4 - Orbit Determination

Dmitry Savransky

Cornell University

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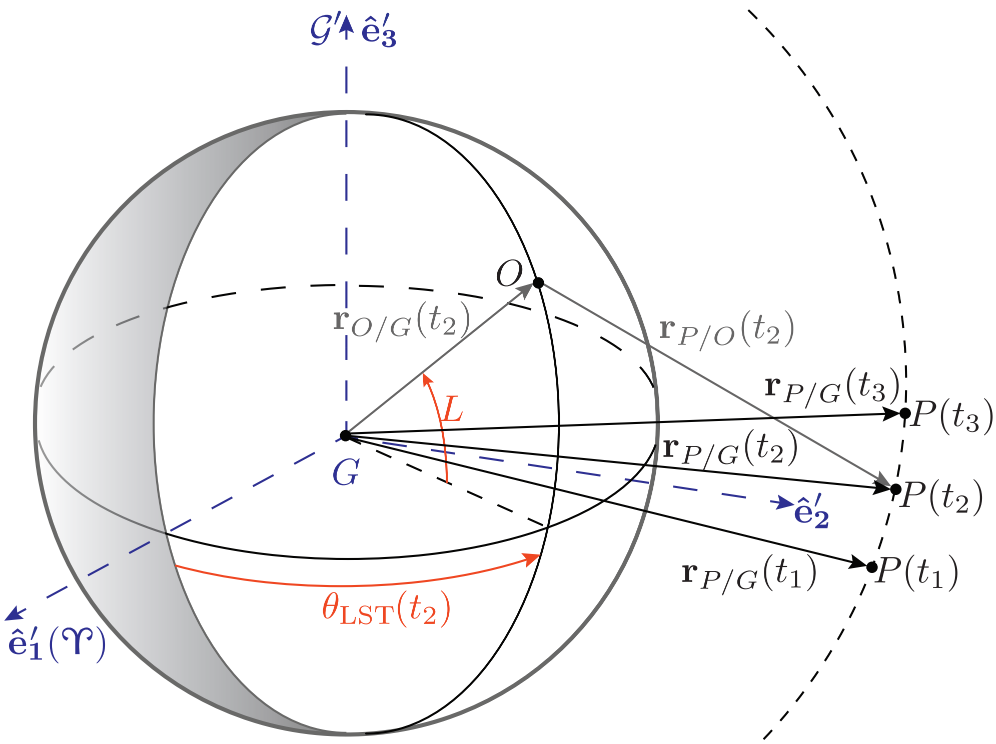
Orbit Determination

A two-body orbit is fully determined by its Keplerian orbital elements, or a simultaneous measurement of the orbiting body's position and velocity with respect to the central body. However, such a measurement is often non-trivial and it is frequently difficult or impossible to accurately establish the distances to distant objects. On the other hand, measuring angles on the sky (which can then be converted to unit vectors of positions and velocities) is much easier, and so there exist multiple methods for using multiple position vectors (or unit vectors) to estimate an orbit's parameters.

Gauss's Method



$$\mathbf{r}_i \triangleq \mathbf{r}_{P/G}(t_i) \qquad r_i \triangleq \|\mathbf{r}_i\|$$

$$\boldsymbol{\rho}_i \triangleq \mathbf{r}_{P/O}(t_i) \qquad \rho_i \triangleq \|\boldsymbol{\rho}_i\|$$

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = 0$$

$$c_1\left(\mathbf{r}_1 \times \mathbf{r}_3\right) = -c_2\left(\mathbf{r}_2 \times \mathbf{r}_3\right)$$

$$c_3\left(\mathbf{r}_1 \times \mathbf{r}_3\right) = -c_2\left(\mathbf{r}_1 \times \mathbf{r}_2\right)$$

$$\mathbf{r}_1 = f_1\mathbf{r}_2 + g_1\mathbf{v}_2$$

$$\mathbf{r}_3 = f_3\mathbf{r}_2 + g_3\mathbf{v}_2$$

$$c_2 \triangleq -1$$

$$c_1 = \frac{g_3}{f_1 g_3 - f_3 g_1}$$

$$c_3 = \frac{g_1}{f_3 g_1 - f_1 g_3}$$

Recall The Series Solutions to f and g Functions

$$\sigma \triangleq \frac{\mu}{r^3} \quad \Longrightarrow \quad \begin{cases} f = 1 - \dfrac{\sigma}{2}(\Delta t)^2 \dots \\ g = \Delta t - \dfrac{\sigma}{6}(\Delta t)^3 \dots \end{cases}$$

