbf{x}_{i+1})\right) \Delta t \end{array}$$

Two-Body Hamiltonian

$$H_{2-\text{body}} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_1}{2m_1} + \frac{\mathbf{p}_2 \cdot \mathbf{p}_2}{2m_2} - \frac{Gm_1 m_2}{\|\mathbf{q}_1 - \mathbf{q}_2\|} \qquad T_{2-\text{body}} = \frac{m_1}{2}(\dot{\mathbf{q}}_1 \cdot \dot{\mathbf{q}}_1) + \frac{m_2}{2}(\dot{\mathbf{q}}_2 \cdot \dot{\mathbf{q}}_2)$$

Change of coordinates to COM & relative position (orbital radius)

$$\left.\begin{aligned} \mathbf{Q}_1 &\equiv \mathbf{r}_{\text{COM}} = \frac{m_1 \mathbf{q}_1 + m_2 \mathbf{q}_2}{m_1 + m_2} \\ \mathbf{Q}_2 &\equiv \mathbf{r} = \mathbf{q}_2 - \mathbf{q}_1 \end{aligned}\right\} \mathbf{P}_i = \frac{\partial L}{\partial \dot{\mathbf{Q}}_i} \equiv \frac{\partial T}{\partial \dot{\mathbf{Q}}_i} = \begin{cases} \mathbf{P}_1 = (m_1 + m_2)\dot{\mathbf{Q}}_1 = \mathbf{p}_1 + \mathbf{p}_2 \\ \mathbf{P}_2 = \dfrac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{Q}}_2 = \frac{m_1 \mathbf{p}_2 - m_2 \mathbf{p}_1}{m_1 + m_2} \end{cases}$$

$$\boxed{H_{2-\text{body}} = \underbrace{\frac{\mathbf{P}_1 \cdot \mathbf{P}_1}{2(m_1 + m_2)}}_{H_{\text{COM}}} + \underbrace{\frac{\mathbf{P}_2 \cdot \mathbf{P}_2}{2m_1 m_2}(m_1 + m_2) - \frac{Gm_1 m_2}{\|\mathbf{Q}_2\|}}_{H_{\text{Kepler}}}}$$

NB: Cyclic in $\mathbf{Q}_1$—COM coords are ignorable and $\mathbf{P}_1$ is a constant of motion.

Jacobi Coordinates

$$H_{\text{n-body}} = \sum_{i=0}^{n-1} \frac{\mathbf{p}_i \cdot \mathbf{p}_i}{2m_i} - \sum_{i=0}^{n-1} \sum_{j>i}^{n-1} \frac{Gm_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|} = H_{\text{Kepler}} + H_{\text{interaction}}$$

$$\boxed{\begin{array}{lll} \mathbf{r}'_i \triangleq \begin{cases} \dfrac{\sum_i m_i \mathbf{q}_i}{\sum_i m_i} \equiv \mathbf{c}_{n-1} & i = 0 \\ \mathbf{q}_i - \mathbf{c}_{i-1} & i > 0 \end{cases} & \mathbf{c}_j \triangleq \dfrac{1}{\eta_j} \displaystyle\sum_{k=0}^{j} m_k \mathbf{q}_k & \eta_j \triangleq \displaystyle\sum_{k=0}^{j} m_k \\ \mathbf{p}'_i = m'_i \dot{\mathbf{r}}'_i & m'_i \triangleq \dfrac{\eta_{i-1}}{\eta_i} m_i & m'_0 \triangleq \eta_{n-1} \end{array}}$$

$$H = \underbrace{\frac{\mathbf{p}'_0 \cdot \mathbf{p}'_0}{2m'_0}}_{H_{\text{COM}}} + \underbrace{\sum_{i=1}^{n-1} \left(\frac{\mathbf{p}'_i \cdot \mathbf{p}'_i}{2m'_i} - \frac{Gm_i m_0}{\|\mathbf{r}'_i\|} \right)}_{H_{\text{Kepler}}} + \underbrace{\sum_{i=1}^{n-1} \left(\frac{Gm_i m_0}{\|\mathbf{r}'_i\|} - \frac{Gm_i m_0}{\|\mathbf{q}_i - \mathbf{q}_0\|} \right) - \sum_{i=1}^{n-1} \sum_{j>1} \frac{Gm_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|}}_{H_{\text{interaction}}}$$

