12 - Secular and Resonant Perturbations

Dmitry Savransky

Cornell University

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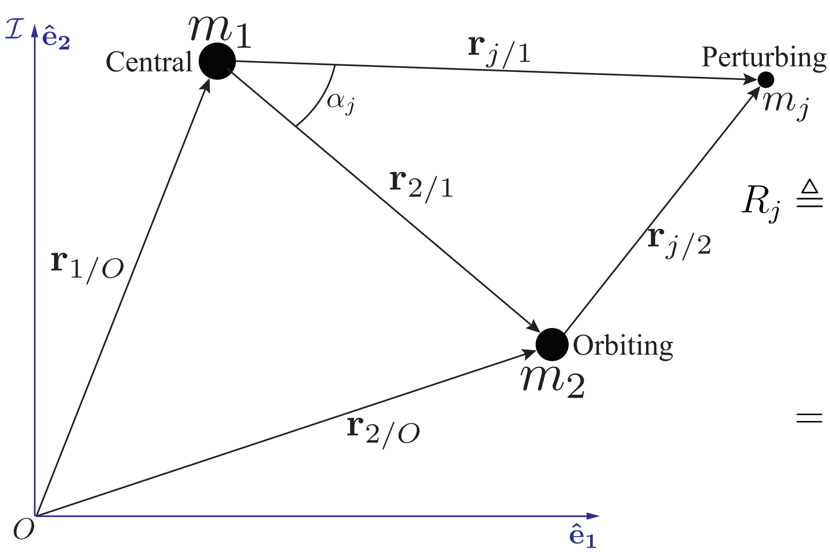
Secular and Resonant Perturbations

For our final set of topics, we will return to N-body interactions, as encoded by the disturbing function, to see what more analytical tractability can be teased out of these systems. In particular, we will see how low-order (in eccentricity and inclination) expansions of the disturbing function can yield analytical, and even linear, expressions for orbital element variations. These are directly applicable to systems of nearly co-planar, nearly circular orbits—exactly like our own solar system.

Recall the Disturbing Function

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/1} + \frac{G(m_1+m_2)}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = \underbrace{-G\sum_{j=3}^{n} m_j \left(\frac{\mathbf{r}_{j/1}}{\|\mathbf{r}_{j/1}\|^3} + \frac{\mathbf{r}_{2/j}}{\|\mathbf{r}_{2/j}\|^3}\right)}_{\triangleq\, \mathbf{f} \,=\, \boldsymbol{\nabla}_2 R}$$

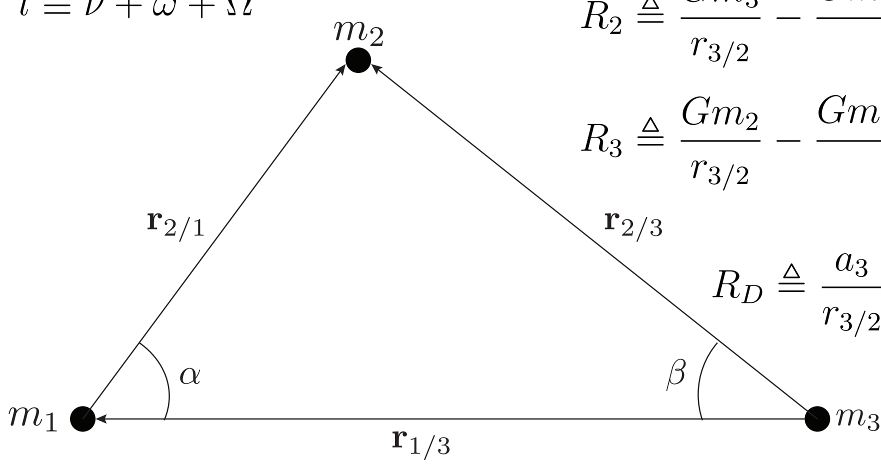


$$R_j \triangleq Gm_j \left(\underbrace{\frac{1}{r_{2/j}}}_{\text{Direct}} - \underbrace{\frac{\mathbf{r}_{2/1}\cdot\mathbf{r}_{j/1}}{r_{j/1}^3}}_{\text{Indirect}}\right)$$

$$= \frac{Gm_j}{r_{j/1}}\left[1 + \sum_{k=2}^{\infty}\left(\frac{r_{2/1}}{r_{j/1}}\right)^k P_k\left(\cos\alpha_j\right)\right]$$

Direct and Indirect Disturbing Function Terms

$$l \equiv \nu + \omega + \Omega$$



$$R_2 \triangleq \frac{Gm_3}{r_{3/2}} - \frac{Gm_3\mathbf{r}_{2/1}\cdot\mathbf{r}_{3/1}}{r_{3/1}^3} = \frac{\mu_3}{a_3}R_D + \frac{\mu_3}{a_3}\left(\frac{a_2}{a_3}\right)R_E$$

$$R_3 \triangleq \frac{Gm_2}{r_{3/2}} - \frac{Gm_2\mathbf{r}_{2/1}\cdot\mathbf{r}_{3/1}}{r_{2/1}^3} = \frac{\mu_2}{a_3}R_D + \frac{\mu_2}{a_3}\left(\frac{a_3}{a_2}\right)^2 R_I$$

$$R_D \triangleq \frac{a_3}{r_{3/2}} \qquad R_E \triangleq -\left(\frac{r_{2/1}}{a_2}\right)\left(\frac{a_3}{r_{3/1}}\right)^2\cos\alpha$$

$$R_I \triangleq -\left(\frac{r_{3/1}}{a_3}\right)\left(\frac{a_2}{r_{2/1}}\right)^2\cos\alpha$$

$$r_{3/2}^{-1} = \left[r_{2/1}^2 + r_{3/1}^2 - 2r_{2/1}r_{3/1}\left(\cos(l_2-l_3)+\Psi\right)\right]^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}\left(\frac{r_{2/1}r_{3/1}}{2}\Psi\right)^n r_0^{-(2n+1)}$$

$$\Psi \triangleq \cos\alpha - \cos(l_2-l_3) \qquad r_0 \triangleq \left[r_{2/1}^2 + r_{3/1}^2 - 2r_{2/1}r_{3/1}\cos(l_2-l_3)\right]^{\frac{1}{2}}$$

Laplace Coefficients

$$(1-xz)^{-s}(1-xz^{-1})^{-s} = \frac{1}{2}\sum_{j=-\infty}^{\infty} b_s^{(j)}(x) z^j \qquad s = i + \frac{1}{2}; i \in \mathbb{Z}^+$$

$$\frac{1}{2}b_s^{(j)}(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(j\psi)}{(1-2x\cos\psi + x^2)^s}\,\mathrm{d}\psi = \frac{s(s+1)\dots(s+j-1)}{j!}x^j\,{}_2F_1(s, s+j, j+1; x^2)$$

$$b_s^{(-j)} = b_s^{(j)} \qquad \frac{\mathrm{d}}{\mathrm{d}x}b_s^{(j)} = s\left(b_{s+1}^{(j-1)} - 2xb_{s+1}^{(j)} + b_{s+1}^{(j+1)}\right)$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}b_s^{(j)} = s\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}b_{s+1}^{(j-1)} - 2x\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}b_{s+1}^{(j)} + \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}b_{s+1}^{(j+1)} - 2(n-1)\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}b_{s+1}^{(j)}\right)$$

$$\begin{aligned} x^n\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}b_s^{(j)} - \frac{\mathrm{d}^n}{\mathrm{d}x^n}b_s^{(j-2)}\right) = &-(j+n-1)x^{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}b_s^{(j)} - (j-n-1)x^{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}b_s^{(j-2)} \\ &+ 2(j-1)\left[x^n\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}b_s^{(j-1)} + (n-1)x^{n-1}\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}b_s^{(j-1)}\right] \end{aligned}$$

