5 - Perturbations from Circular and Elliptic Orbits

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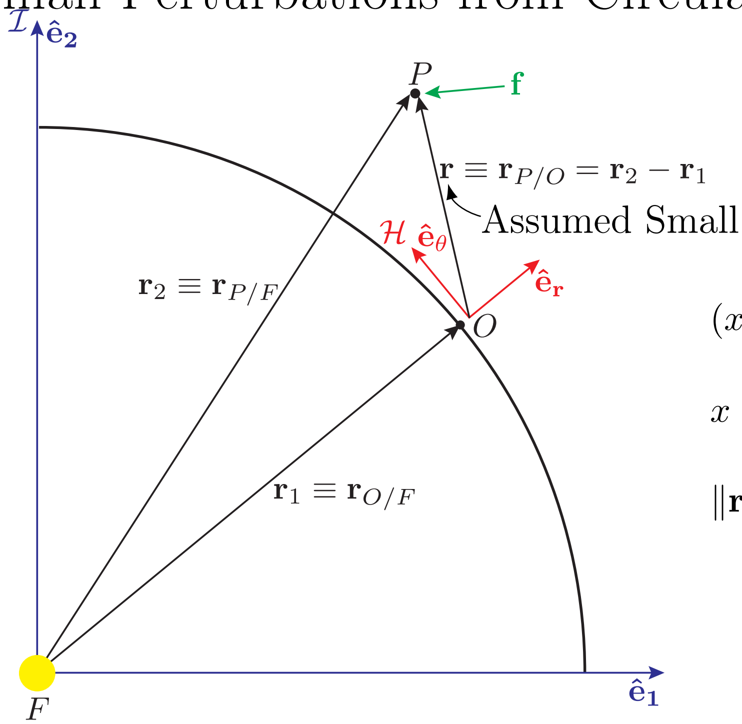
Perturbations from Circular and Elliptic Orbits

Now that we have deeply explored two-body orbits, it is time to go beyond them. While two-body orbits are an excellent initial approximation to many real systems, we can gain a lot of fidelity by incorporating additional effects as perturbations to the two-body model—that is, allowing the orbital elements that are constants in two-body systems to gradually evolve in time in response to various additional gravitational and non-gravitational forces. As an initial step towards a completely general treatment of perturbations, we will consider small deviations from circular orbits. We will also review the basic impulsive model of orbital control (i.e., the instantaneous change in orbital velocity while preserving orbital radius) that serves as a key tool in preliminary orbital maneuver design.

Small Perturbations from Circular Orbits



$$ {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \underbrace{\frac{\mu}{\|\mathbf{r}_1\|^3}}_{n^2}\left(\mathbf{r}_1 - \left(\frac{\|\mathbf{r}_1\|}{\|\mathbf{r}_2\|}\right)^3 \mathbf{r}_2\right) + \mathbf{f} $$

$$ \|\mathbf{r}_2\|^{-3} = \left[(\mathbf{r}_1 + \mathbf{r})\cdot(\mathbf{r}_1 + \mathbf{r})\right]^{-\frac{3}{2}} $$

$$ (x+y)^r = \sum_{k=0}^{\infty}\binom{r}{k} x^{r-k}y^k = x^r + r x^{r-1} y + \ldots $$

$$ x = \|\mathbf{r}_1\|^2 \qquad y = 2\mathbf{r}\cdot\mathbf{r}_1 + \|\mathbf{r}\|^2 \qquad r = -\frac{3}{2} $$

$$ \|\mathbf{r}_2\|^{-3} = \|\mathbf{r}_1\|^{-3}\left(1 - \frac{3}{2}\left(\frac{2\mathbf{r}\cdot\mathbf{r}_1}{\|\mathbf{r}_1\|^2}\right) + \mathcal{O}\left(\mathbf{r}^2\right)\right) $$

$$ \boxed{{}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} \approx n^2\left(-\mathbf{r} + 3\frac{\mathbf{r}_1\cdot\mathbf{r}}{\|\mathbf{r}_1\|^2}\mathbf{r}_1\right) + \mathbf{f}} $$

Euler-Hill/Clohessy-Wiltshire Equations

$$ \boxed{{}^{\mathcal{H}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} \approx -2n\hat{\mathbf{e}}_3 \times {}^{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} - n^2\left(\hat{\mathbf{e}}_3 \times \left(\hat{\mathbf{e}}_3 \times \mathbf{r}\right)\right) - n^2\left(\mathbf{r} - 3\left(\hat{\mathbf{e}}_\mathbf{r}\cdot\mathbf{r}\right)\hat{\mathbf{e}}_\mathbf{r}\right) + \mathbf{f}} $$

