6 - The Three Body Problem

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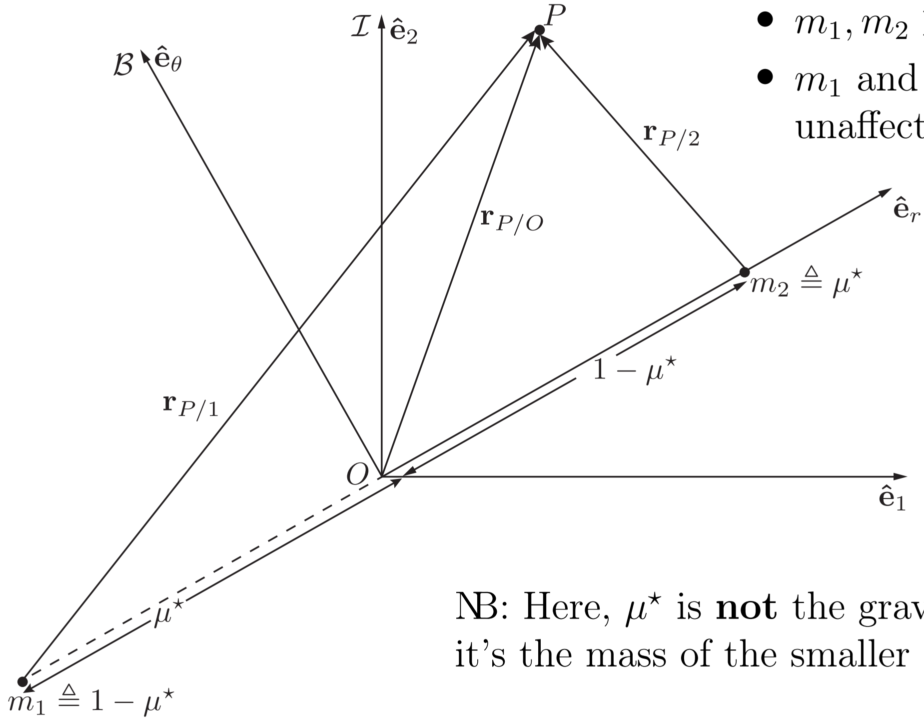
MAE 6720/ASTRO 6579, Spring 2022

The Three-Body Problem and N-Body Problem

As soon as any additional body is added to the two-body problem, we lose the ability to write down a fully analytical solution. We have previously dealt with this by treating additional bodies as perturbers to a two-body system, slowly modifying the Keplerian elements of the two-body orbit. There are cases, however (e.g., when operating near the boundaries of spheres of influence) where this model ceases being useful, as the two-body elements evolve rapidly and significantly over the course of a single orbit. Here, we must explicitly deal with (at least) three co-orbiting bodies, which drives us towards numerical integration in order to accurately predict how an orbit evolves. However, in the case of three bodies, if we make certain additional assumptions, we can still find a conserved quantity that allows us to predict some of the system's behavior (although not the exact trajectories). Study of the three-body problem is incredibly important in modern astrodynamics, as three-body orbital design allows us to create incredibly fuel-efficient trajectories in cases where our spacecraft are in close proximity to multiple bodies (for example, the moon systems of Jupiter and Saturn). Three-body analysis also opens the possibility of creating stable, periodic orbits about empty points in space!

The Circular Restricted Three-Body Problem (CR3BP)



- $m_1, m_2 \gg m_p$
- m_1 and m_2 in mutual circular orbits, unaffected by m_p

$$\mathbf{r}_{1/O} = -\mu^\star \hat{\mathbf{e}}_\mathbf{r}$$
$$\mathbf{r}_{2/O} = (1-\mu^\star)\hat{\mathbf{e}}_\mathbf{r}$$

NB: Here, $\mu^\star$ is **not** the gravitational parameter—it's the mass of the smaller major body

Canonical units for the CR3BP are defined such that $G = 1$
1 MU $= m_1 + m_2$, 1 DU $= \|\mathbf{r}_{1/2}\|$, and 2π TU $= T_{p,1,2}$ (the orbital period of m_1 and m_2).