Direct Disturbing Function

$$\boxed{\begin{aligned} R_D = \sum_{n=0}^{\infty}\Bigg[&\frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\frac{r_{2/1}}{a_2}\frac{r_{3/1}}{a_3}\Psi\right)^n \frac{a_2^n a_3^{n+1}}{2} \\ &\times \sum_{j=-\infty}^{\infty}\left(\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}\varepsilon_2^k\varepsilon_3^{m-k}A_{n,j,k,m-k}\right)\cos\left(j(l_2-l_3)\right)\Bigg] \end{aligned}}$$

$$\varepsilon_i \triangleq \frac{r_{i/1}}{a_i} - 1$$

$$A_{n,j,k,m-k} \triangleq a_2^k a_3^{m-k}\frac{\partial^m}{\partial a_2^k \partial a_3^{m-k}}\left(a_3^{-(2n+1)}b_{n+\frac{1}{2}}^{(j)}\left(\frac{a_2}{a_3}\right)\right)$$

Expansion of $\cos(\alpha)$

$$[\mathbf{r}]_{\mathcal{I}} = r\begin{bmatrix}\cos(\Omega)\cos(\nu+\omega) - \sin(\Omega)\sin(\nu+\omega)\cos(I)\\ \sin(\Omega)\cos(\nu+\omega) + \sin(\nu+\omega)\cos(I)\cos(\Omega)\\ \sin(I)\sin(\nu+\omega)\end{bmatrix}_{\mathcal{I}}$$

$$\begin{aligned}\cos\alpha = \frac{\mathbf{r}_{2/1}\cdot\mathbf{r}_{3/1}}{r_{2/1}r_{3/1}} &= \sin(\nu_2+\omega_2)\sin(\nu_3+\omega_3)\cos(I_2)\cos(I_3)\cos(\Omega_2-\Omega_3)\\ &+ \sin(I_2)\sin(I_3)\sin(\nu_2+\omega_2)\sin(\nu_3+\omega_3) - \sin(\Omega_2-\Omega_3)\sin(\nu_2+\omega_2)\cos(I_2)\cos(\nu_3+\omega_3)\\ &+ \sin(\Omega_2-\Omega_3)\sin(\nu_3+\omega_3)\cos(I_3)\cos(\nu_2+\omega_2) + \cos(\Omega_2-\Omega_3)\cos(\nu_2+\omega_2)\cos(\nu_3+\omega_3)\end{aligned}$$

$$\sin\nu = 2\sqrt{1-e^2}\sum_{s=1}^{\infty}\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}e}J_s(se)\sin(sM)$$

$$\cos\nu = -e + 2\frac{1-e^2}{e}\sum_{s=1}^{\infty}J_s(se)\cos(sM)$$

$$J_s(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+s+1)}\left(\frac{x}{2}\right)^{2m+s}$$

Direct Disturbing Terms to 2^{nd} order in e_i, I_i

$$\Delta \triangleq \frac{a_2}{a_3} \qquad D^n \triangleq \frac{\mathrm{d}^n}{\mathrm{d}\Delta^n} \qquad s_i \triangleq \sin\frac{I_i}{2} \qquad \lambda_i = M_i + \varpi_i$$

$$\begin{aligned}R_D \approx \sum_{j=-\infty}^{\infty}\Bigg[&\left(\frac{1}{2}b_{\frac{1}{2}}^{(j)} + \frac{1}{8}(e_2^2+e_3^2)\left(-4j^2+2\Delta D+\Delta^2D^2\right)b_{\frac{1}{2}}^{(j)} + \frac{1}{4}(s_2^2+s_3^2)\left(-\Delta b_{\frac{3}{2}}^{(j-1)} - \Delta b_{\frac{3}{2}}^{(j+1)}\right)\right)\cos(j\lambda_3 - j\lambda_2)\\ &+ \left(\frac{1}{4}e_2e_3\left(2+6j+4j^2-2\Delta D-\Delta^2D^2\right)b_{\frac{1}{2}}^{(j+1)}\right)\cos(j\lambda_3 - j\lambda_2 + \varpi_3 - \varpi_2)\\ &+ \left(s_2s_3\Delta b_{\frac{3}{2}}^{(j+1)}\right)\cos(j\lambda_3 - j\lambda_2 + \Omega_3 - \Omega_2) + \left(\frac{1}{2}e_2(-2j-\Delta D)b_{\frac{1}{2}}^{(j)}\right)\cos(j\lambda_3 + (1-j)\lambda_2 - \varpi_2)\\ &+ \left(\frac{1}{2}e_3(-1+2j+\Delta D)b_{\frac{1}{2}}^{(j-1)}\right)\cos(j\lambda_3 + (1-j)\lambda_2 - \varpi_3) + \left(\frac{1}{2}s_3^2\Delta b_{\frac{3}{2}}^{(j-1)}\right)\cos(j\lambda_3 + (2-j)\lambda_2 - 2\Omega_3)\\ &+ \left(\frac{1}{8}e_2^2(-5j+4j^2-2\Delta D+4j\Delta D+\Delta^2D^2)b_{\frac{1}{2}}^{(j)}\right)\cos(j\lambda_3 + (2-j)\lambda_2 - 2\varpi_2)\\ &+ \left(\frac{1}{4}e_3e_2(-2+6j-4j^2+2\Delta D-4j\Delta D-\Delta^2D^2)b_{\frac{1}{2}}^{(j-1)}\right)\cos(j\lambda_3 + (2-j)\lambda_2 - \varpi_2 - \varpi_3)\\ &+ \left(\frac{1}{8}e_3^2(2-7j+4j^2-2\Delta D+4j\Delta D+\Delta^2D^2)b_{\frac{1}{2}}^{(j-2)}\right)\cos(j\lambda_3 + (2-j)\lambda_2 - 2\varpi_3)\\ &+ \left(\frac{1}{2}s_2^2\Delta b_{\frac{3}{2}}^{(j-1)}\right)\cos(j\lambda_3 + (2-j)\lambda_2 - 2\Omega_2) + \left(-s_2s_3\Delta b_{\frac{3}{2}}^{(j-1)}\right)\cos(j\lambda_3 + (2-j)\lambda_2 - \Omega_2 - \Omega_3)\Bigg]\end{aligned}$$