$$ \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{\mathcal{H}} = \underbrace{\begin{bmatrix}2n\dot{y}\\ -2n\dot{x}\\ 0\end{bmatrix}_{\mathcal{H}} + \begin{bmatrix}n^2x\\ n^2y\\ 0\end{bmatrix}_{\mathcal{H}}}_{\text{Rotating Frame}} - \underbrace{\begin{bmatrix}n^2x\\ n^2y\\ n^2z\end{bmatrix}_{\mathcal{H}} + \begin{bmatrix}3n^2x\\ 0\\ 0\end{bmatrix}_{\mathcal{H}}}_{\text{Gravity Perturbations}} + \underbrace{[\mathbf{f}]_{\mathcal{H}}}_{\text{Other Perturbations}} $$

$$ \boxed{\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2x &= \mathbf{f}\cdot\hat{\mathbf{e}}_\mathbf{r} \triangleq f_x \\ \ddot{y} + 2n\dot{x} &= \mathbf{f}\cdot\hat{\mathbf{e}}_\theta \triangleq f_y \\ \ddot{z} + n^2z &= \mathbf{f}\cdot\hat{\mathbf{e}}_3 \triangleq f_z \end{aligned}} $$

Natural Motion

$$\ddot{x} - 2n\dot{y} - 3n^2 x = 0 \qquad X(s) \triangleq \mathcal{L}\{x(t)\}$$
$$\ddot{y} + 2n\dot{x} = 0 \qquad \dot{Y}(s) \triangleq \mathcal{L}\{\dot{y}(t)\}$$
$$\ddot{z} + n^2 z = 0$$

$$\mathcal{L}\left\{\begin{bmatrix}\ddot{x} - 2n\dot{y} - 3n^2 x \\ \ddot{y} + 2n\dot{x}\end{bmatrix} = 0\right\} \Longrightarrow \underbrace{\begin{bmatrix} s^2 - 3n^2 & -2n \\ 2ns & s\end{bmatrix}}_{\triangleq A} \begin{bmatrix} X(s) \\ \dot{Y}(s)\end{bmatrix} = 0 - \text{Initial Conditions}$$

$$\det A = s(s^2 - 3n^2) + 4n^2 s = 0 \Longrightarrow s = 0, \pm in$$

$$x(t) = 4x_0 - 3x_0\cos(nt) + \frac{\dot{x}_0}{n}\sin(nt) + 2\frac{\dot{y}_0}{n} - 2\frac{\dot{y}_0\cos(nt)}{n}$$
$$y(t) = -6x_0 nt + 6x_0\sin(nt) + 2\cos(nt)\frac{\dot{x}_0}{n} - 2\frac{\dot{x}_0}{n} + \frac{\dot{y}_0}{n}\left(4\sin(nt) - 3nt\right) + y_0$$
$$z(t) = z_0\cos(nt) + \frac{\dot{z}_0}{n}\sin(nt)$$

Mode 1: $s = 0$

$$A = \begin{bmatrix} s^2 - 3n^2 & -2n \\ 2ns & s\end{bmatrix} = \begin{bmatrix} -3n^2 & -2n \\ 0 & 0\end{bmatrix}$$

$x_0 =$ arbitrary
$y_0 =$ arbitrary
$\dot{x}_0 =$ arbitrary (often set to 0)
$\dot{y}_0 = \dfrac{-3nx_0}{2}$