Mixed Variable Symplectic Integrators

Wisdom & Holman (1991)

$$H_{\text{n}-\text{body}} = \cancel{H_{\text{COM}}} + H_{\text{Kepler}} + H_{\text{Interaction}}$$

Decoupled, Ignorable

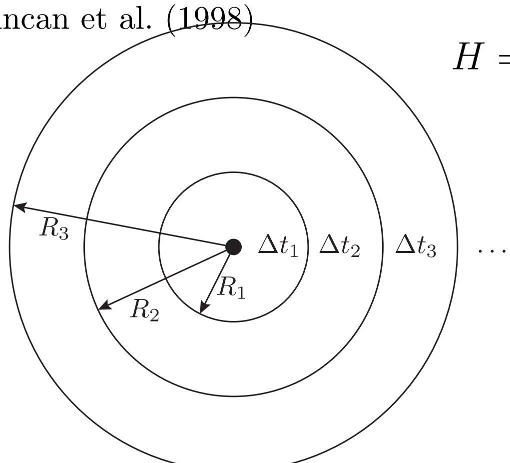
$$S_2(\Delta t) = S_{\text{Kepler}}\left(\frac{\Delta t}{2}\right) S_{\text{interaction}}\left(\Delta t\right) S_{\text{Kepler}}\left(\frac{\Delta t}{2}\right)$$

Algorithm

1. Drift all bodies on Keplerian orbits for ${}^{\Delta t}/_{2}$
2. Kick each body's momentum due to $H_{\text{interaction}}$ for Δt
3. Drift all bodies on Keplerian orbits for ${}^{\Delta t}/_{2}$

Multiple Time Step Symplectic Methods

Duncan et al. (1998)



$$H = H_T + \underbrace{H_V}_{} = \sum_k V_k \quad \Longrightarrow \quad \mathbf{F} = \sum_k \mathbf{F}_k = -\sum_k \frac{\partial V_k}{\partial \mathbf{q}}$$

1. $\mathbf{F}(\mathbf{q}) = \sum_{k=0}^{\infty} \mathbf{F}_k(\mathbf{q}) \quad \forall \mathbf{q}$
2. $\mathbf{F}_k = 0$ for $\|\mathbf{q}\| > R_k$ (except $\mathbf{F}_0$)
3. $\dfrac{\Delta t_k}{\Delta t_{k+1}} \in \mathbb{N}$ (frequently set to constant M)

$$\left.\begin{aligned} S_i\left(\Delta t\right) &\triangleq e^{\Delta t D_{V_i}} \\ S_{\Sigma_i}\left(\Delta t\right) &\triangleq e^{\Delta t D_H} \end{aligned}\right\} S_{\Sigma_0}(\Delta t_0) \approx S_0\left(\frac{\Delta t_0}{2}\right) \underbrace{S_{\Sigma_1}(\Delta t_0)} S_0\left(\frac{\Delta t_0}{2}\right)$$

$$\approx \left[S_1\left(\frac{\Delta t_1}{2}\right) \underbrace{S_{\Sigma_2}(\Delta t_1)}_{\cdots} S_1\left(\frac{\Delta t_1}{2}\right)\right]^{\frac{\Delta t_1}{\Delta t_2}}$$