R_E and R_I Terms to 2^{nd} order in e_i, I_i

$$\begin{aligned}\mathcal{R}_E \approx &\left(-1+\frac{1}{2}e_2^2+\frac{1}{2}e_3^2+s_2^2+s_3^2\right)\cos(\lambda_3-\lambda_2) - e_2 e_3\cos(2\lambda_3-2\lambda_2-\varpi_3+\varpi_2)\\
&-2s_3s_2\cos(\lambda_3-\lambda_2-\Omega_3+\Omega_2)-\frac{1}{2}e_2\cos(\lambda_3-2\lambda_2+\varpi_2)+\frac{3}{2}e_2\cos(\lambda_3-\varpi_2)-2e_3\cos(2\lambda_3-\lambda_2-\varpi_3)\\
&-\frac{3}{8}e_2^2\cos(\lambda_3-3\lambda_2+2\varpi_2)-\frac{1}{8}e_2^2\cos(\lambda_3+\lambda_2-2\varpi_2)+3e_2e_3\cos(2\lambda_3-\varpi_3-\varpi_2)-\frac{1}{8}e_3^2\cos(\lambda_3+\lambda_2-2\varpi_3)\\
&-\frac{27}{8}e_3^2\cos(3\lambda_3-\lambda_2-2\varpi_3)-s_2^2\cos(\lambda_3+\lambda_2-2\Omega_2)+2s_2s_3\cos(\lambda_3+\lambda_2-\Omega_3-\Omega_2)-s_3^2\cos(\lambda_3+\lambda_2-2\Omega_3)\end{aligned}$$

$$\begin{aligned}\mathcal{R}_I \approx &\left(-1+\frac{1}{2}e_2^2+\frac{1}{2}e_3^2+s_2^2+s_3^2\right)\cos(\lambda_3-\lambda_2) - e_2 e_3\cos(2\lambda_3-2\lambda_2-\varpi_3+\varpi_2)\\
&-2s_3s_2\cos(\lambda_3-\lambda_2-\Omega_3+\Omega_2)-2e_2\cos(\lambda_3-2\lambda_2+\varpi_2)+\frac{3}{2}e_3\cos(\lambda_2-\varpi_3)-\frac{1}{2}e_3\cos(2\lambda_3-\lambda_2-\varpi_3)\\
&-\frac{27}{8}e_2^2\cos(\lambda_3-3\lambda_2+2\varpi_2)-\frac{1}{8}e_2^2\cos(\lambda_3+\lambda_2-2\varpi_2)+3e_2e_3\cos(2\lambda_2-\varpi_3-\varpi_2)-\frac{1}{8}e_3^2\cos(\lambda_3+\lambda_2-2\varpi_3)\\
&-\frac{3}{8}e_3^2\cos(3\lambda_3-\lambda_2-2\varpi_3)-s_2^2\cos(\lambda_3+\lambda_2-2\Omega_2)+2s_2s_3\cos(\lambda_3+\lambda_2-\Omega_3-\Omega_2)-s_3^2\cos(\lambda_3+\lambda_2-2\Omega_3)\end{aligned}$$

Slight Modification to Lagrange's Planetary Equations

$$\lambda = M+\omega+\Omega = n(t-t_p)+\varpi = nt+\epsilon$$

$$\begin{aligned}\dot{a} &= \frac{2}{na}\frac{\partial R}{\partial \epsilon}\\
\dot{e} &= -\frac{\sqrt{1-e^2}}{na^2e}\left(\left(1-\sqrt{1-e^2}\right)\frac{\partial R}{\partial \epsilon}+\frac{\partial R}{\partial \varpi}\right)\\
\dot{I} &= \frac{1}{na^2\sqrt{1-e^2}\sin(I)}\left[(\cos(I)-1)\left(\frac{\partial R}{\partial \epsilon}+\frac{\partial R}{\partial \varpi}\right)-\frac{\partial R}{\partial \Omega}\right]\\
\dot{\epsilon} &= \frac{1}{na}\left(-2\frac{\partial R}{\partial a}+\frac{\sqrt{1-e^2}+e^2-1}{ae}\frac{\partial R}{\partial e}+\frac{\tan\left(\frac{I}{2}\right)}{a\sqrt{1-e^2}}\frac{\partial R}{\partial I}\right)\\
\dot{\varpi} &= \frac{1}{na^2}\left(\frac{\sqrt{1-e^2}}{e}\frac{\partial R}{\partial e}-\frac{\tan\left(\frac{I}{2}\right)}{\sqrt{1-e^2}}\frac{\partial R}{\partial I}\right)\\
\dot{\Omega} &= \frac{1}{na^2\sqrt{1-e^2}\sin(I)}\frac{\partial R}{\partial I}\end{aligned}$$

Linearized Lagrange's Planetary Equations

- $m_2 \ll m_1, m_3$
- $I_3 = 0$
- First order in e, I (body 2)

$$\dot{a} = \frac{2}{na}\frac{\partial \left(\overline{R}\right)_{\mathrm{av}}}{\partial \lambda}$$

$$\dot{e} = -\frac{1}{na^2 e}\frac{\partial \left(\overline{R}\right)_{\mathrm{av}}}{\partial \varpi}$$

$$\dot{\varpi} = \frac{1}{na^2}\frac{\partial \left(\overline{R}\right)_{\mathrm{av}}}{\partial e}$$

$$\dot{\Omega} = \frac{1}{na^2 \sin(I)}\frac{\partial \left(\overline{R}\right)_{\mathrm{av}}}{\partial I}$$

2^{nd} order Expansion Secular Terms

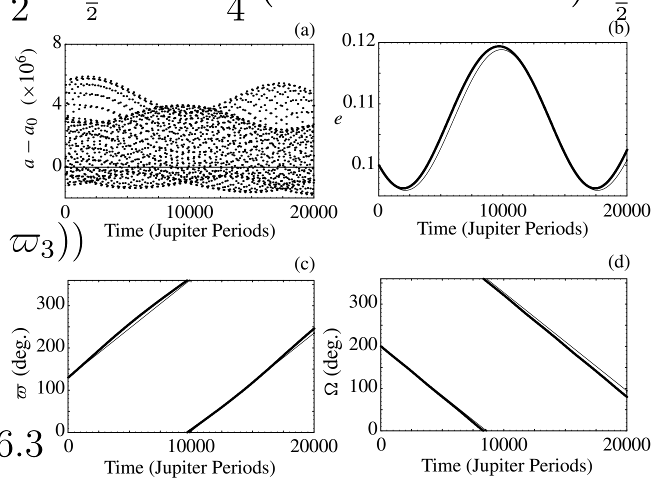
$$(R_D)_{\mathrm{sec}} = \underbrace{C_0}_{\frac{1}{2}b_{\frac{1}{2}}^{(0)}} + \underbrace{C_1}_{\frac{1}{8}\left(2\Delta D + \Delta^2 D^2\right) b_{\frac{1}{2}}^{(0)}}\left(e_2^2 + e_3^2\right) + \underbrace{C_2}_{-\frac{1}{2}\Delta b_{\frac{3}{2}}^{(1)}} \sin^2\left(\frac{I_2}{2}\right) + \underbrace{C_3}_{\frac{1}{4}\left(2 - 2\Delta D - \Delta^2 D^2\right) b_{\frac{1}{2}}^{(1)}} e_2 e_3 \cos\left(\varpi_2 - \varpi_3\right)$$

$$(\dot{a}_2)_{\mathrm{sec}} = 0$$

$$(\dot{e}_2)_{\mathrm{sec}} = \frac{C_3 e_3 \mu_3}{a_2^2 a_3 n_2}\sin\left(\varpi_2 - \varpi_3\right)$$

$$(\dot{\varpi}_2)_{\mathrm{sec}} = \frac{\mu_3}{a_2^2 a_3 e_2 n_2}\left(2C_1 e_2 + C_3 e_3 \cos\left(\varpi_2 - \varpi_3\right)\right)$$

$$\left(\dot{\Omega}_2\right)_{\mathrm{sec}} = \frac{C_2 \mu_3}{2a_2^2 a_3 n_2}$$



Murray & Dermott (1999) Fig. 6.3

2$^{\text{nd}}$ order Expansion Resonant Terms

$$(R_D)_{\text{res}} = \underbrace{\frac{1}{2}\left(-4-\Delta D\right) b_{\frac{1}{2}}^{(2)}}_{\triangleq\, C_4} e_2 \cos\left(2\lambda_3-\lambda_2-\varpi_2\right) + \underbrace{\frac{1}{2}\left(3+\Delta D\right) b_{\frac{1}{2}}^{(1)}}_{\triangleq\, C_5} e_3 \cos\left(2\lambda_3-\lambda_2-\varpi_3\right)$$