$\xRightarrow{\dot{x}_0 = 0}$

$x(t) = x_0$
$y(t) = -\dfrac{3}{2}x_0 nt + y_0$

Body is on a circular orbit of radius $\|\mathbf{r}_1\| + x_0$

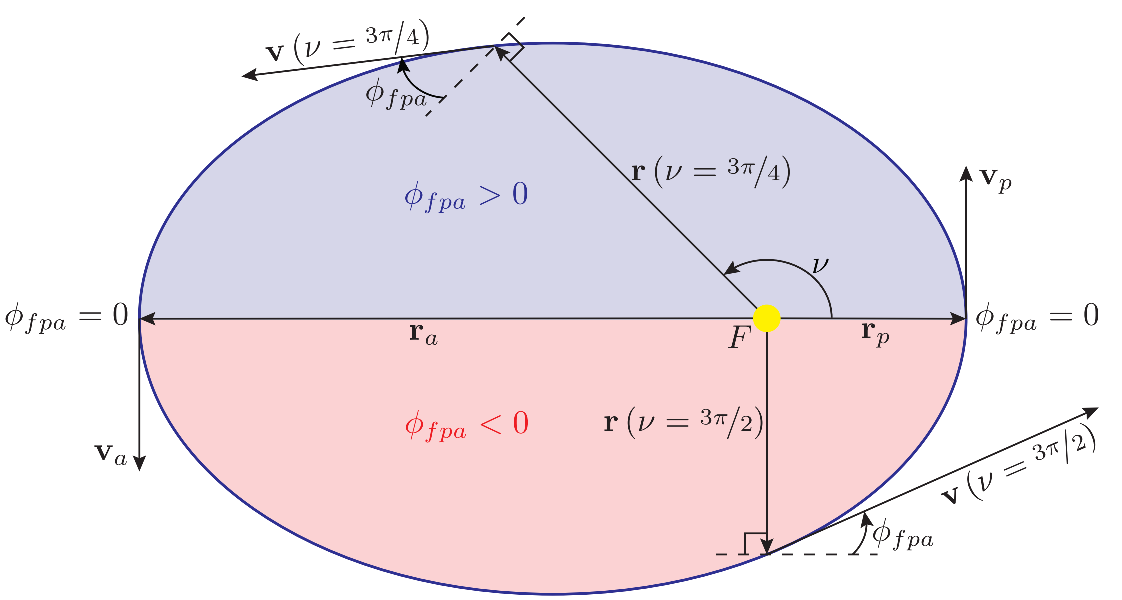
Modes 2/3: $s = \pm in$

$$A = \begin{bmatrix} s^2 - 3n^2 & -2n \\ 2ns & s \end{bmatrix} = \begin{bmatrix} -n^2 - 3n^2 & -2n \\ \pm 2in^2 & \pm in \end{bmatrix}$$

$$\begin{aligned} x_0 &= \text{arbitrary} \\ y_0 &= \text{arbitrary} \\ \dot{x}_0 &= \text{arbitrary (often set to 0)} \\ \dot{y}_0 &= -2nx_0 \end{aligned} \quad \overset{\dot{x}_0 = 0}{\Longrightarrow} \quad \begin{aligned} x(t) &= x_0 \cos(nt) \\ y(t) &= -2x_0 \sin(nt) + y_0 \end{aligned}$$

Oscillatory motion about O in the rotating frame

Flight Path Angle



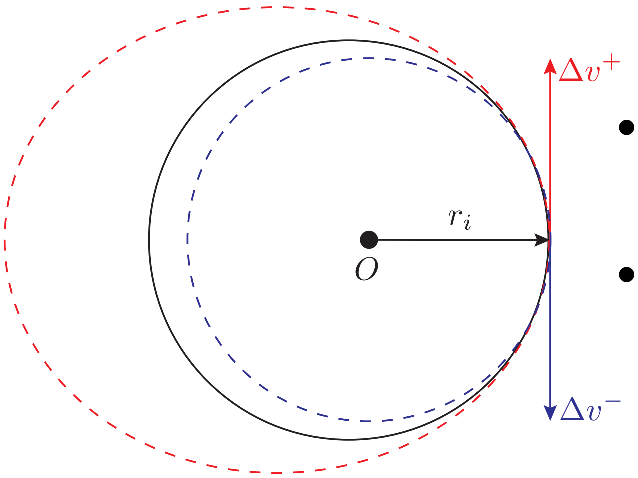
ϕ_{fpa}: The angle between the local horizontal and the velocity vector such that

$$\begin{aligned} h &= \|\mathbf{r} \times \mathbf{v}\| \\ &= rv\cos(\phi_{fpa}) \end{aligned}$$

$$\cos\phi_{fpa} = \frac{r\dot{\nu}}{v} = \frac{1 + e\cos\nu}{\sqrt{1 + 2e\cos\nu + e^2}} \qquad \sin\phi_{fpa} = \frac{\dot{r}}{v} = \frac{e\sin\nu}{\sqrt{1 + 2e\cos\nu + e^2}}$$

Tangential Burns

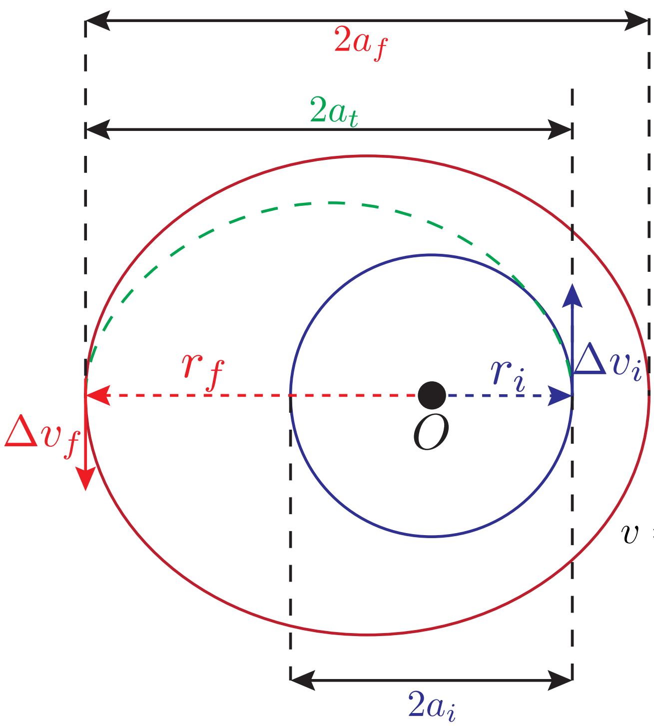
- $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad \Longrightarrow \quad a = \frac{\mu r}{2\mu - rv^2}$
- $e^2 = \mathbf{e}\cdot\mathbf{e} = \left\|\frac{\mathbf{v}\times\mathbf{h}}{\mu} - \frac{\mathbf{r}}{r}\right\|^2 = \frac{\left(rv^2-\mu\right)^2}{\mu^2} + \frac{(\mathbf{r}\cdot\mathbf{v})^2 v^2}{\mu^2} - \frac{2(\mathbf{r}\cdot\mathbf{v})^2\left(rv^2-\mu\right)}{\mu^2 r}$



