2 - The Two Body Problem

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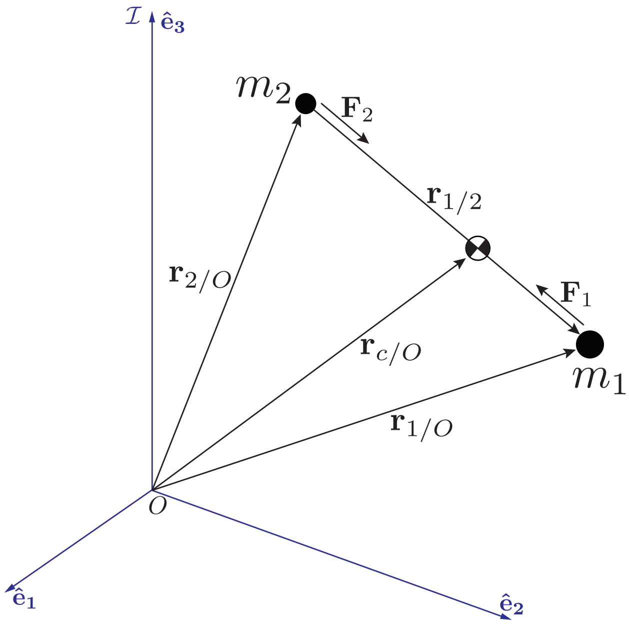
The Two-Body Problem

The two-body problem (two point masses interacting via gravity, with no other forces present) is the fundamental building block of celestial mechanics. In fact, the two-body problem is the only orbital mechanics problem with an exact solution, allowing you to express the positions of both bodies in the past, present, and future via a single analytical expression. Although in practice you are unlikely to ever deal with an exact two-body system, many complex systems (including the solar system) behave like collections of two-body orbits that gradually change over time, making two-body concepts broadly applicable to a variety of other cases.

Newton's Law of Gravity and the Two Body Problem



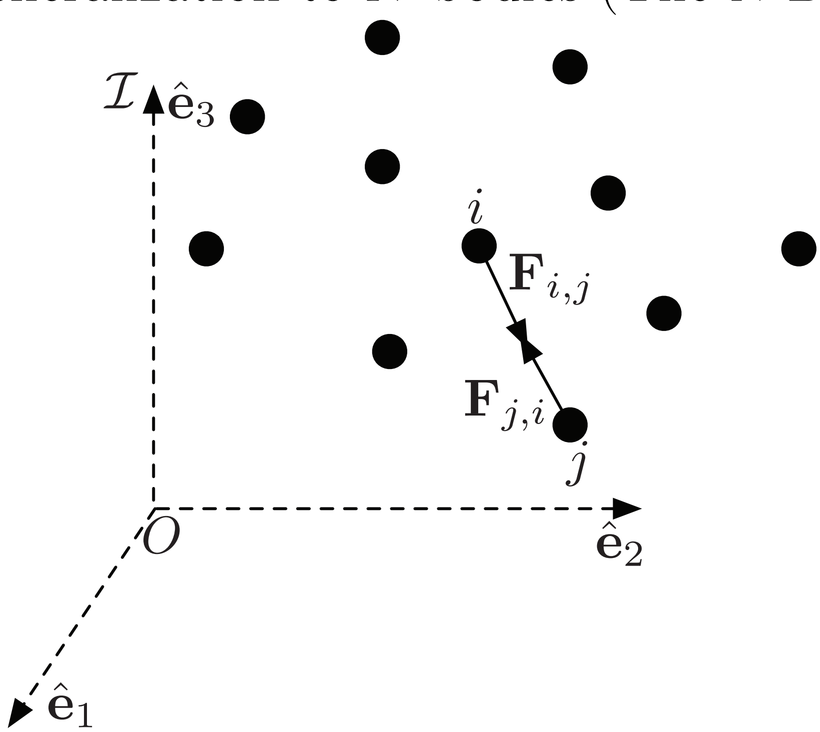
Gravitational Constant

$$\mathbf{F}_1 = -\mathbf{F}_2 = -\frac{G\, m_1 m_2}{\|\mathbf{r}_{1/2}\|^3}\mathbf{r}_{1/2}$$