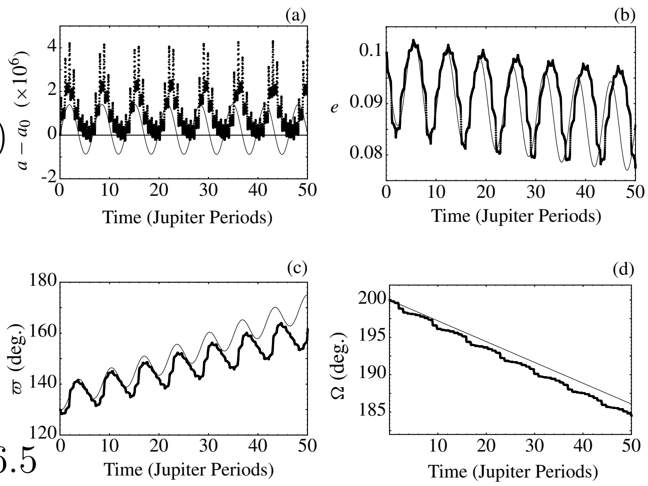
$$(R_E)_{\text{res}} = -2e_3 \cos\left(2\lambda_3-\lambda_2-\varpi_3\right)$$

$$(\dot{a}_2)_{\text{res}} = \frac{2\mu_3}{a_2 a_3^2 n_2}\left(C_4 a_3 e_2 \sin\left(2\lambda_3-\lambda_2-\varpi_2\right) + e_3\left(C_5 a_3 - 2a_2\right)\sin\left(2\lambda_3-\lambda_2-\varpi_3\right)\right)$$

$$(\dot{e}_2)_{\text{res}} = \frac{C_4\mu_3}{a_2^2 a_3 n_2}\sin\left(2\lambda_3-\lambda_2-\varpi_2\right)$$

$$(\dot{\varpi}_2)_{\text{res}} = \frac{C_4\mu_3}{a_2^2 a_3 e_2 n_2}\cos\left(2\lambda_3-\lambda_2-\varpi_2\right)$$

$$\left(\dot{\Omega}_2\right)_{\text{res}} = 0$$



Murray & Dermott (1999) Fig. 6.5

D'Alembert Relation

$$R = \mu_3 \sum S(a_2, a_3, e_2, e_3, I_2, I_3)\cos(\phi)$$

$$\phi = \underbrace{(l_2 - 2p_3 + q_3)}_{c_1}\lambda_3 \underbrace{-(l_2 - 2p_2 + q_2)}_{c_2}\lambda_2 \underbrace{-q_3}_{c_3}\varpi_3 + \underbrace{q_2}_{c_4}\varpi_2 + \underbrace{(m_2 - l_2 + 2p_3)}_{c_5}\Omega_3 \underbrace{-(m_2 - l_2 + 2p_2)}_{c_6}\Omega_2$$

$$l_2, m_2, p_2, p_3, q_2, q_3 \in \mathbb{Z}$$

$$\boxed{\sum_{i=1}^{6} c_i = 0}$$

$$S \approx \frac{f(\Delta)}{a_3} e_2^{|q_2|} e_3^{|q_3|} s_2^{|m_2-l_2+2p_2|} s_3^{|m_2-l_2+2p_3|}$$

Secular Perturbations of 2 Bodies

$$\Delta \triangleq \frac{a_2}{a_3} \qquad D^n \triangleq \frac{\mathrm{d}^n}{\mathrm{d}\Delta^n} \qquad s_i \triangleq \sin\frac{I_i}{2}$$

$$\begin{aligned}\left(\bar{R}_D\right)_{\text{sec}} =& \frac{1}{8}\left(2\Delta D + \Delta^2 D^2\right) b_{\frac{1}{2}}^{(0)}\left(e_2^2 + e_3^2\right) - \frac{1}{2}\Delta b_{\frac{3}{2}}^{(1)}\left(s_2^2 + s_3^2\right) \\ &+ \frac{1}{4}\left(2 - 2\Delta D - \Delta^2 D^2\right) b_{\frac{1}{2}}^{(1)} e_2 e_3 \cos\left(\varpi_2 - \varpi_3\right) + \Delta b_{\frac{3}{2}}^{(1)} s_2 s_3 \cos(\Omega_2 - \Omega_3)\end{aligned}$$

$$j = 2,3 \qquad p \triangleq j-1 \qquad\qquad k = 3,2 \qquad q \triangleq k-1$$

$$R_j = n_j a_j^2\left(\frac{A_{pp}}{2}e_j^2 + A_{pq}e_2e_3\cos(\varpi_2 - \varpi_3) + \frac{B_{pp}}{2}I_j^2 + B_{pq}I_2I_3\cos(\Omega_2 - \Omega_3)\right)$$

$$\begin{aligned}A_{pq} &= (-1)^{1-\delta_{pq}}\frac{n_p}{4}\frac{m_{(3-p)}}{m_1 + m_p}\Delta\left(\Delta\right)^{2-p} b_{\frac{3}{2}}^{(2-\delta_{pq})}(\Delta) \\ B_{pq} &= (-1)^{\delta_{pq}}\frac{n_p}{4}\frac{m_{(3-p)}}{m_1 + m_p}\Delta\left(\Delta\right)^{2-p} b_{\frac{3}{2}}^{(1)}(\Delta)\end{aligned}$$

New Coordinates to Avoid Singularities

$$\begin{bmatrix}\dot{e}_j \\ \dot{\varpi}_j\end{bmatrix} = \frac{1}{n_j a_j^2 e_j}\begin{bmatrix}-\dfrac{\partial R_j}{\partial \varpi_j} \\ \dfrac{\partial R_j}{\partial e_j}\end{bmatrix} \qquad \begin{bmatrix}\dot{I}_j \\ \dot{\Omega}_j\end{bmatrix} = \frac{1}{n_j a_j^2 I_j}\begin{bmatrix}-\dfrac{\partial R_j}{\partial \Omega_j} \\ \dfrac{\partial R_j}{\partial I_j}\end{bmatrix}$$

$$\begin{bmatrix}c_j \\ d_j\end{bmatrix} \triangleq e_j\begin{bmatrix}\sin\varpi_j \\ \cos\varpi_j\end{bmatrix} \qquad \begin{bmatrix}u_j \\ v_j\end{bmatrix} \triangleq I_j\begin{bmatrix}\sin\Omega_j \\ \cos\Omega_j\end{bmatrix}$$

$$R_j = n_j a_j^2\left(\frac{A_{pp}}{2}(c_j^2 + d_j^2) + A_{pq}(c_jc_k + d_jd_k) + \frac{B_{pp}}{2}(u_j^2 + v_j^2) + B_{pq}(u_ju_k + v_jv_k)\right)$$

$$\Longrightarrow \quad \begin{bmatrix}\dot{c}_j \\ \dot{d}_j\end{bmatrix} = \frac{1}{n_j a_j^2}\begin{bmatrix}\dfrac{\partial R_j}{\partial d_j} \\ -\dfrac{\partial R_j}{\partial c_j}\end{bmatrix} \qquad \begin{bmatrix}\dot{u}_j \\ \dot{v}_j\end{bmatrix} = \frac{1}{n_j a_j^2}\begin{bmatrix}\dfrac{\partial R_j}{\partial v_j} \\ -\dfrac{\partial R_j}{\partial u_j}\end{bmatrix}$$