$$\Delta t_1 \triangleq t_1 - t_2 \qquad \Delta t_3 \triangleq t_3 - t_2$$

$$c_1 = \underbrace{\frac{\Delta t_3}{\Delta t_3 - \Delta t_1}}_{\triangleq\, a_1} + \underbrace{\frac{\Delta t_3\left((\Delta t_3 - \Delta t_1)^2 - \Delta t_3^2\right)}{6(\Delta t_3 - \Delta t_1)}}_{\triangleq\, b_1}\sigma + \mathcal{O}(\Delta t_i^3)$$

$$c_3 = \underbrace{\frac{-\Delta t_1}{\Delta t_3 - \Delta t_1}}_{\triangleq\, a_3} + \underbrace{\frac{-\Delta t_1\left((\Delta t_3 - \Delta t_1)^2 - \Delta t_1^2\right)}{6(\Delta t_3 - \Delta t_1)}}_{\triangleq\, b_3}\sigma + \mathcal{O}(\Delta t_i^3)$$

Back to Gauss's Method

$$\mathbf{r}_{P/G}(t_i) = \mathbf{r}_{P/O}(t_i) + \mathbf{r}_{O/G}(t_i) \implies \mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{r}_{O/G}(t_i)$$

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = 0 \implies c_1\boldsymbol{\rho}_1 + c_2\boldsymbol{\rho}_2 + c_3\boldsymbol{\rho}_3 = -\left(c_1\mathbf{r}_{O/G}(t_1) + c_2\mathbf{r}_{O/G}(t_2) + c_3\mathbf{r}_{O/G}(t_3)\right)$$

$$\underbrace{\begin{bmatrix}\hat{\boldsymbol{\rho}}_1 & \hat{\boldsymbol{\rho}}_2 & \hat{\boldsymbol{\rho}}_3\end{bmatrix}}_{\triangleq A} \begin{bmatrix} c_1\rho_1 \\ c_2\rho_2 \\ c_3\rho_3 \end{bmatrix} = \begin{bmatrix}\mathbf{r}_{O/G}(t_1) & \mathbf{r}_{O/G}(t_2) & \mathbf{r}_{O/G}(t_3)\end{bmatrix} \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix}$$

$$\begin{bmatrix} c_1\rho_1 \\ c_2\rho_2 \\ c_3\rho_3 \end{bmatrix} = \underbrace{A^{-1}\begin{bmatrix}\mathbf{r}_{O/G}(t_1) & \mathbf{r}_{O/G}(t_2) & \mathbf{r}_{O/G}(t_3)\end{bmatrix}}_{\triangleq B} \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix}$$

$$c_2 = -1 \implies \rho_2 = \underbrace{B_{21}a_1 - B_{22} + B_{23}a_3}_{\triangleq d_1} + (\underbrace{B_{21}b_1 + B_{23}b_3}_{\triangleq d_2})\sigma$$

$$\boxed{r_2^8 = \left(d_1^2 + 2d_1\hat{\boldsymbol{\rho}}_2 \cdot \mathbf{r}_{O/G}(t_2) + \|\mathbf{r}_{O/G}(t_2)\|^2\right) r_2^6 + 2\mu\left(d_2\hat{\boldsymbol{\rho}}_2 \cdot \mathbf{r}_{O/G}(t_2) + d_1 d_2\right) r_2^3 + \mu^2 d_2^2}$$

Gibbs Method

$$\sum_i c_i\mathbf{r}_i = 0 \begin{cases} c_2(\mathbf{r}_1 \times \mathbf{r}_2) = c_3(\mathbf{r}_3 \times \mathbf{r}_1) \\ c_1(\mathbf{r}_1 \times \mathbf{r}_2) = c_3(\mathbf{r}_2 \times \mathbf{r}_3) \\ c_1(\mathbf{r}_3 \times \mathbf{r}_1) = c_2(\mathbf{r}_2 \times \mathbf{r}_3) \end{cases} \qquad \left(\sum_i c_i\mathbf{r}_i\right) \cdot \mathbf{e} = 0 \\ = c_1(\ell - r_1) + c_2(\ell - r_2) + c_3(\ell - r_3)$$

$$\ell(\underbrace{\mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1}_{\triangleq \mathbf{d}}) = \underbrace{r_3(\mathbf{r}_1 \times \mathbf{r}_2) + r_1(\mathbf{r}_2 \times \mathbf{r}_3) + r_2(\mathbf{r}_3 \times \mathbf{r}_1)}_{\triangleq \mathbf{n} = \ell\mathbf{d}}$$