Orbital Radius: $\mathbf{r} \equiv \mathbf{r}_{1/2}$ (or $\mathbf{r}_{2/1}$)
Gravitational Parameter: $\mu \triangleq G(m_1 + m_2)$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = 0$$

Generalization to N bodies (The N-Body Problem)



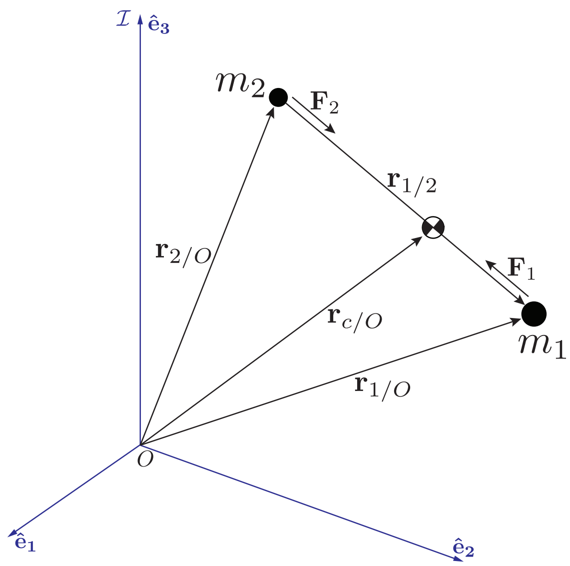
$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{i/O} = -G\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j}{\|\mathbf{r}_{i/j}\|^3}\mathbf{r}_{i/j}$$

$$\mathbf{r}_{G/O} \triangleq \frac{1}{m_G}\sum_{i=1}^{N} m_i \mathbf{r}_{i/O}$$

$$m_G \triangleq \sum_{i=1}^{N} m_i$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{G/O} = 0$$

Conserved Quantities in the Two Body Problem



Specific Angular Momentum: $\mathbf{h} \triangleq \mathbf{r} \times \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r}$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = 0 \Rightarrow \frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} \times \mathbf{h} = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{\|\mathbf{r}\|}\mathbf{r}\right)$$

$$\Rightarrow \quad \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{\|\mathbf{r}\|} + \mathbf{e}\right)$$

Constant of Integration

$$\boxed{r \triangleq \|\mathbf{r}\| = \frac{h^2/\mu}{1 + e\cos(\nu)}}$$

$$h \triangleq \|\mathbf{h}\| \qquad e \triangleq \|\mathbf{e}\| \qquad \mathbf{r} \cdot \mathbf{e} = re\cos\nu$$

$\mathbf{e}$ is the eccentricity (Laplace–Runge–Lenz) vector. ν is the angle between $\mathbf{e}$ and $\mathbf{r}$.

The Perifocal Frame

$$\mathcal{P} \triangleq \left(O, \hat{\mathbf{e}}, \hat{\mathbf{q}}, \hat{\mathbf{h}}\right)$$