Democratic Heliocentric (Mixed-Center) Coordinates

Duncan et al. (1998)

$$\text{Define new coordinates: } \mathbf{Q}_i \triangleq \begin{cases} \mathbf{q}_i - \mathbf{q}_0 & i \neq 0 \text{ (heliocentric)} \\ \dfrac{1}{\sum_i m_i} \sum_{j=0} m_j \mathbf{q}_j & i = 0 \text{ (barycentric)} \end{cases}$$

$$F_3\left(\mathbf{p}_i, \mathbf{Q}_i\right) = -\mathbf{p}_0 \cdot \left(\mathbf{Q}_0 - \frac{1}{\sum_i m_i} \sum_{j=1}^{n-1} m_j \mathbf{Q}_j \right) - \sum_{i=1}^{n-1} \left[\mathbf{p}_i \cdot \left(\mathbf{Q}_i + \mathbf{Q}_0 - \frac{1}{\sum_i m_i} \sum_{j=1}^{n-1} m_j \mathbf{Q}_j \right) \right]$$

$$\boxed{\mathbf{q}_i = -\frac{\partial F_3}{\partial \mathbf{p}_i} = \begin{cases} \mathbf{Q}_i + \mathbf{Q}_0 - \dfrac{\sum_{j=1}^{n-1} m_j \mathbf{Q}_j}{\sum_i m_i} \\ \mathbf{Q}_0 - \dfrac{1}{\sum_i m_i} \displaystyle\sum_{j=1}^{n-1} m_j \mathbf{Q}_j \end{cases} \qquad \mathbf{P}_i = -\frac{\partial F_3}{\partial \mathbf{Q}_i} = \begin{cases} \mathbf{p}_i - m_i \dfrac{\sum_{j=0}^{n-1} m_j \mathbf{p}_j}{\sum_j m_j} & i \neq 0 \\ \displaystyle\sum_{j=0}^{n-1} \mathbf{p}_j & i = 0 \end{cases}}$$

Integrating Mixed-Center Coordinates

Duncan et al. (1998)

$$H = \underbrace{\cancel{\frac{\mathbf{P}_0 \cdot \mathbf{P}_0}{2 \sum_i m_i}}}_{H_{\text{COM}}} + \underbrace{\frac{1}{2m_0} \left\| \left(\sum_{i=1}^{n-1} \mathbf{P}_i \right) \right\|^2}_{H_{\text{Central Body}}} + \underbrace{\sum_{i=1}^{n-1} \left(\frac{\mathbf{P}_i \cdot \mathbf{P}_i}{2m_i} - \frac{G m_i m_0}{\|\mathbf{Q}_i\|} \right)}_{H_{\text{Kepler}}} - \underbrace{\sum_{i=1}^{n-1} \sum_{j>1} \frac{G m_i m_j}{\|\mathbf{Q}_i - \mathbf{Q}_j\|}}_{H_{\text{interaction}}}$$

Decoupled, Ignorable

$$S_2(\Delta t) = S_{\text{CB}} \left(\frac{\Delta t}{2} \right) S_{\text{interaction}} \left(\frac{\Delta t}{2} \right) S_{\text{Kepler}} \left(\Delta t \right) S_{\text{interaction}} \left(\frac{\Delta t}{2} \right) S_{\text{CB}} \left(\frac{\Delta t}{2} \right)$$

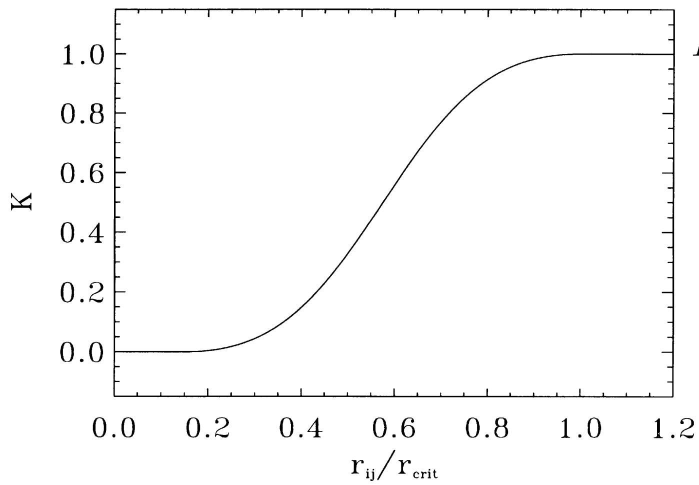
Algorithm

1. Drift all bodies ${}^{\Delta t}/_{2m_0} \sum_{i=1}^{n} \mathbf{P}_i$
2. Kick each body's momentum due to $H_{\text{interaction}}$ for ${}^{\Delta t}/_{2}$
3. Evolve each body along its Keplerian orbit for Δt
4. Repeat (2) and (1)