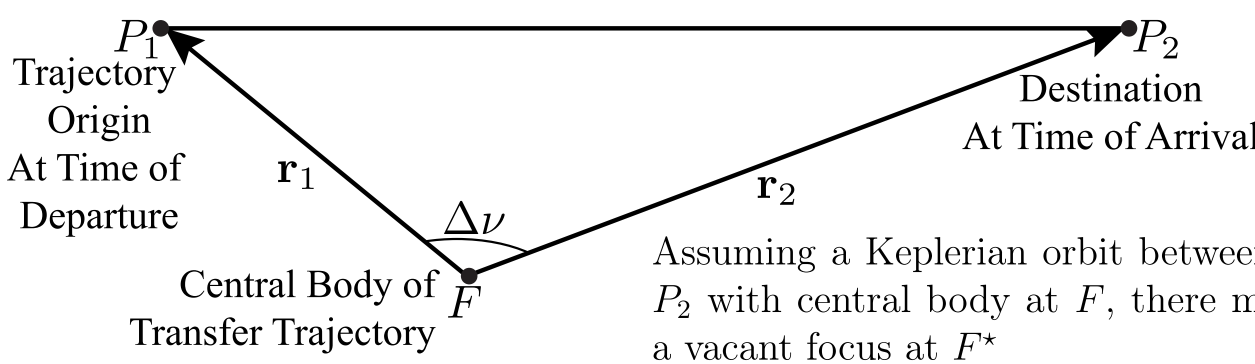
$$\mathbf{n} \times \mathbf{e} = \ell[\underbrace{(r_2 - r_3)\mathbf{r}_1 + (r_3 - r_1)\mathbf{r}_2 + (r_1 - r_2)\mathbf{r}_3}_{\triangleq \mathbf{s}}]$$

$$\boxed{\mathbf{n} \parallel \mathbf{d} \parallel \hat{\mathbf{h}} \qquad \mathbf{s} \parallel \hat{\mathbf{q}} \qquad e = \frac{\|\mathbf{s}\|}{\|\mathbf{d}\|} \quad \ell = \frac{\|\mathbf{n}\|}{\|\mathbf{d}\|}}$$

Orbit Boundary Value Problem

Unlike the previous two methods, Lambert's problem, which is a boundary value problem between two points in space and time, does not yield a unique solution. Rather, it allows us to explore *all* of the possible two-body orbits about a central body that connect these points in space. As such, Lambert solvers (codes to solve arbitrary Lambert problems) are an incredibly important aspect of trajectory design.

Lambert's Problem



Assuming a Keplerian orbit between P_1 and P_2 with central body at F, there must exist a vacant focus at $F^\star$

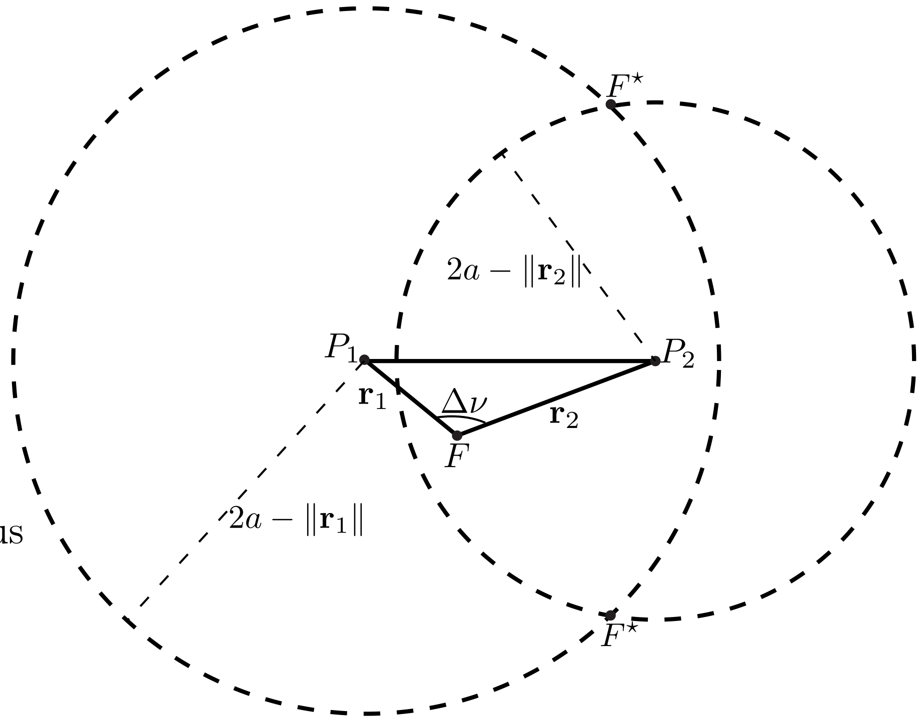
If the transfer between P_1 and P_2 is an ellipse, we know $r' + r = 2a$ so:

$$\left.\begin{array}{l}\overline{P_1F} + \overline{P_1F^\star} \\ \overline{P_2F} + \overline{P_2F^\star}\end{array}\right\} = 2a \quad \Longrightarrow \quad \begin{array}{l}\overline{P_1F^\star} = 2a - \|\mathbf{r}_1\| \\ \overline{P_2F^\star} = 2a - \|\mathbf{r}_2\|\end{array}$$