CR3BP Dynamics

$$\mathbf{F}_P = -\frac{G m_1 m_P}{\|\mathbf{r}_{P/1}\|^3}\mathbf{r}_{P/1} - \frac{G m_2 m_P}{\|\mathbf{r}_{P/2}\|^3}\mathbf{r}_{P/2} \qquad {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = n\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3$$

$${}^{\mathcal{B}}\mathbf{a}_{P/O} + 2\hat{\mathbf{e}}_3 \times {}^{\mathcal{B}}\mathbf{v}_{P/O} + \hat{\mathbf{e}}_3 \times \left(\hat{\mathbf{e}}_3 \times \mathbf{r}_{P/O}\right) = -G\left(\frac{m_1}{\|\mathbf{r}_{P/1}\|^3}\mathbf{r}_{P/1} + \frac{m_2}{\|\mathbf{r}_{P/2}\|^3}\mathbf{r}_{P/2}\right)$$

$$\left[\mathbf{r}_{P/O}\right]_{\mathcal{B}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}}$$

$$\mathbf{F}_P = -\nabla V \quad \text{where} \quad V = -\left(\frac{1-\mu^\star}{r_1} + \frac{\mu^\star}{r_2}\right)$$

$$r_1 \triangleq \|\mathbf{r}_{P/1}\| = \sqrt{(x+\mu^\star)^2 + y^2 + z^2}$$
$$r_2 \triangleq \|\mathbf{r}_{P/2}\| = \sqrt{(x-(1-\mu^\star))^2 + y^2 + z^2}$$

$$\boxed{\begin{aligned} \ddot{x} - 2\dot{y} - x &= -\frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} - y &= -\frac{\partial V}{\partial y} \\ \ddot{z} &= -\frac{\partial V}{\partial z} \end{aligned}}$$

Remember that, just as in the Clohessy-Wiltshire equations, x, y, z are rotating frame coordinates.

A New Potential

$$\text{Define}: U \triangleq -\frac{1}{2}(x^2+y^2) - \left(\frac{1-\mu^\star}{r_1} + \frac{\mu^\star}{r_2}\right)$$

$$\left.\begin{aligned} \ddot{x} - 2\dot{y} - x &= -\frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} - y &= -\frac{\partial V}{\partial y} \\ \ddot{z} &= -\frac{\partial V}{\partial z} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \ddot{x} &= -\frac{\partial U}{\partial x} + 2\dot{y} \\ \ddot{y} &= -\frac{\partial U}{\partial y} - 2\dot{x} \\ \ddot{z} &= -\frac{\partial U}{\partial z} \end{aligned}}$$

$$\text{NB}: \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = -\frac{\mathrm{d}U}{\mathrm{d}t}$$

The Jacobi Constant

$$\frac{1}{2}\left({}^{\mathcal{B}}\mathbf{v}_{P/O} \cdot {}^{\mathcal{B}}\mathbf{v}_{P/O}\right) + U(x,y,z) = C \triangleq \text{Jacobi Constant}$$

$${}^{\mathcal{B}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{P/O} - \hat{\mathbf{e}}_3 \times \mathbf{r}_{P/O}$$

$$\underbrace{\frac{1}{2}\left({}^{\mathcal{I}}\mathbf{v}_{P/O} \cdot {}^{\mathcal{I}}\mathbf{v}_{P/O}\right) - \left(\frac{1-\mu^\star}{r_1} + \frac{\mu^\star}{r_2}\right)}_{\text{KE+PE}} - \underbrace{{}^{\mathcal{I}}\mathbf{v}_{P/O} \cdot \left(\hat{\mathbf{e}}_3 \times \mathbf{r}_{P/O}\right)}_{\hat{\mathbf{e}}_3 \cdot (\mathbf{r}_{P/O} \times {}^{\mathcal{I}}\mathbf{v}_{P/O}) = \hat{\mathbf{e}}_3 \cdot {}^{\mathcal{I}}\mathbf{h}_{P/O}} = C$$