- But at $\phi_{fpa} = 0$, $\mathbf{r}\cdot\mathbf{v} = 0$ so:
  $e = \frac{|rv^2 - \mu|}{\mu}$
- Increasing velocity at turning points also increases the semi-major axis and eccentricity

NB: As eccentricity cannot go below zero, burning from a circular orbit will always result in an *increase* in eccentricity, regardless of whether the semi-major axis increases or decreases.

Hohmann Transfers



A Hohmann transfer requires two tangential burns.

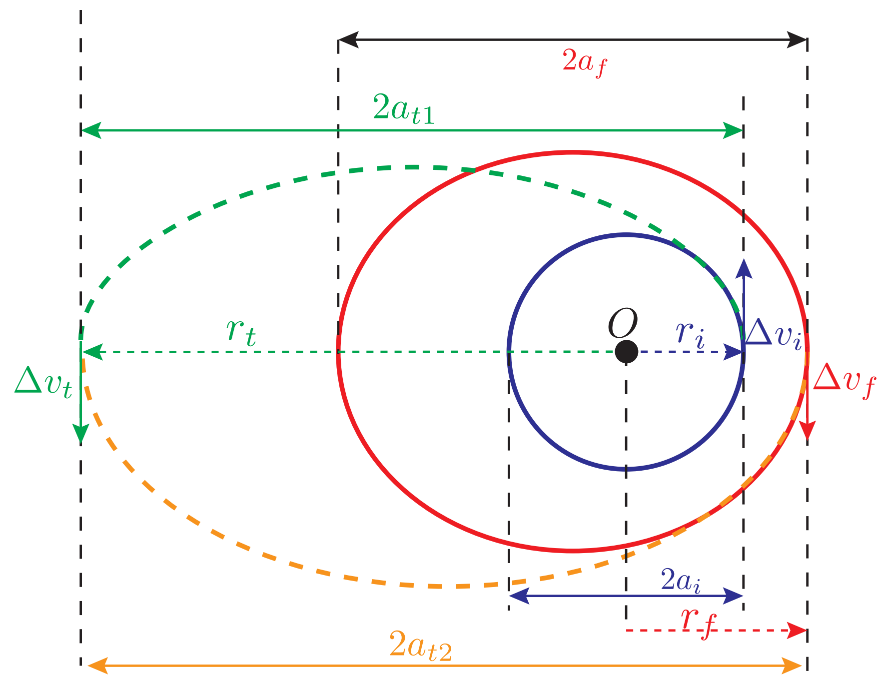
$$a_{\text{transfer}} \equiv a_t = \frac{r_i + r_f}{2}$$

$$t_{\text{transfer}} = \frac{1}{2}T_P^{\text{transfer}} = \pi\sqrt{\frac{a_t^3}{\mu}}$$

$$\Delta v = |\Delta v_i| + |\Delta v_f|$$

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} \begin{cases} \Delta v_i = \underbrace{\sqrt{\frac{2\mu}{r_i} - \frac{\mu}{a_t}}}_{v_{t_i}} - \underbrace{\sqrt{\frac{2\mu}{r_i} - \frac{\mu}{a_i}}}_{v_i} \\ \Delta v_f = \underbrace{\sqrt{\frac{2\mu}{r_f} - \frac{\mu}{a_f}}}_{v_f} - \underbrace{\sqrt{\frac{2\mu}{r_f} - \frac{\mu}{a_t}}}_{v_{t_f}} \end{cases}$$