$$\mathbf{r} = r\cos(\nu)\hat{\mathbf{e}} + r\sin(\nu)\hat{\mathbf{q}}$$

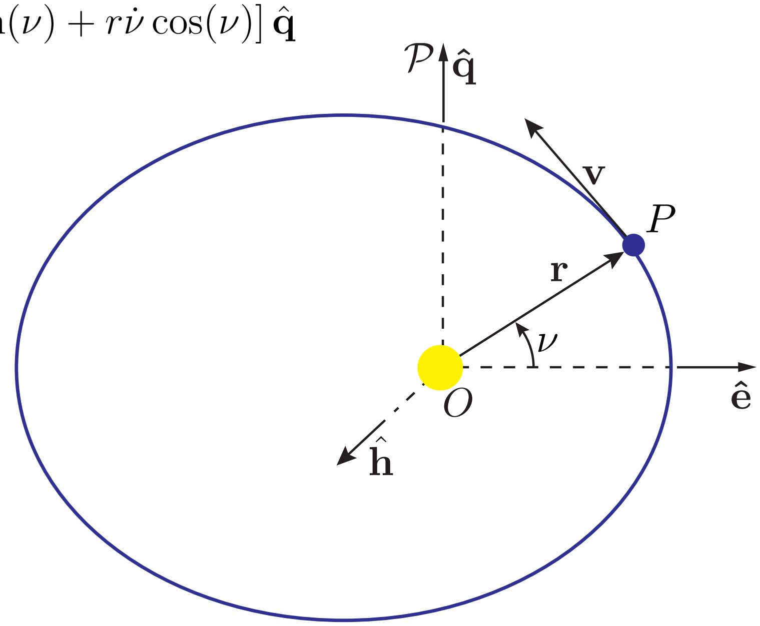
$$\mathbf{v} \triangleq \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \left[\dot{r}\cos(\nu) - r\dot{\nu}\sin(\nu)\right]\hat{\mathbf{e}} + \left[\dot{r}\sin(\nu) + r\dot{\nu}\cos(\nu)\right]\hat{\mathbf{q}}$$

$$= \frac{\mu}{h}\left[-\sin(\nu)\hat{\mathbf{e}} + (e + \cos(\nu))\hat{\mathbf{q}}\right]$$

$$r = \|\mathbf{r}\| = \frac{h^2/\mu}{1 + e\cos(\nu)}$$

$$\left.\begin{aligned} e &= \|\mathbf{e}\| = \left\|\frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}\right\| \\ h &= \|\mathbf{h}\| = \|\mathbf{r} \times \mathbf{v}\| \end{aligned}\right\}\text{Constants}$$

$$\boxed{h = r^2\dot{\nu}}$$

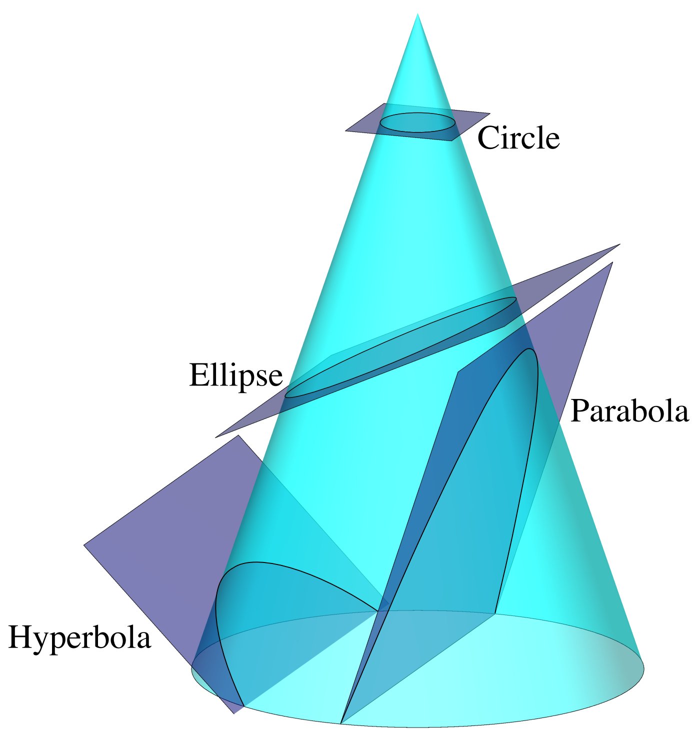


$\hat{\mathbf{e}}, \hat{\mathbf{q}}, \mathbf{r}$, and $\mathbf{v}$ all lie within the perifocal plane