Hybrid Symplectic Integrator (Chambers, 1999)

$$H = H_A + H_B \qquad\qquad H_A \gg H_B$$

$$H_A = \sum_{i=1}^{n-1} \left(\frac{\mathbf{p}_i \cdot \mathbf{p}_i}{2m_i} - \frac{Gm_i m_0}{\|\mathbf{q}_i - \mathbf{q}_0\|} \right) - \sum_{i=1}^{n-1} \sum_{j>1} \frac{Gm_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|} \left(1 - K(\|\mathbf{q}_i - \mathbf{q}_j\|)\right)$$



$$H_B = -\sum_{i=1}^{n-1} \sum_{j>1} \frac{Gm_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|} K(\|\mathbf{q}_i - \mathbf{q}_j\|)$$

$$K = \begin{cases} 0 & y < 0 \\ \frac{y^2}{2y^2 - 2y + 1} & 0 < y < 1 \\ 1 & y > 1 \end{cases}$$

$$y = \frac{\|\mathbf{q}_i - \mathbf{q}_j\| - 0.1 r_{\text{crit}}}{0.9 r_{\text{crit}}}$$

Numerical Methods for Trajectory Optimization

Betts (1998)

- Split trajectory into N phases.
- Phase k has independent var t, s.t. $t_0^{(k)} \le t \le t_f^{(k)}$.
- Define state $\mathbf{z} = \begin{bmatrix} \mathbf{x}^{(k)}(t) \\ \mathbf{u}^{(k)}(t) \end{bmatrix}$ $\begin{array}{l} \text{State Variables} \\ \text{Control Inputs} \end{array}$
- $\exists$ parameters $\mathbf{p}$ that may not be t-dependent, as well as disturbances $\mathbf{w}$
- Dynamics of the system are $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}, \mathbf{w}, t)$
- Have initial and desired final conditions ψ_0, ψ_f
- Can have constraints: $\mathbf{g}_l \le \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}, t) \le \mathbf{g}_u$
- Can have state constraints: $\begin{array}{l} \mathbf{x}_l \le \mathbf{x}(t) \le \mathbf{x}_u \\ \mathbf{u}_l \le \mathbf{u}(t) \le \mathbf{u}_u \end{array}$

$$\boxed{\min_{\mathbf{u}^{(k)}, \mathbf{p}} J : \quad J = \phi\left(\left\{ \mathbf{x}(t_0^{(i)}), t_0^{(i)}, \mathbf{x}(t_f^{(i)}), t_f^{(i)}, \mathbf{p}^{(i)}, \mathbf{w}^{(i)} \right\}_{i=0}^{N} \right)}$$

Nonlinear Programming (NLP)

Step Length Search Direction

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \alpha\, \mathbf{q}$$

Recall Newton's Method: $\mathbf{x} : \mathbf{f}(\mathbf{x}) = 0;\ \mathbf{x} \in \mathbb{R}^n \quad \Longrightarrow \quad A(\mathbf{x})\mathbf{q} = -\mathbf{f}(\mathbf{x})$

$$A \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

- Step length can be used to control for divergence by selecting α s.t. $\|\mathbf{f}(\mathbf{x}_{n+1})\| \leq \|\mathbf{f}(\mathbf{x}_n)\|$
- Unconstrained optimization: find $\mathbf{x}$ that minimizes scalar function $f(\mathbf{x})$: $H(\mathbf{x})\mathbf{q} = -\nabla_{\mathbf{x}} f$
- H is the Hessian: $[H]_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$

Imposing Constraints

Equality Constraints

Find $\mathbf{x} \in \mathbb{R}^n$ to minimize $f(\mathbf{x})$ s.t. $\mathbf{c}(\mathbf{x}) = 0 \in \mathbb{R}^m$ for $m < n$