Bi-Elliptic Transfers

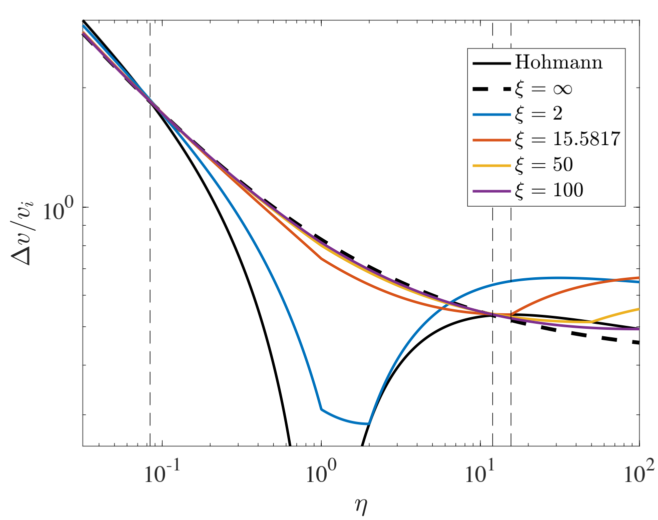


Hohmann vs. Bi-Elliptic

$$\text{Hohmann}: \quad \frac{\|\Delta v_i\| + \|\Delta v_f\|}{v_i} = \left|\sqrt{\frac{2\eta}{1+\eta}} + \sqrt{\frac{1}{\eta}} - \left(1 + \sqrt{\frac{2}{\eta(1+\eta)}}\right)\right|$$

$$\text{Bi} - \text{Elliptic}: \frac{\|\Delta v_i\| + \|\Delta v_t\| + \|\Delta v_f\|}{v_i} = \left|\sqrt{\frac{2\xi}{1+\xi}} - 1\right| + \left|\sqrt{\frac{2\eta}{\xi(\eta+\xi)}} - \sqrt{\frac{2}{\xi(1+\xi)}}\right| + \left|\sqrt{\frac{1}{\eta}} - \sqrt{\frac{2\xi}{\eta(\eta+\xi)}}\right|$$

$$\Rightarrow \lim_{r_t \to \infty} \left(\frac{\|\Delta v_i\| + \|\Delta v_t\| + \|\Delta v_f\|}{v_i}\right) = \sqrt{2} - 1 + \left|\sqrt{\frac{1}{\eta}} - \sqrt{\frac{2}{\eta}}\right|$$