Lambert's Problem: Location of the Vacant Focus

- Vacant focus must be at intersection of two circles centered at P_1 and P_2 with radii of $2a - \|\mathbf{r}_1\|$ and $2a - \|\mathbf{r}_2\|$, respectively

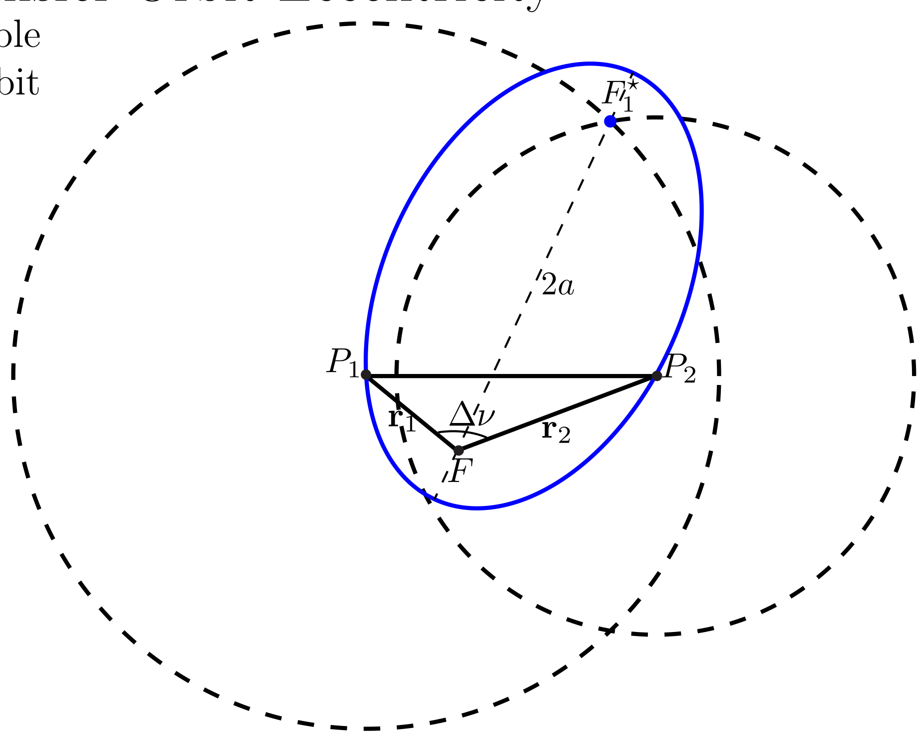
- Selecting transfer orbit a determines the possible locations of the vacant focus and sets the transfer orbit specific energy ($\mathcal{E} = -\dfrac{\mu}{2a}$) and period



Lambert's Problem: Transfer Orbit Eccentricity

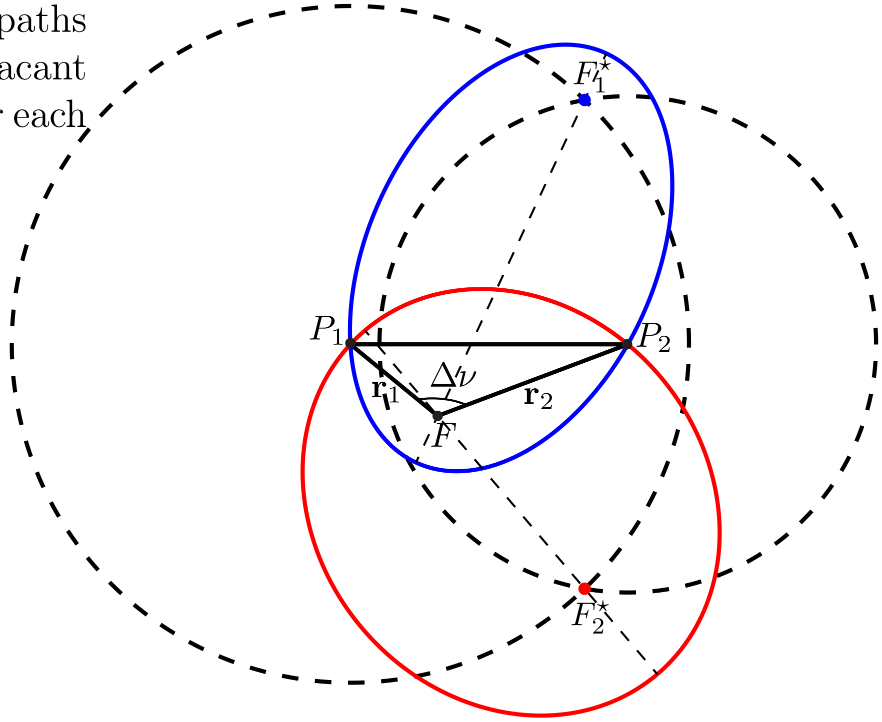
Selecting one of the two possible vacant foci sets the transfer orbit eccentricity:

$$\overline{FF^\star} = 2ae$$



Lambert's Problem: Closed Transfer Orbits

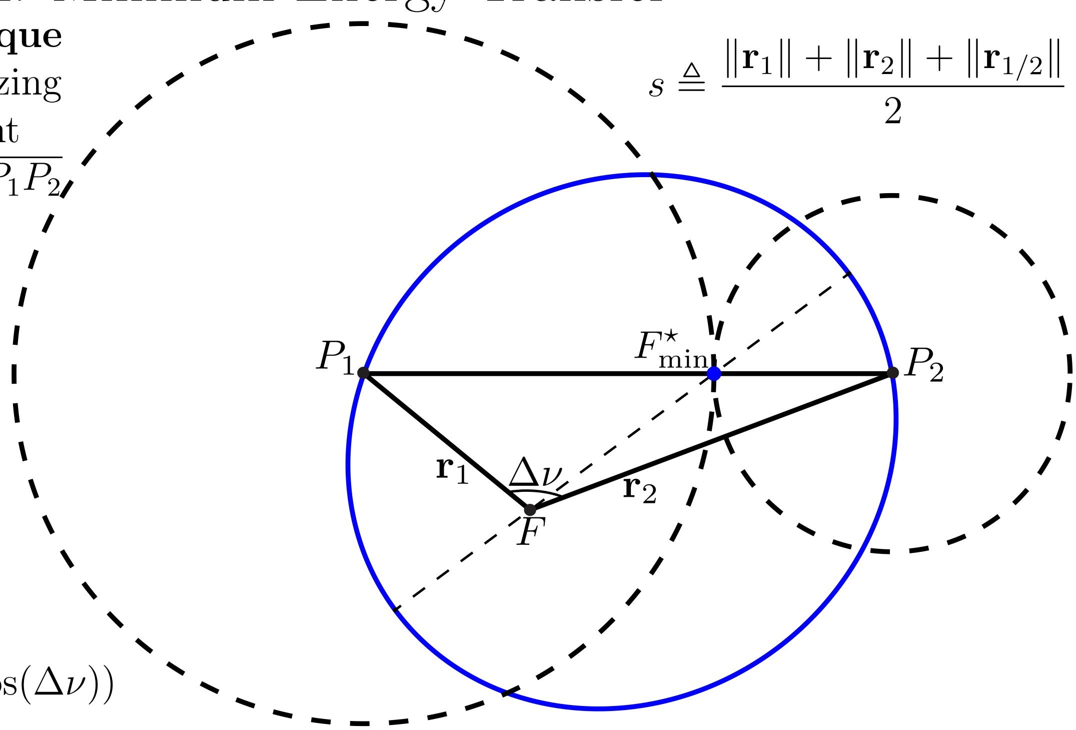
There are four possible transfer paths for each semi-major axis: 2 vacant foci, and 2 directions of travel for each



Lambert's Problem: Minimum Energy Transfer

There is always a **unique** elliptical orbit minimizing energy, where the vacant focus lies on the chord $\overline{P_1P_2}$

$$s \triangleq \frac{\|\mathbf{r}_1\| + \|\mathbf{r}_2\| + \|\mathbf{r}_{1/2}\|}{2}$$



$$a_{\min} = \frac{s}{2}$$

$$e_{\min} = \sqrt{1 - \frac{2\ell_{\min}}{s}}$$

$$\ell_{\min} = \frac{\|\mathbf{r}_1\|\|\mathbf{r}_2\|}{\|\mathbf{r}_{1/2}\|}(1 - \cos(\Delta\nu))$$