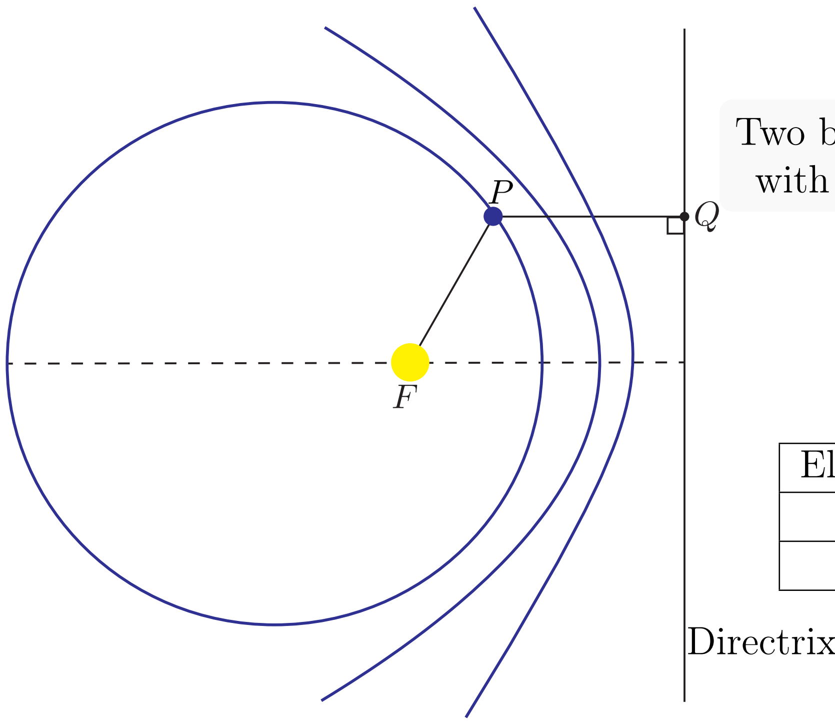
Kepler's Laws of Planetary Motion

1. The orbit of a planet is an ellipse (conic section) with the Sun at a focus
2. A line segment joining a planet and the Sun sweeps out equal areas in equal time
3. The square of the orbital period is proportional to the cube of the semi-major axis