Laplace-Lagrange Secular Solution

$$\begin{bmatrix}\dot{c}_2\\ \dot{c}_3\end{bmatrix} = \underbrace{\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}}_{\triangleq A}\begin{bmatrix}d_2\\ d_3\end{bmatrix};\quad \begin{bmatrix}\dot{d}_2\\ \dot{d}_3\end{bmatrix} = -A\begin{bmatrix}c_2\\ c_3\end{bmatrix};\quad \begin{bmatrix}\dot{u}_2\\ \dot{u}_3\end{bmatrix} = \underbrace{\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}}_{\triangleq B}\begin{bmatrix}v_2\\ v_3\end{bmatrix};\quad \begin{bmatrix}\dot{v}_2\\ \dot{v}_3\end{bmatrix} = -B\begin{bmatrix}u_2\\ u_3\end{bmatrix}$$

$$A\mathbf{f}_i = \lambda_i\mathbf{f}_i \Leftrightarrow AF = F\Lambda \qquad\qquad B\mathbf{g}_i = \gamma_i\mathbf{g}_i \Leftrightarrow BG = G\Gamma$$

$$\begin{bmatrix}c_2\\ c_3\end{bmatrix} = \bar{F}\begin{bmatrix}\sin(\lambda_1 t+\alpha_1)\\ \sin(\lambda_2 t+\alpha_2)\end{bmatrix} \qquad \begin{bmatrix}u_2\\ u_3\end{bmatrix} = \bar{G}\begin{bmatrix}\sin(\gamma_1 t+\beta_1)\\ \sin(\gamma_2 t+\beta_2)\end{bmatrix}$$

$$\begin{bmatrix}d_2\\ d_3\end{bmatrix} = \bar{F}\begin{bmatrix}\cos(\lambda_1 t+\alpha_1)\\ \cos(\lambda_2 t+\alpha_2)\end{bmatrix} \qquad \begin{bmatrix}v_2\\ v_3\end{bmatrix} = \bar{G}\begin{bmatrix}\cos(\gamma_1 t+\beta_1)\\ \cos(\gamma_2 t+\beta_2)\end{bmatrix}$$

Normalized to Initial Conditions

Free and Forced Elements

Add a particle with orbital elements $(a, e, I, \varpi, \Omega, n)$ to the three-body system. The net disturbing function due to m_2 and m_3 is:

$$R_p = na^2\left[\frac{A_p}{2}e^2 + \frac{B_p}{2}I^2 + \sum_{j=2}^{3}A_j e e_j\cos(\varpi-\varpi_j) + \sum_{j=2}^{3}B_j I I_j\cos(\Omega-\Omega_j)\right]$$

$$A_p \triangleq \frac{n}{4}\sum_{j=2}^{3}\frac{m_j}{m_1}\left(\frac{a_j}{a}\right)^{(2(a_j<a)-1)}\left(\frac{a}{a_j}\right)^{(a_j>a)} b_{\frac{3}{2}}^{(1)}\left(\left(\frac{a_j}{a}\right)^{(2(a_j<a)-1)}\right)$$

$$B_p \triangleq -\frac{n}{4}\sum_{j=2}^{3}\frac{m_j}{m_1}\left(\frac{a_j}{a}\right)^{(2(a_j<a)-1)}\left(\frac{a}{a_j}\right)^{(a_j>a)} b_{\frac{3}{2}}^{(1)}\left(\left(\frac{a_j}{a}\right)^{(2(a_j<a)-1)}\right)$$

$$A_j \triangleq -\frac{n}{4}\frac{m_j}{m_1}\left(\frac{a_j}{a}\right)^{(2(a_j<a)-1)}\left(\frac{a}{a_j}\right)^{(a_j>a)} b_{\frac{3}{2}}^{(2)}\left(\left(\frac{a_j}{a}\right)^{(2(a_j<a)-1)}\right)$$

$$B_j \triangleq \frac{n}{4}\frac{m_j}{m_1}\left(\frac{a_j}{a}\right)^{(2(a_j<a)-1)}\left(\frac{a}{a_j}\right)^{(a_j>a)} b_{\frac{3}{2}}^{(1)}\left(\left(\frac{a_j}{a}\right)^{(2(a_j<a)-1)}\right)$$

Free and Forced Elements (2)

$$\begin{bmatrix} c \\ d \end{bmatrix} \triangleq e \begin{bmatrix} \sin\varpi \\ \cos\varpi \end{bmatrix} \qquad \begin{bmatrix} u \\ v \end{bmatrix} \triangleq I \begin{bmatrix} \sin\Omega \\ \cos\Omega \end{bmatrix} \qquad \Longrightarrow$$

$$R_p = na^2 \left[\frac{A_p}{2}\left(c^2 + d^2\right) + \frac{B_p}{2}\left(u^2 + v^2\right) + \sum_{j=2}^{3} A_j \left(cc_j + dd_j\right) + \sum_{j=2}^{3} B_j I \left(uu_j + vv_j\right) \right]$$

$$\begin{bmatrix} \dot{c} \\ \dot{d} \end{bmatrix} = \frac{1}{na^2} \begin{bmatrix} \partial R_p / \partial d \\ -\partial R_p / \partial c \end{bmatrix} = A_p \begin{bmatrix} d \\ -c \end{bmatrix} + \underbrace{\begin{bmatrix} d_2 & d_3 \\ -c_2 & -c_3 \end{bmatrix}}_{\begin{bmatrix} \cos(\lambda_1 t + \alpha_1) & \cos(\lambda_2 t + \alpha_2) \\ -\sin(\lambda_1 t + \alpha_1) & -\sin(\lambda_2 t + \alpha_2) \end{bmatrix} \bar{F}^T} \begin{bmatrix} A_2 \\ A_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \frac{1}{na^2} \begin{bmatrix} \partial R_p / \partial v \\ -\partial R_p / \partial u \end{bmatrix} = B_p \begin{bmatrix} v \\ -u \end{bmatrix} + \underbrace{\begin{bmatrix} v_2 & v_3 \\ -u_2 & -u_3 \end{bmatrix}}_{\begin{bmatrix} \cos(\gamma_1 t + \beta_1) & \cos(\gamma_2 t + \beta_2) \\ -\sin(\gamma_1 t + \beta_1) & -\sin(\gamma_2 t + \beta_2) \end{bmatrix} \bar{G}^T} \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$$