- Solved by applying Euler-Lagrange Equations to constraint function with Lagrange multipliers: $g = f + \lambda c(\mathbf{x})$.
- Define $L(\mathbf{x}, \boldsymbol{\lambda}) \triangleq f(\mathbf{x}) - \boldsymbol{\lambda}^T \mathbf{c}(\mathbf{x})$
- Find critical points: $\begin{array}{l} \nabla_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}) = \nabla_{\mathbf{x}} f - \mathbf{G}^T(\mathbf{x})\boldsymbol{\lambda} = 0 \\ \nabla_{\boldsymbol{\lambda}} L(\mathbf{x}, \boldsymbol{\lambda}) = -\mathbf{c}(\mathbf{x}) = 0 \end{array}$
- Solve by Newton's Method:

$$\boxed{\begin{bmatrix} \mathbf{H}_L & \mathbf{G}^T \\ \mathbf{G} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\nabla_{\mathbf{x}} f \\ -\mathbf{c} \end{bmatrix} \quad \text{where} \quad \mathbf{H}_L = \nabla^2_{\mathbf{x}} f - \sum_{i=1}^{m} \lambda_i \nabla^2_{\mathbf{x}} c_i}$$

Karush-Kuhn-Tucker System

Optimal Control

Problem Statement

$$\arg\min_{\mathbf{u}} J = \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u}, t)\, \mathrm{d}t$$

$$J = \begin{cases} \int_{t_0}^{t_f} 1\, \mathrm{d}t \\ \int_{t_0}^{t_f} \dot{m}\, \mathrm{d}t \qquad \dot{m} = L(\mathbf{x}(t), \mathbf{u}(t)) \\ (\mathbf{x}(t_f) - \mathbf{x}_D)^T W (\mathbf{x}(t_f) - \mathbf{x}_D) \qquad \mathbf{x}_D = \text{constant} \\ \int_{t_0}^{t_f} \mathbf{u}^T W \mathbf{u}\, \mathrm{d}t \\ (\mathbf{x}(t_f) - \mathbf{x}_D)^T W (\mathbf{x}(t_f) - \mathbf{x}_D) + \int_{t_0}^{t_f} \left(\begin{bmatrix} \mathbf{x}^T & \mathbf{u}^T \end{bmatrix} \begin{bmatrix} Q & M \\ M^T & R \end{bmatrix} \begin{bmatrix} \mathbf{x} & \mathbf{u} \end{bmatrix} \right) \mathrm{d}t \end{cases}$$

Optimal Control (continued)

The Dynamics are a Constraint

$\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{p}, \mathbf{w}, t) - \dot{\mathbf{x}} = 0$ must be continuously satisfied from t_0 to t_f

$$J_A \triangleq \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \Big(L(\mathbf{x}, \mathbf{u}, t) + \underbrace{\boldsymbol{\lambda}^T}_{\text{Lagrange Multipliers (co-states)}} (\mathbf{f} - \dot{\mathbf{x}}) \Big)\, \mathrm{d}t$$

$$H \triangleq L(\mathbf{x}, \mathbf{u}, t) + \boldsymbol{\lambda}^T \mathbf{f} \implies J_A = \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t)\, \mathrm{d}t - \underbrace{\int_{t_0}^{t_f} \boldsymbol{\lambda}^T \dot{\mathbf{x}}\, \mathrm{d}t}_{\text{integrate by parts}}$$

$$J_A = \phi(\mathbf{x}(t_f), t_f) + \big(\boldsymbol{\lambda}^T(t_0) \mathbf{x}(t_0) - \boldsymbol{\lambda}^T(t_f) \mathbf{x}(t_f) \big) + \int_{t_0}^{t_f} \Big(H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) + \dot{\boldsymbol{\lambda}}^T \mathbf{x} \Big)\, \mathrm{d}t$$

Necessary Conditions for Optimality

$$\text{Define: } \delta\cdot \triangleq \frac{\partial\cdot}{\partial\mathbf{u}}\delta\mathbf{u} + \frac{\partial\cdot}{\partial\mathbf{x}}\delta\mathbf{x}(\delta\mathbf{u})$$