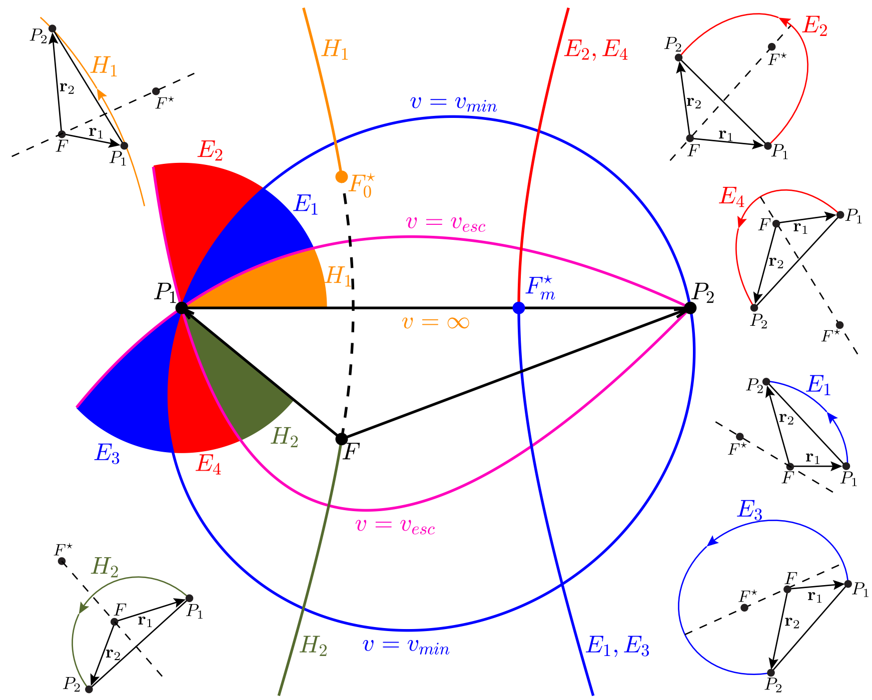
Lambert's Problem: Location of Vacant Focus

Location of the vacant focus is given by the hyperbola:

$$a_F = -\left|\frac{\|\mathbf{r}_1\| - \|\mathbf{r}_2\|}{2}\right|$$

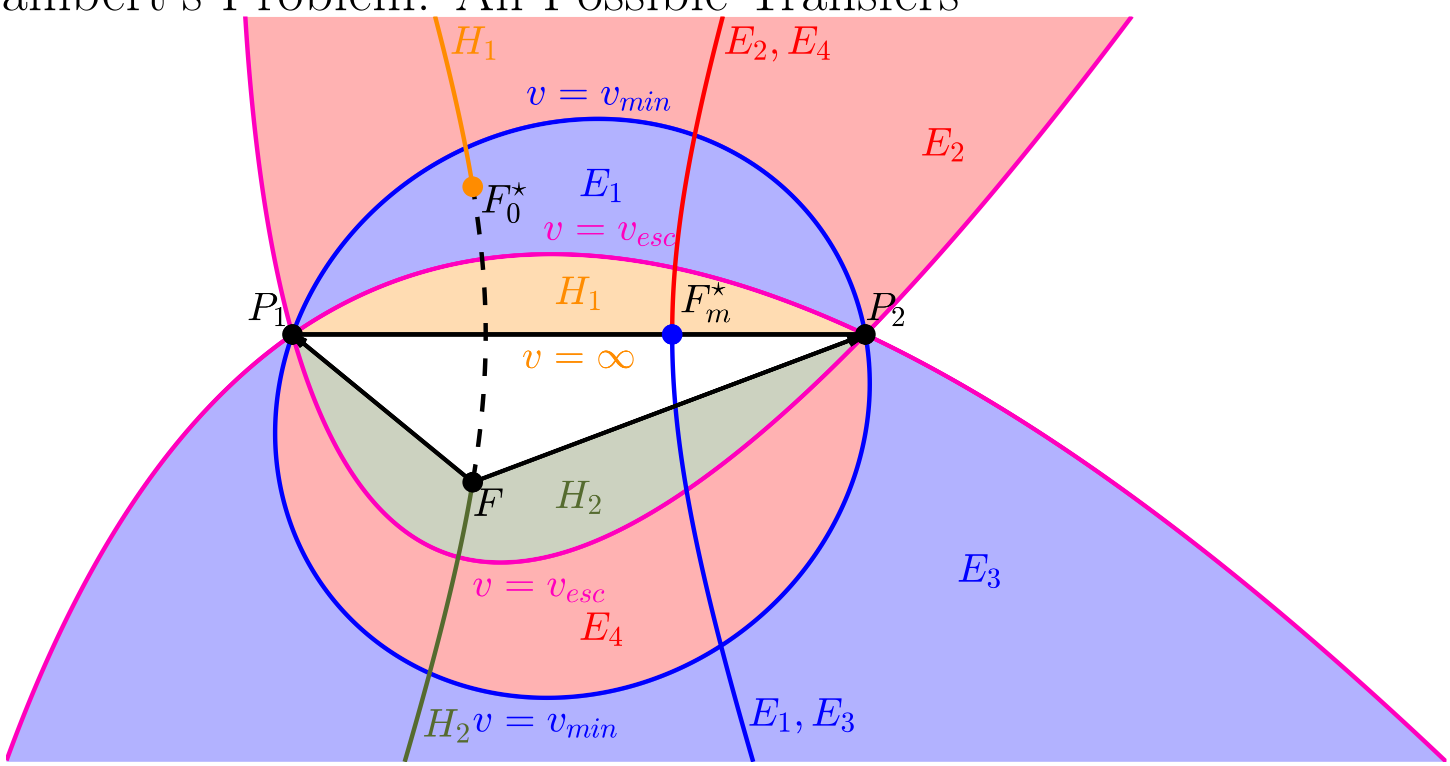
$$e_F = \left|\frac{\|\mathbf{r}_{1/2}\|}{\|\mathbf{r}_1\| - \|\mathbf{r}_2\|}\right|$$