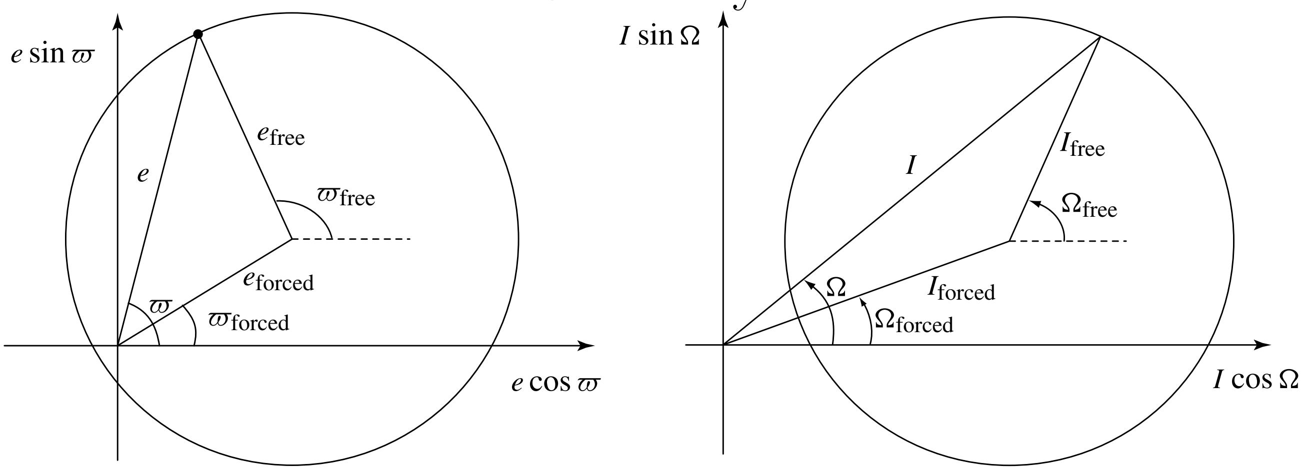
Free and Forced Elements (3)

$$\begin{bmatrix} \ddot{c} \\ \ddot{d} \end{bmatrix} = -A_p^2 \begin{bmatrix} c \\ d \end{bmatrix} - \begin{bmatrix} (A_p + \lambda_1)\sin(\lambda_1 t + \alpha_1) & (A_p + \lambda_2)\sin(\lambda_2 t + \alpha_2) \\ (A_p + \lambda_1)\cos(\lambda_1 t + \alpha_1) & (A_p + \lambda_2)\cos(\lambda_2 t + \alpha_2) \end{bmatrix} \bar{F}^T \begin{bmatrix} A_2 \\ A_3 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{u} \\ \ddot{v} \end{bmatrix} = -B_p^2 \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} (B_p + \gamma_1)\sin(\gamma_1 t + \beta_1) & (B_p + \gamma_2)\sin(\gamma_2 t + \beta_2) \\ (B_p + \gamma_1)\cos(\gamma_1 t + \beta_1) & (B_p + \gamma_2)\cos(\gamma_2 t + \beta_2) \end{bmatrix} \bar{G}^T \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$$

$$\boxed{\left.\begin{aligned} c &= e_{\text{free}}\sin(A_p t + \alpha_p) + c_0(t) \\ d &= e_{\text{free}}\cos(A_p t + \alpha_p) + d_0(t) \\ u &= I_{\text{free}}\sin(B_p t + \beta_p) + u_0(t) \\ v &= I_{\text{free}}\cos(B_p t + \beta_p) + v_0(t) \end{aligned}\right\} \begin{aligned} \begin{bmatrix} c_0(t) \\ d_0(t) \end{bmatrix} &= -\begin{bmatrix} \frac{\sin(\lambda_1 t + \alpha_1)}{(A_p - \lambda_1)} & \frac{\sin(\lambda_2 t + \alpha_2)}{(A_p - \lambda_2)} \\ \frac{\cos(\lambda_1 t + \alpha_1)}{(A_p - \lambda_1)} & \frac{\cos(\lambda_2 t + \alpha_2)}{(A_p - \lambda_2)} \end{bmatrix} \bar{F}^T \begin{bmatrix} A_2 \\ A_3 \end{bmatrix} \\ \begin{bmatrix} u_0(t) \\ v_0(t) \end{bmatrix} &= -\begin{bmatrix} \frac{\sin(\gamma_1 t + \beta_1)}{(B_p - \gamma_1)} & \frac{\sin(\gamma_2 t + \beta_2)}{(B_p - \gamma_2)} \\ \frac{\cos(\gamma_1 t + \beta_1)}{(B_p - \gamma_1)} & \frac{\cos(\gamma_2 t + \beta_2)}{(B_p - \gamma_2)} \end{bmatrix} \bar{G}^T \begin{bmatrix} B_2 \\ B_3 \end{bmatrix} \end{aligned}}$$

Free and Forced Element Geometry



Murray & Dermott (1999) Figs. 7.2-7.3

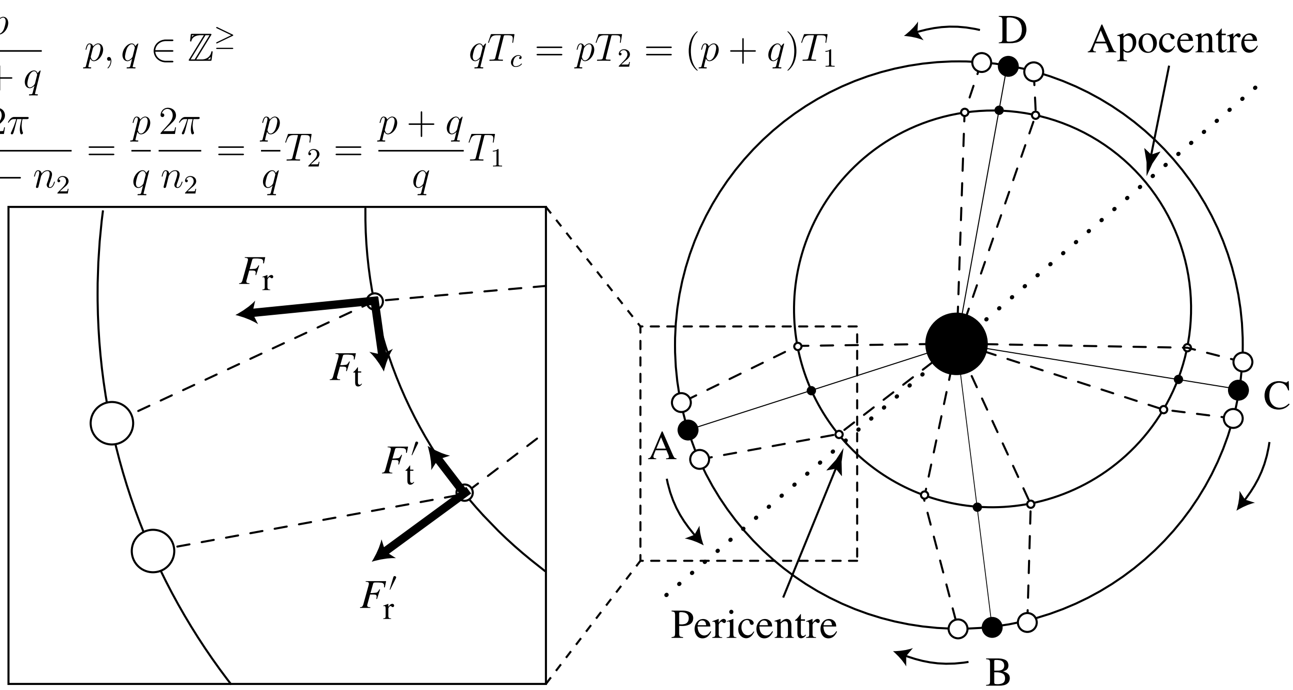
$$e_{\text{forced}} \triangleq \sqrt{c_0^2 + d_0^2} \qquad \varpi_{\text{free}} = A_p t + \alpha_p$$

$$I_{\text{forced}} \triangleq \sqrt{u_0^2 + v_0^2} \qquad \Omega_{\text{free}} = B_p t + \beta_p$$

Resonant Perturbations

$$\frac{n_2}{n_1} = \frac{p}{p+q} \quad p, q \in \mathbb{Z}^{\geq} \qquad qT_c = pT_2 = (p+q)T_1$$

$$T_c = \frac{2\pi}{n_1 - n_2} = \frac{p}{q}\frac{2\pi}{n_2} = \frac{p}{q}T_2 = \frac{p+q}{q}T_1$$