Conic Sections



Kepler's First Law

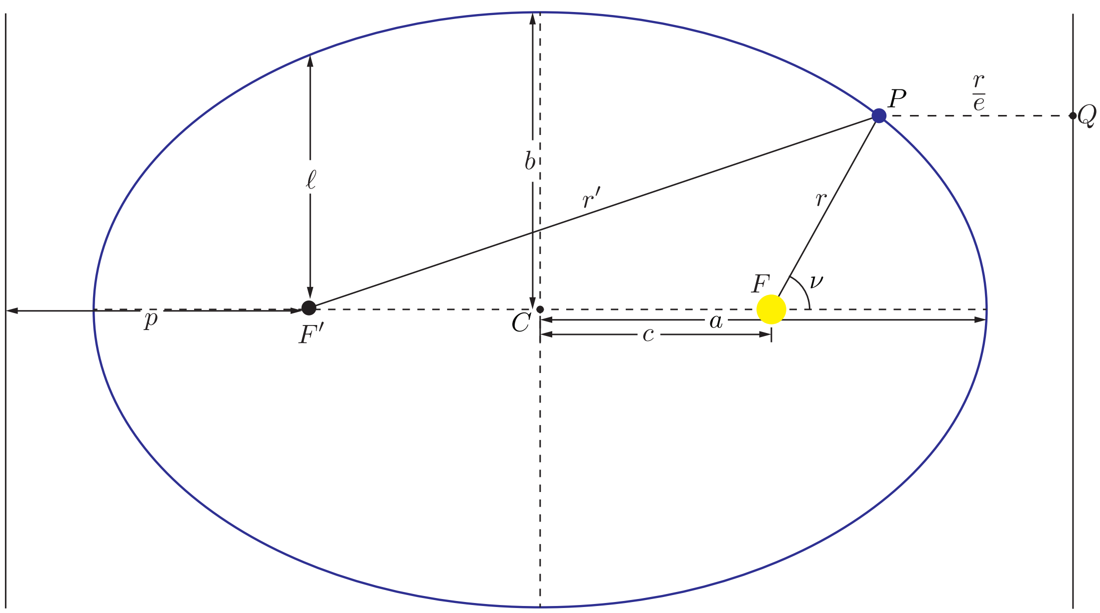


Two body orbits are conic sections with the central body at a focus

$$\overline{FP} = e\overline{PQ}$$

Ellipse (Circle)	$0 < e < 1$
Parabola	$e = 1$
Hyperbola	$e > 1$

Elliptical Orbits

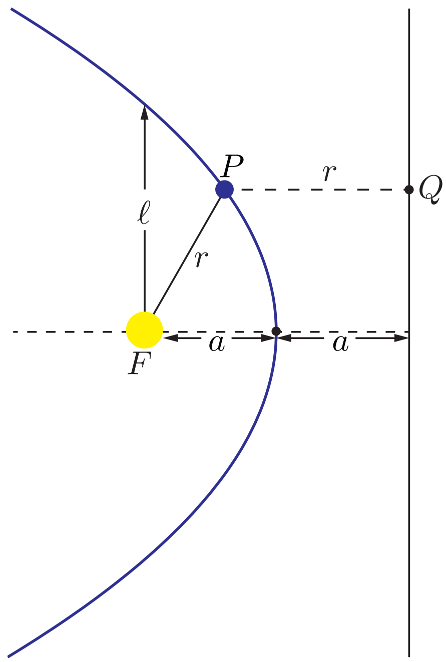


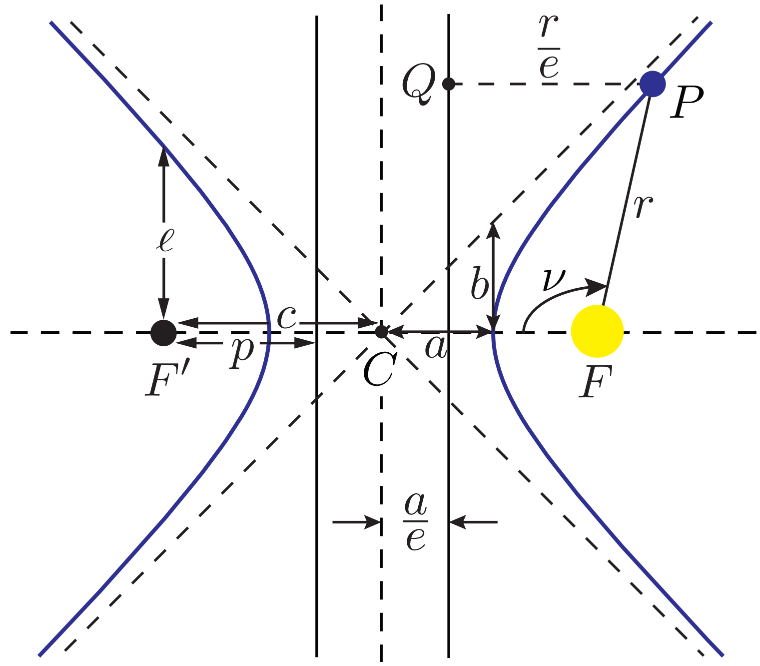
$$r' + r = 2a$$

$$\begin{aligned} r &= \frac{a(1-e^2)}{1+e\cos(\nu)} \\ &= \frac{h^2/\mu}{1+e\cos(\nu)} \\ &= \frac{\ell}{1+e\cos(\nu)} \end{aligned}$$

*This last equation applies for all conic sections.