NB: This is **not** a transfer orbit itself.



Based on Kaplan (1976)

Lambert's Problem: All Possible Transfers



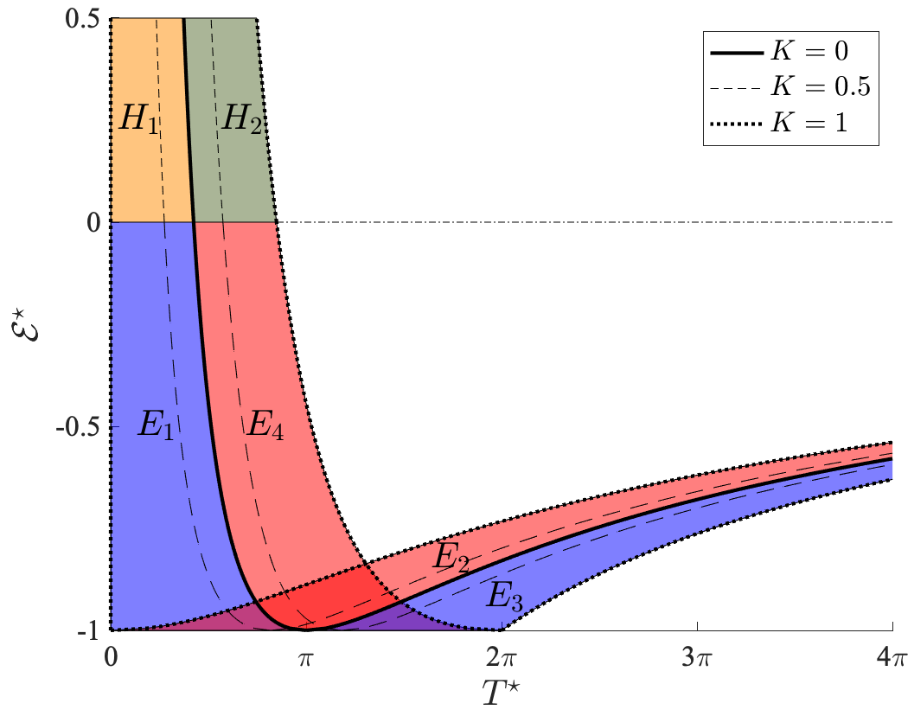
Lambert's Time of Flight Theorem

	$t = \frac{1}{\sqrt{\mu}} \int_{s-c}^{s} \frac{r}{\sqrt{2r - r^2/a}}\,\mathrm{d}r$	ℓ
$E_{1/4}$	$\frac{T_P}{2\pi}\left[(\alpha - \sin\alpha) \mp (\beta - \sin\beta)\right]$	$\frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\frac{\alpha \pm \beta}{2}\right)$
$E_{2/3}$	$T_P - \frac{T_P}{2\pi}\left[(\alpha - \sin\alpha) \pm (\beta - \sin\beta)\right]$	$\frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\frac{\alpha \mp \beta}{2}\right)$
Parabolas	$\frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{\frac{3}{2}} \mp (s-c)^{\frac{3}{2}}\right]$	$\frac{4(s-r_1)(s-r_2)}{c^2}\left[\sqrt{\frac{s}{2}} \pm \sqrt{\frac{s-c}{2}}\right]^2$
$H_{1/2}$	$\sqrt{\frac{-a^3}{\mu}}\left[(\sinh\gamma - \gamma) \mp (\sinh\delta - \delta)\right]$	$\frac{-4a(s-r_1)(s-r_2)}{c^2}\sinh^2\left(\frac{\gamma \pm \delta}{2}\right)$

$$c = \|\mathbf{r}_{1/2}\| \qquad r_1 = \|\mathbf{r}_1\| \qquad r_2 = \|\mathbf{r}_2\| \qquad 2s = \|\mathbf{r}_1\| + \|\mathbf{r}_2\| + \|\mathbf{r}_{1/2}\|$$