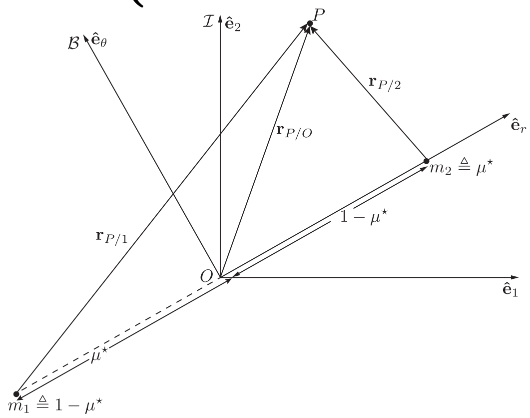
Based on Murray & Dermott (1999) Fig. 8.5

Resonance in the CR3BP

$$\overline{R} = \frac{Gm_2}{a_2}\left[\frac{1}{8}\left(2\Delta D + \Delta^2 D^2\right) b_{\frac{1}{2}}^{(0)} e^2 + f_d(\Delta) e^{|c_4|} \cos \underbrace{\left(c_1\lambda_2 + c_2\lambda + c_4\varpi\right)}_{\triangleq \phi}\right]$$

$$\Delta \triangleq \frac{a}{a_2},\; D^n \triangleq \frac{\mathrm{d}^n}{\mathrm{d}\Delta^n}$$

$$f_d = \begin{cases} \frac{1}{2}\left(-2j - \Delta D\right) b_{\frac{1}{2}}^{(j)} & c_1 = j, c_2 = 1 - j, c_4 = -1 \\ \frac{1}{8}\left(-5j + 4j^2 - 2\Delta D + 4j\Delta D + j^2 D^2\right) b_{\frac{1}{2}}^{(j)} & c_1 = j, c_2 = 2 - j, c_4 = -2 \end{cases}$$



$$\dot{n} = 3c_2 c_r n e^{|c_4|} \sin\phi \qquad \dot{\varpi} = 2c_s + |c_4| c_r e^{|c_4|-2} \cos\phi$$

$$\dot{e} = c_4 c_r e^{|c_4|-1} \sin\phi \qquad \dot{\epsilon} = c_s e^2 + \frac{|c_4| c_r}{2} e^{|c_4|} \cos\phi$$

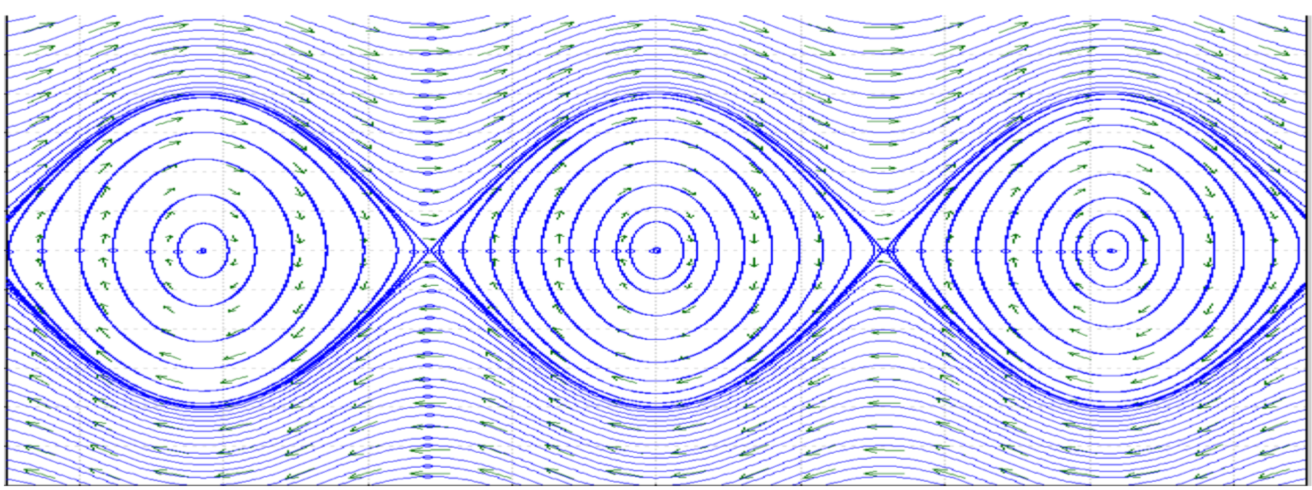
$$c_r \triangleq \left(\frac{m_2}{m_1}\right) n\Delta f_d(\Delta)$$

$$c_s \triangleq \left(\frac{m_2}{m_1}\right) \frac{n\Delta}{8}\left(2\Delta D + \Delta^2 D^2\right) b_{\frac{1}{2}}^{(0)}$$

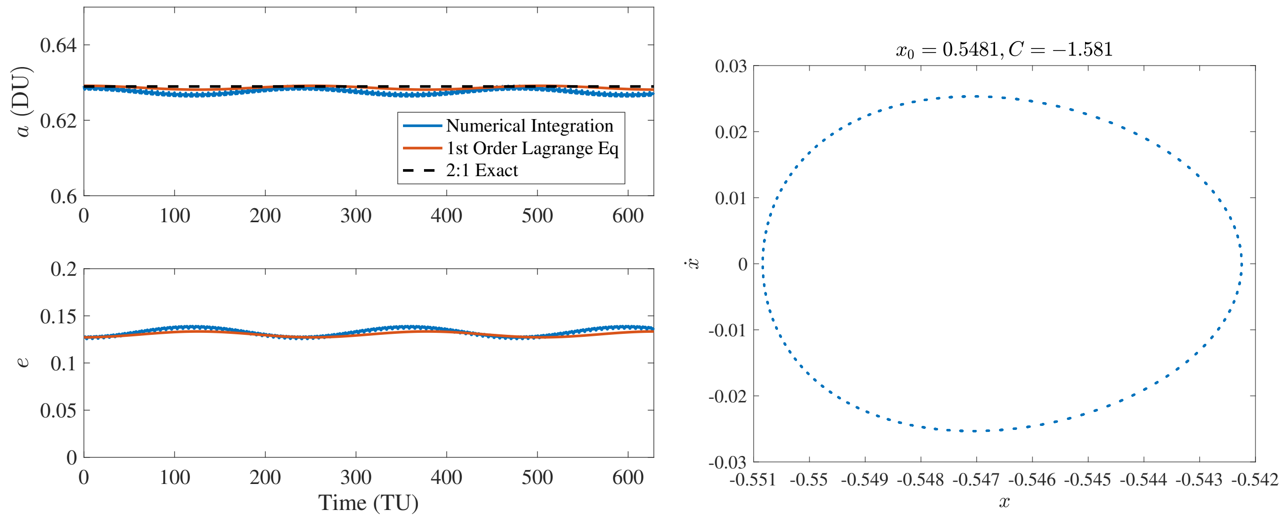
Libration

$$\phi = c_1\lambda_2 + c_2\lambda + c_4\varpi \quad \Rightarrow \quad \dot{\phi} = c_1 n_2 + c_2(n + \dot{\epsilon}) + c_4\dot{\varpi} \quad \Rightarrow \quad \ddot{\phi} = c_2\dot{n} + c_2\ddot{\epsilon} + c_4\ddot{\varpi}$$