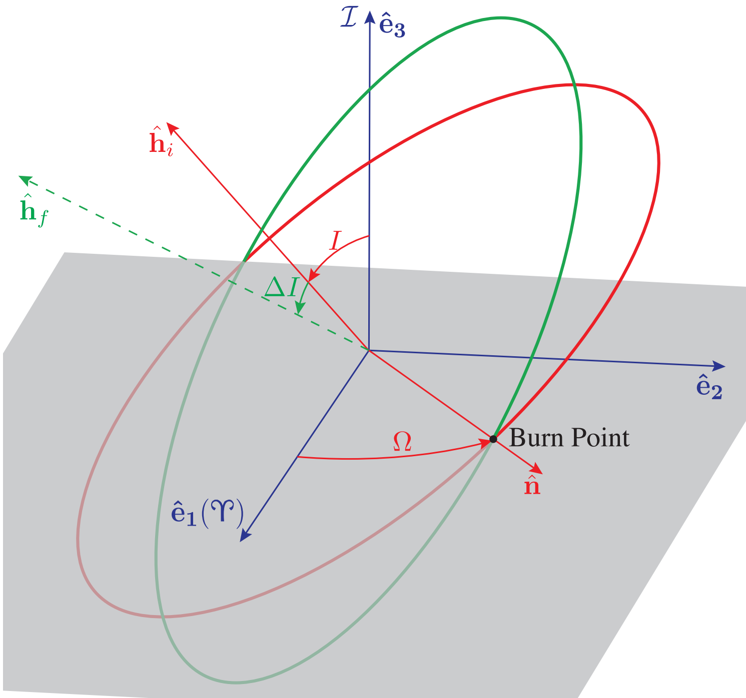
$$\eta \triangleq \frac{a_f}{a_i}$$

$$\xi \triangleq \frac{r_t}{a_i}$$

Hohmann maximum (for $\eta > 1$) occurs at $\eta = 15.5817$

Hohmann and $\eta = \infty$ intersect at $\eta = 11.93876^{\pm 1}$

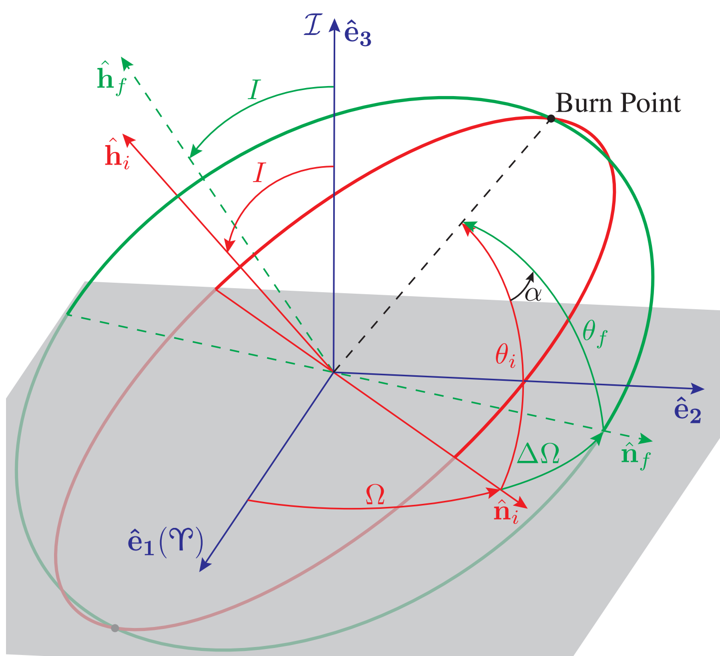
Inclination Changes (Super Costly!)



$$\Delta v = 2 v_i \cos(\phi_{fpa}) \sin\left(\frac{\Delta I}{2}\right)$$

- NB: Δv $\propto v_i$. For elliptical orbits, one of the two nodes will be less costly
- For $\Delta I = 60^\circ$, $\Delta v = v_i$
- To leave Ω unchanged, burn must occur on the line of nodes

Ascending Node Change



- In general, elliptical orbits require multiple burns to change **only** Ω, but circular orbits can do it in one
- The burn occurs on the original orbit at argument of latitude $\theta_i = \omega_i + \nu_i$ resulting with the spacecraft on the final orbit at θ_f, with a burn angle α

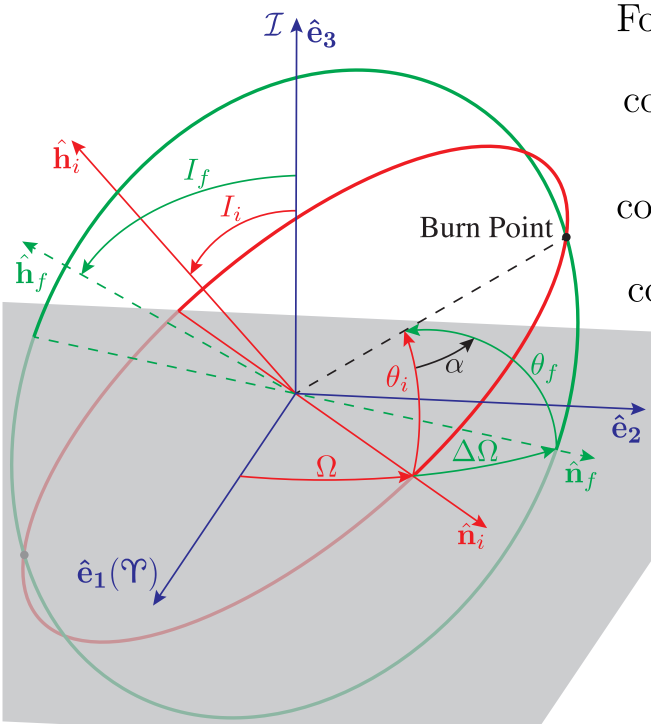
$$\cos(\theta_i) = \tan I \left(\frac{\cos(\Delta\Omega) - \cos\alpha}{\sin\alpha}\right)$$

$$\cos(\theta_f) = \cos I \sin I \left(\frac{1 - \cos(\Delta\Omega)}{\sin\alpha}\right)$$

$$\cos(\alpha) = \cos^2 I + \sin^2 I \cos(\Delta\Omega)$$

$$\Delta v^{\text{circ}} = 2 v_i \sin\left(\frac{\alpha}{2}\right)$$

Ascending Node and Inclination Change



For circular orbits:

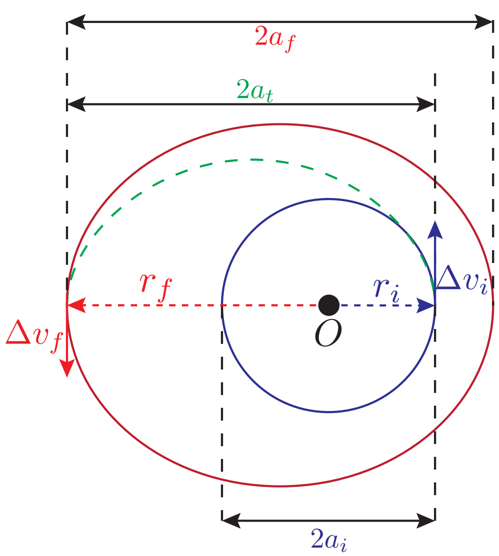
$$\cos(\theta_i) = \frac{\sin(I_f)\cos(\Delta\Omega) - \cos(\alpha)\sin(I_i)}{\sin(\alpha)\cos(I_i)}$$

$$\cos(\theta_f) = \frac{\cos(I_i)\sin(I_f) - \sin(I_i)\cos(I_f)\cos(\Delta\Omega)}{\sin(\alpha)}$$

$$\cos(\alpha) = \cos(I_i)\cos(I_f) + \sin(I_i)\sin(I_f)\cos(\Delta\Omega)$$

$$\Delta v^{\text{circ}} = 2 v_i \sin\left(\frac{\alpha}{2}\right)$$

Hohmann Transfer + Inclination Change



For a total inclination change of ΔI:

- Change by $x\Delta I$ on initial burn
- Change by $(1-x)\Delta I$ on final burn

- Select x to minimize total Δv:

$$\sin(x\Delta I) = \frac{\Delta v_i v_f v_{t_f} \sin((1-x)\Delta I)}{\Delta v_f v_i v_{t_i}}$$

- A good approximation is:

$$x \approx \frac{1}{\Delta I}\tan^{-1}\left(\frac{\sin(\Delta I)}{\frac{v_i v_{t_i}}{v_f v_{t_f}} + \cos(\Delta I)}\right)$$

- Δvs for the combined maneuvers are:

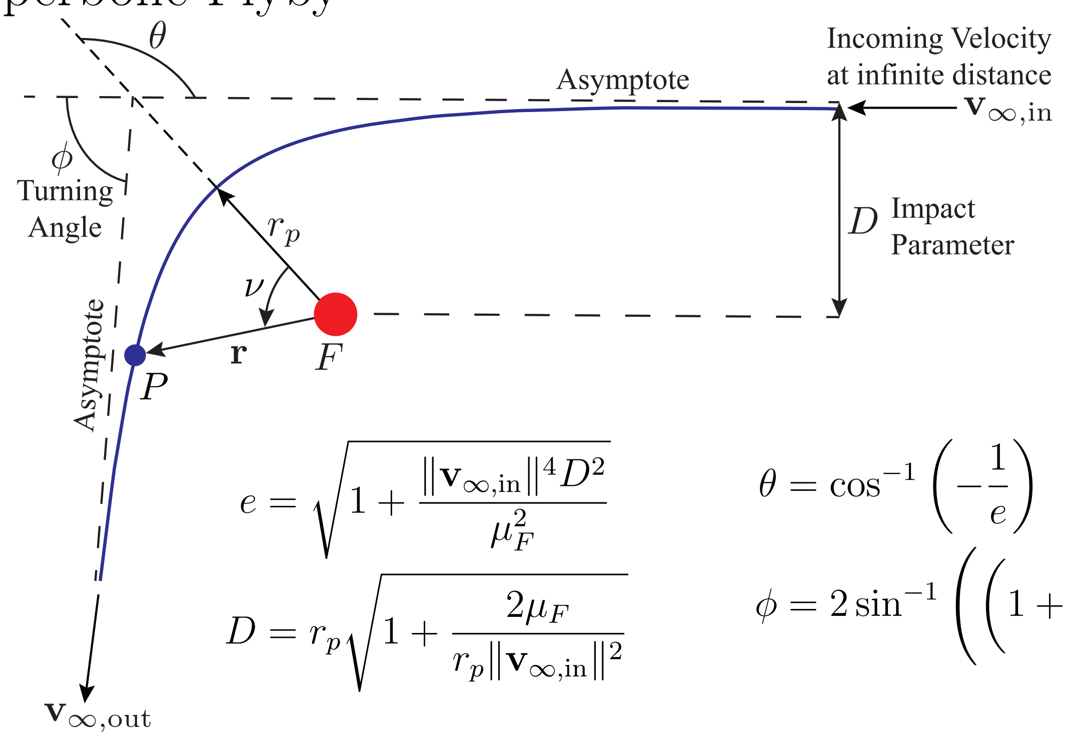
$$\Delta v_i = \sqrt{v_i^2 + v_{t_i}^2 - 2 v_i v_{t_i} \cos(x\Delta I)}$$

$$\Delta v_f = \sqrt{v_f^2 + v_{t_f}^2 - 2 v_f v_{t_f} \cos((1-x)\Delta I)}$$

General Impulsive Maneuvers

- Remember: $a, e, I, \omega, \Omega, \nu(t) \Longleftrightarrow \mathbf{r}(t), \mathbf{v}(t)$
- Before Burn: $\left.\begin{matrix}\mathbf{r}_i \\ \mathbf{v}_i\end{matrix}\right\} a_i, e_i, I_i, \omega_i, \Omega_i, \nu_i(t)$
- After Burn: $\left.\begin{matrix}\mathbf{r}_f \equiv \mathbf{r}_i \\ \mathbf{v}_f = \mathbf{v}_i + \boldsymbol{\Delta}\mathbf{v}\end{matrix}\right\} a_f, e_f, I_f, \omega_f, \Omega_f, \nu_f(t)$
- You can always solve for the $\boldsymbol{\Delta}\mathbf{v}$ to produce the desired change in orbital elements as long as the initial and final orbits intersect at the burn location
- These maneuvers are not guaranteed to be feasible or optimal
- Typical approach is numerical optimization