Initial control has no effect on initial state

$$\delta J_A = \cancel{\left(\boldsymbol{\lambda}^T\delta\mathbf{x}(\delta\mathbf{u})\right)\Big|_{t_0}} + \left(\underbrace{\left(\frac{\partial\phi}{\partial\mathbf{x}} - \boldsymbol{\lambda}^T\right)}_{}\delta\mathbf{x}(\delta\mathbf{u})\right)\Bigg|_{t_f} + \int_{t_0}^{t_f}\left(\underbrace{\left(\frac{\partial H}{\partial\mathbf{x}} + \dot{\boldsymbol{\lambda}}^T\right)}_{}\delta\mathbf{x}(\delta\mathbf{u}) + \underbrace{\frac{\partial H}{\partial\mathbf{u}}}_{}\delta\mathbf{u}\right)\mathrm{d}t$$

$$\boxed{\boldsymbol{\lambda}(t_f) = \left(\frac{\partial\phi}{\partial\mathbf{x}}\right)^T\Bigg|_{t=t_f} \qquad \underbrace{\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial H}{\partial\mathbf{x}}\right)^T \qquad \left(\frac{\partial H}{\partial\mathbf{u}}\right)^T = 0}_{t\in[t_0,t_f]}}$$

Shooting Methods

$$\text{BVP: } \ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\dot{\mathbf{x}},t),\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\text{Corresponding IVP: } \ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\dot{\mathbf{x}},t),\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \dot{\mathbf{x}}(t_0) = \dot{\mathbf{x}}_0 \text{ has solution: } \mathbf{x}(\dot{\mathbf{x}}_0,t)$$

If $\mathbf{g}(\dot{\mathbf{x}}_0) \triangleq \mathbf{x}(\dot{\mathbf{x}}_0,t_f) - \mathbf{x}_f$ has a root of $\dot{\mathbf{x}}_0$, i.e., $\mathbf{g}(\dot{\mathbf{x}}_0) = 0$, then the solution $\mathbf{x}(\dot{\mathbf{x}}_0,t)$ is a solution of the original BVP.

Direct Shooting: Assume a functional form for control $\mathbf{u}(t) = \sum_{i=1}^{m}\mathbf{g}_i\phi_i(t)$ where ϕ_i are known functions and $\mathbf{g}_i$ are unknown parameters. Set up NLPs to solve for $\mathbf{g}_i$ such that cost is minimized.

Indirect Shooting: Assume some initial (final) guess for unknown state and co-state elements, numerically integrate EL equations forward (backward) in time, and minimize difference between final (initial) states.

Fuel-Optimal Trajectories

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \\ m \end{bmatrix} \begin{array}{l} \text{Position} \\ \text{Velocity} \\ \text{Spacecraft Mass} \end{array} \qquad \dot{\mathbf{x}} = \begin{bmatrix} \mathbf{v} \\ -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \frac{T_{\max} u}{m}\boldsymbol{\alpha} \\ -\frac{T_{\max} u}{I_{\mathrm{sp}} g_0} \end{bmatrix} \quad \mathbf{x}(t_0) = \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \\ 1 \end{bmatrix} \quad \mathbf{x}(t_f) = \begin{bmatrix} \mathbf{r}_f \\ \mathbf{v}_f \\ ? \end{bmatrix}$$

$T_{\max}$ = Max Thrust, $u \in [0, 1]$ = Thrust Ratio, $\boldsymbol{\alpha}$ = Thrust Direction

$$J \triangleq \frac{T_{\max}}{I_{\mathrm{sp}} g_0} \int_{t_0}^{t_f} u \, \mathrm{d}t \quad \Longrightarrow \quad H = \boldsymbol{\lambda}^T \begin{bmatrix} \mathbf{v} \\ -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \frac{T_{\max} u}{m}\boldsymbol{\alpha} \\ -\frac{T_{\max} u}{I_{\mathrm{sp}} g_0} \end{bmatrix} + \frac{T_{\max} u}{I_{\mathrm{sp}} g_0} \qquad \boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_r \\ \boldsymbol{\lambda}_v \\ \lambda_m \end{bmatrix}$$