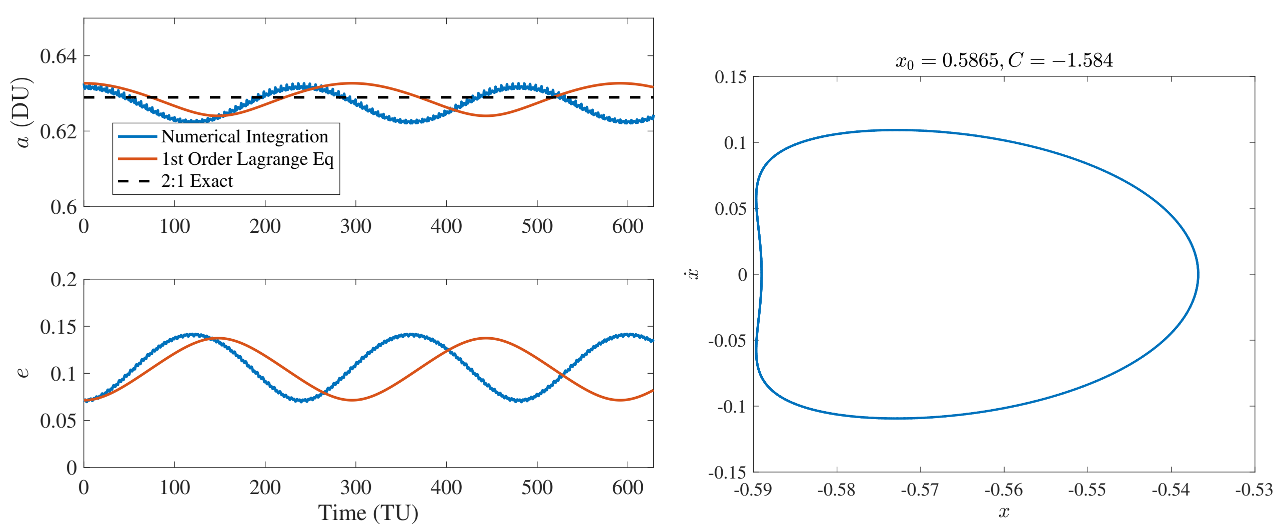
$$\left.\begin{aligned} \ddot{\epsilon} &= 2c_s e\dot{e} + |c_4| c_r \left(\tfrac{\mathrm{d}}{\mathrm{d}e}\left(\tfrac{e^{|c_4|}}{2}\right)\dot{e}\cos\phi - \left(\tfrac{e^{|c_4|}}{2}\right)\dot{\phi}\sin\phi\right) \\ \ddot{\varpi} &= |c_4| c_r \left(\tfrac{\mathrm{d}}{\mathrm{d}e}\left(e^{|c_4|-2}\right)\dot{e}\cos\phi - \left(e^{|c_4|-2}\right)\dot{\phi}\sin\phi\right) \end{aligned}\right\} \ddot{\phi} \approx 3c_2^2 c_r n e^{|c_4|} \sin\phi$$



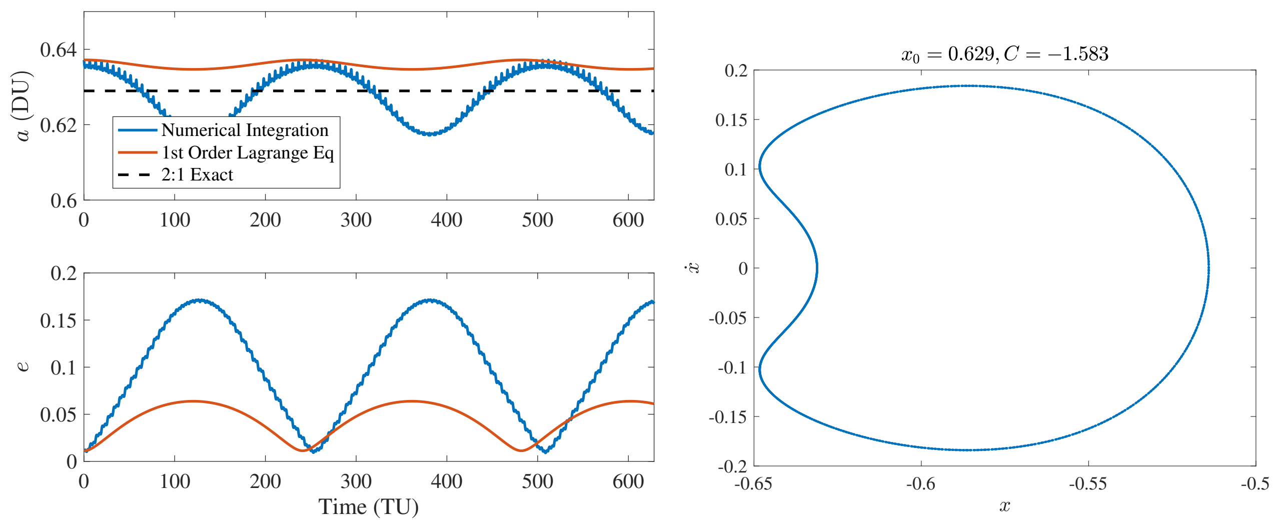
CR3BP 2:1 Resonance Exact



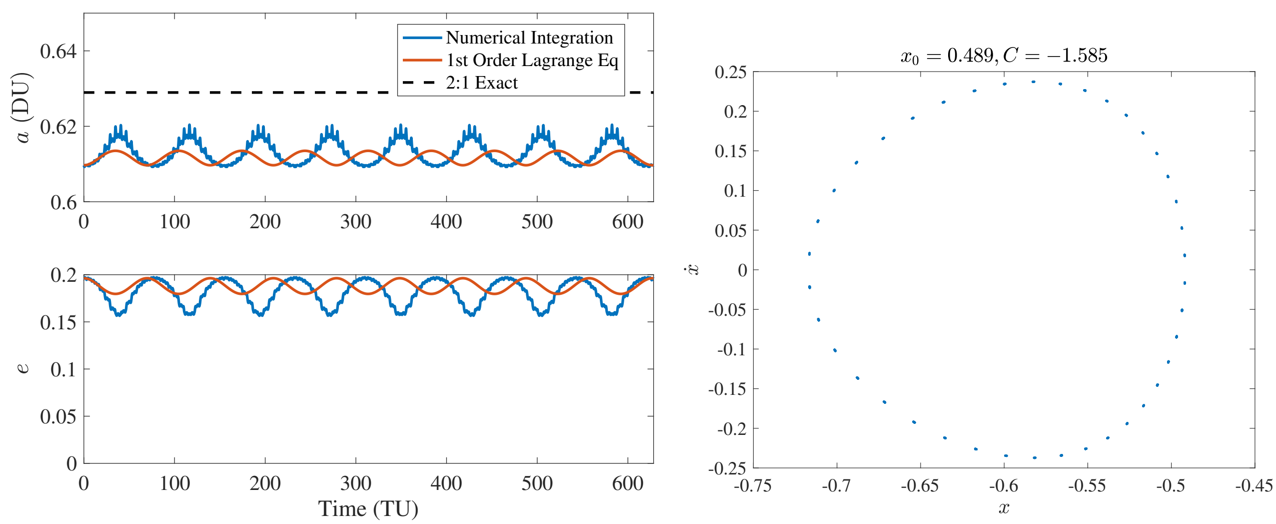
CR3BP 2:1 Resonance Medium Libration



CR3BP 2:1 Resonance Large Libration



CR3BP 2:1 Resonance Outer Circulation



Chaos in the CR3BP