Hyperbolic Flyby



$$e = \sqrt{1 + \frac{\|\mathbf{v}_{\infty,\text{in}}\|^4 D^2}{\mu_F^2}} \qquad \theta = \cos^{-1}\left(-\frac{1}{e}\right)$$

$$D = r_p\sqrt{1 + \frac{2\mu_F}{r_p\|\mathbf{v}_{\infty,\text{in}}\|^2}} \qquad \phi = 2\sin^{-1}\left(\left(1 + \frac{r_p\|\mathbf{v}_{\infty,\text{in}}\|^2}{\mu_F}\right)^{-1}\right)$$

Gravity Assist Effects

- The $\mathbf{v}_\infty$ vectors are with respect to the flyby body ($\mathbf{v}_\infty \equiv \mathbf{v}_{\infty/F}$)
- For a flyby occurring between times t_1 and t_2:

$$\mathbf{v}_{\infty,\text{in}/\odot} = \mathbf{v}_{\infty,\text{in}} + \mathbf{v}_{F/\odot}(t_1) \quad \text{and} \quad \mathbf{v}_{\infty,\text{out}/\odot} = \mathbf{v}_{\infty,\text{out}} + \mathbf{v}_{F/\odot}(t_2)$$

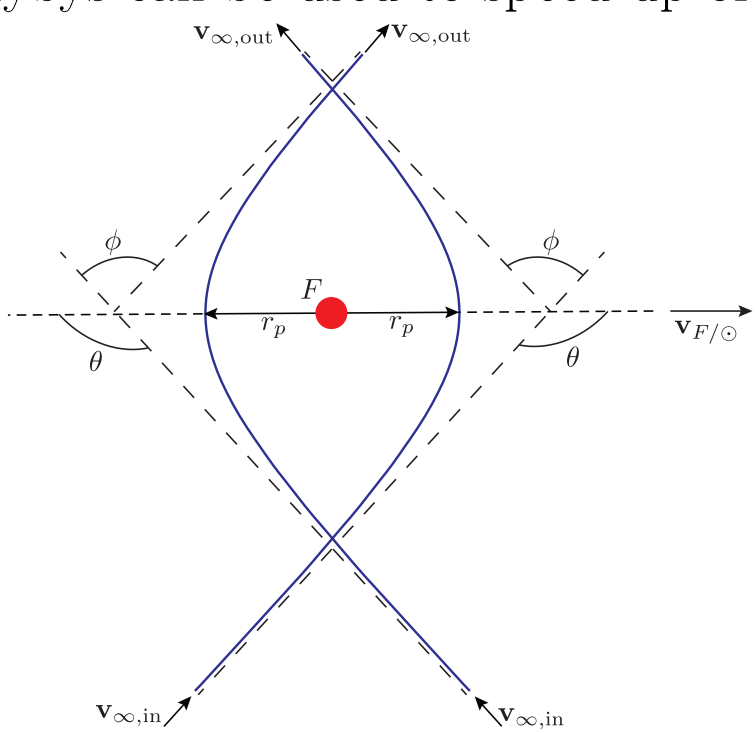
- Assuming $\mathbf{v}_{F/\odot}(t_1) \approx \mathbf{v}_{F/\odot}(t_2)$, the heliocentric Δv is:

$$\Delta v \triangleq \|\mathbf{v}_{\infty,\text{out}/\odot} - \mathbf{v}_{\infty,\text{in}/\odot}\| = 2\|\mathbf{v}_\infty\| \sin\left(\frac{\phi}{2}\right) = \frac{2\|\mathbf{v}_\infty\|}{1 + r_p \|\mathbf{v}_\infty\|^2/\mu_F}$$

- Maximum Δv will be when ${}^{\mathrm{d}\Delta v}/_{\mathrm{d}\|\mathbf{v}_\infty\|} = 0 \implies \phi = 60°$ and $\|\mathbf{v}_\infty\| = \sqrt{\mu_F/r_p}$
- r_p must be greater than the flyby body's radius (R_F) therefore:

$$\boxed{\Delta v_{\max} = \sqrt{\frac{\mu_F}{R_F}} = \frac{v_{\text{esc},F}}{\sqrt{2}}}$$

Flybys can be used to speed up or slow down



- Passing **behind** the flyby body (with respect to its heliocentric velocity) **increases** your heliocentric velocity and specific energy
- Passing **in front** of the flyby body (with respect to its heliocentric velocity) **decreases** your heliocentric velocity and specific energy