Fuel-Optimal Trajectory Solution

$$\boxed{\boldsymbol{\alpha} = -\frac{\boldsymbol{\lambda}_v}{\|\boldsymbol{\lambda}_v\|} \qquad u \begin{cases} = 0 & \rho > 0 \\ = 1 & \rho < 0 \\ \in (0, 1) & \rho = 0 \end{cases} \qquad \rho = 1 - \frac{I_{\mathrm{sp}} g_0 \|\boldsymbol{\lambda}_v\|}{m} - \lambda_m}$$

$$\dot{\boldsymbol{\lambda}}(t) = \left(\frac{\partial H}{\partial \mathbf{x}}\right)^T = \begin{bmatrix} \frac{\mu}{\|\mathbf{r}\|^3}\boldsymbol{\lambda}_v - \frac{3\mu \mathbf{r} \cdot \boldsymbol{\lambda}_v}{\|\mathbf{r}\|^5}\mathbf{r} \\ -\boldsymbol{\lambda}_r \\ -\frac{T_{\max} u}{m^2}\|\boldsymbol{\lambda}_v\| \end{bmatrix}$$

$$\text{Shooting Function:} \quad \boldsymbol{\Phi} \triangleq \begin{bmatrix} \mathbf{r}(t_f) - \mathbf{r}_f \\ \mathbf{v}(t_f) - \mathbf{v}_f \\ \lambda_m(t_f) \end{bmatrix} = 0$$

Energy-Optimal Trajectories (Bertrand and Epenoy, 2002)

$$J \triangleq \lambda_0 \frac{T_{\max}}{I_{\text{sp}} g_0} \int_{t_0}^{t_f} (u - \varepsilon u(1-u))\, \mathrm{d}t \qquad \varepsilon \in [\underset{\text{Mass}}{0},\ \underset{\text{Energy}}{1}]$$

$$H = \boldsymbol{\lambda}^T \begin{bmatrix} \mathbf{v} \\ -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \frac{T_{\max} u}{m}\boldsymbol{\alpha} \\ -\frac{T_{\max} u}{I_{\text{sp}} g_0} \end{bmatrix} + \lambda_0 \frac{T_{\max}}{I_{\text{sp}} g_0}(u - \varepsilon u(1-u)) \qquad \boldsymbol{\alpha} = -\frac{\boldsymbol{\lambda}_v}{\|\boldsymbol{\lambda}_v\|}$$

$$\boxed{u = \begin{cases} 0 & \rho > \varepsilon \\ 1 & \rho < -\varepsilon \\ \frac{1}{2} - \frac{\rho}{2\varepsilon} & |\rho| \leq \varepsilon \end{cases} \qquad \rho = 1 - \frac{I_{\text{sp}} g_0 \|\boldsymbol{\lambda}_v\|}{\lambda_0 m} - \frac{\lambda_m}{\lambda_0} \qquad \boldsymbol{\Phi} \triangleq \begin{bmatrix} \mathbf{r}(t_f) - \mathbf{r}_f \\ \mathbf{v}(t_f) - \mathbf{v}_f \\ \lambda_m(t_f) \\ \|\boldsymbol{\lambda}(t_0)\| - 1 \end{bmatrix} = 0}$$

Optimization Resources

- scipy.optimize (https://docs.scipy.org/doc/scipy/reference/optimize.html)
- MATLAB Optimization Toolbox (https://www.mathworks.com/help/optim/index.html)
- EMTG (https://opensource.gsfc.nasa.gov/projects/emtg; https://github.com/nasa/EMTG)
- GMAT (https://opensource.gsfc.nasa.gov/projects/GMAT; https://sourceforge.net/projects/gmat/)
- PAGMO/PyGMO (https://github.com/esa/pagmo2)
- SNOPT (https://web.stanford.edu/group/SOL/snopt.htm)
- GUROBI (https://www.gurobi.com/products/gurobi-optimizer/)
- CPLEX (https://www.ibm.com/analytics/cplex-optimizer)
- OR-Tools (https://developers.google.com/optimization)