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}} \quad \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{s-c}{2a}} \quad \sinh\left(\frac{\gamma}{2}\right) = \sqrt{\frac{s}{-2a}} \quad \sinh\left(\frac{\delta}{2}\right) = \sqrt{\frac{s-c}{-2a}}$$

Lambert's Problem Non-Dimensionalized



$$\mathcal{E}^\star \triangleq -\frac{a_{\min}}{a}$$

$$T^\star \triangleq \sqrt{\frac{\mu}{a_{\min}^3}}t$$

$$K \triangleq 1 - \frac{\|\mathbf{r}_{1/2}\|}{s}$$

Based on Kaplan (1976)

Recall Universal Variables

$$\left.\begin{aligned}\mathbf{r}_2 &= f\mathbf{r}_1 + g\mathbf{v}_1\\ \mathbf{v}_2 &= \dot{f}\mathbf{r}_1 + \dot{g}\mathbf{v}_1\end{aligned}\right\} f\dot{g} - \dot{f}g = 1 \qquad \begin{aligned} r_1 &\triangleq \|\mathbf{r}_1\| \\ r_2 &\triangleq \|\mathbf{r}_2\|\end{aligned} \qquad \dot{\chi} \triangleq \frac{\sqrt{\mu}}{r} \qquad \psi \triangleq \frac{\chi^2}{a}$$

$$c_2 \triangleq \begin{cases} \dfrac{1-\cos\left(\sqrt{\psi}\right)}{\psi} & \psi \geq 0 \\ \dfrac{1-\cosh\left(\sqrt{-\psi}\right)}{\psi} & \psi < 0 \end{cases} \qquad c_3 \triangleq \begin{cases} \dfrac{\sqrt{\psi}-\sin\left(\sqrt{\psi}\right)}{\sqrt{\psi}^3} & \psi \geq 0 \\ \dfrac{\sinh\left(\sqrt{-\psi}\right)-\sqrt{-\psi}}{\sqrt{-\psi}^3} & \psi < 0 \end{cases}$$

$$\begin{aligned}
f &= 1 - \frac{\chi^2}{r_1}c_2 &&= 1 - \frac{r_2}{\ell}\left(1-\cos(\Delta\nu)\right)\\
g &= \Delta t - \frac{\chi^3}{\sqrt{\mu}}c_3 &&= \frac{r_2 r_1}{\sqrt{\mu \ell}}\sin(\Delta\nu)\\
\dot{f} &= \frac{\sqrt{\mu}}{r_2 r_1}\chi(\psi c_3 - 1) &&= \sqrt{\frac{\mu}{\ell}}\tan\left(\frac{\Delta\nu}{2}\right)\left(\frac{1-\cos(\Delta\nu)}{\ell} - \frac{1}{r_1} - \frac{1}{r_2}\right)\\
\dot{g} &= 1 - \frac{\chi^2}{r_2}c_2 &&= 1 - \frac{r_1}{\ell}\left(1-\cos(\Delta\nu)\right)
\end{aligned}$$

Lambert's Problem: Universal Variables

$$f: \qquad \chi^2 = \frac{r_1 r_2}{\ell c_2}\left(1-\cos(\Delta\nu)\right) = \frac{y}{c_2}$$

$$\dot{f}: \qquad \underbrace{\frac{r_1 r_2}{\ell}\left(1-\cos(\Delta\nu)\right)}_{\triangleq\, y} = r_1 + r_2 + \underbrace{\left(r_1 r_2 \frac{\sin^2(\Delta\nu)}{1-\cos(\Delta\nu)}\right)^{1/2}}_{\triangleq\, A}\frac{\psi c_3 - 1}{\sqrt{c_2}}$$

$$g: \qquad \Delta t - \frac{\chi^3}{\sqrt{\mu}}c_3 = A\sqrt{\frac{y}{\mu}}$$

$$\boxed{\begin{gathered} f = 1 - \frac{y}{r_1} \qquad g = A\sqrt{\frac{y}{\mu}} \qquad \dot{g} = 1 - \frac{y}{r_2} \\ \mathbf{v}_1 = \frac{\mathbf{r}_2 - f\mathbf{r}_1}{g} \qquad \mathbf{v}_2 = \frac{\dot{g}\mathbf{r}_2 - \mathbf{r}_1}{g}\end{gathered}}$$