Parabolic and Hyperbolic Orbits





Conic Section Parameters

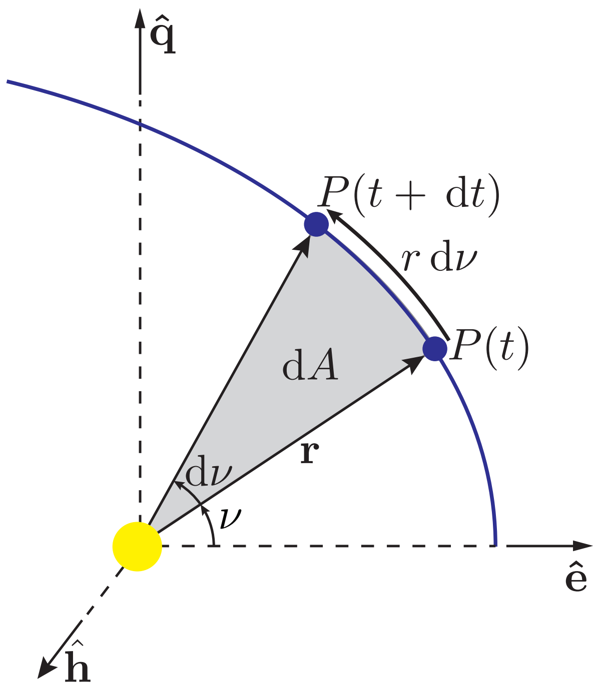
$$\ell = r(\nu = {}^{\pi}/_{2}) = \text{semi-parameter: height above focus}$$
$$c = ae = \text{linear eccentricity: distance from center to focus}$$
$$p = {}^{\ell}/_{e} \text{ focal parameter: distance from focus to directrix}$$

NB: p and ℓ frequently have reversed definitions, depending on the text.

	Definition	e	c	ℓ	p
circle	$x^2 + y^2 = a^2$	0	0	a	∞
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{a^2 - b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2 - b^2}}$
parabola	$y^2 = 4ax$	1	∞	$2a$	$2a^*$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\sqrt{1 + \frac{b^2}{a^2}}$	$\sqrt{a^2 + b^2}$	$\frac{b^2}{\lvert a\rvert}$	$\frac{b^2}{\sqrt{a^2 + b^2}}$

$^*a$ is the focus to vertex distance for a parabola

Kepler's Second and Third Laws



$$r\dot{\nu} = \frac{h}{r} \implies \mathrm{d}A = \frac{1}{2} r (r\,\mathrm{d}\nu) = \frac{1}{2} \underbrace{r^2 \frac{\mathrm{d}\nu}{\mathrm{d}t}}_{h} \mathrm{d}t$$

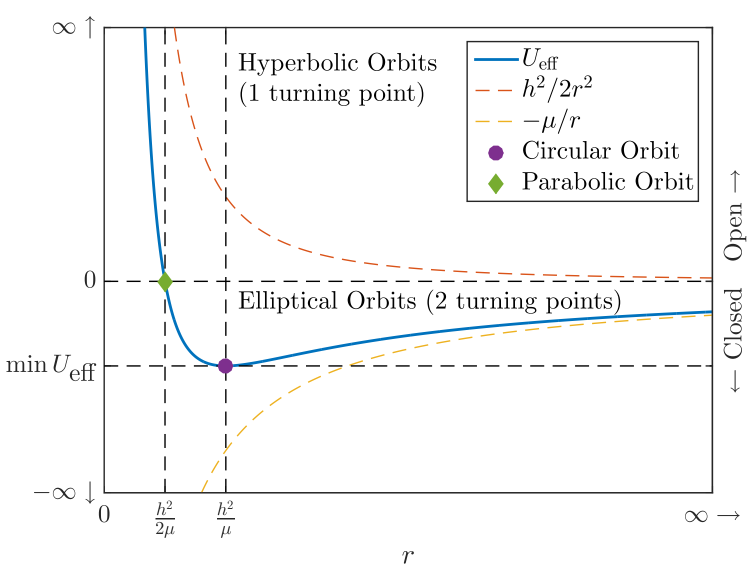
$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{h}{2}}$$

$$\int_0^{T_P} \frac{\mathrm{d}A}{\mathrm{d}t}\,\mathrm{d}t = \int_0^{T_P} \frac{h}{2}\,\mathrm{d}t \implies A = \frac{h}{2} T_P$$

For an ellipse: $A = \pi a b = \pi a^{\frac{3}{2}} \sqrt{\ell} = \dfrac{h}{2} T_P$

$$\boxed{T_P = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}}$$

Specific Energy and Effective Potential



$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = \text{constant}$$

$$\mathcal{E} = -\frac{\mu}{2a}$$

The Vis-Viva Equation

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)$$

$$\mathcal{E} = \frac{\dot{r}^2}{2} + \underbrace{U(r) + \frac{h^2}{2r^2}}_{\triangleq U_{\mathrm{eff}}}$$