$$\boxed{\mathcal{E} - h\cos(I) = C}$$

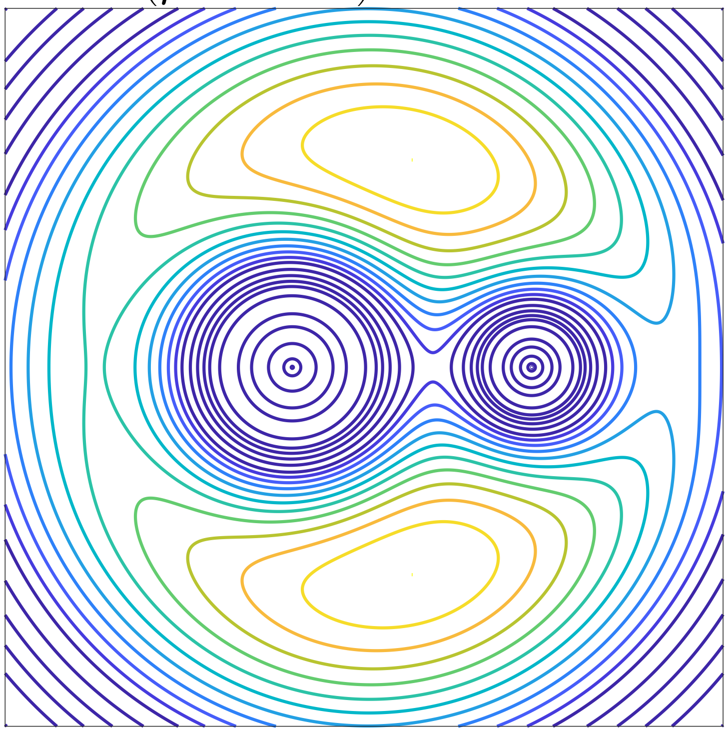
Set by initial conditions

Total Energy of Mass P

Angular Momentum of Mass P

Angle between orbit of mass P and $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ plane

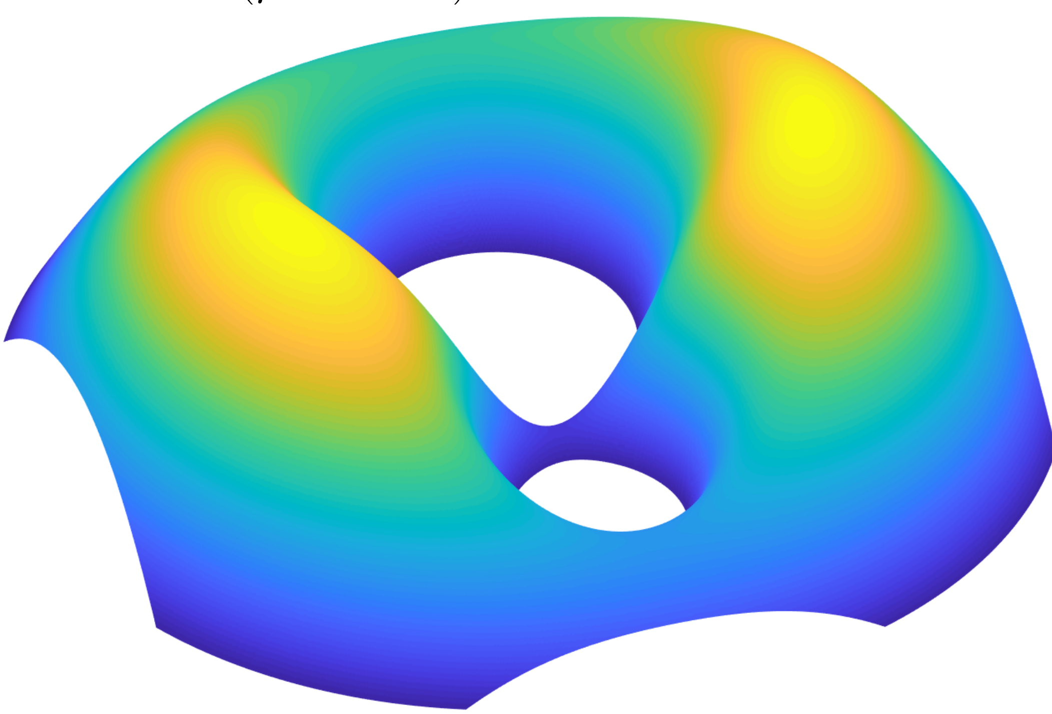
Hill Curves ($\mu^\star = 0.3$)



$$U(x, y) = U(x, -y)$$

$$U(x, y) \neq U(-x, y)$$

Hill Curves ($\mu^\star = 0.3$)



$$\lim_{x,y\to\infty} U = -\frac{x^2 + y^2}{2}$$

$$\lim_{\substack{x\to-\mu^\star \\ y\to 0}} U = -\frac{1-\mu^\star}{r_1}$$

$$\lim_{\substack{x\to 1-\mu^\star \\ y\to 0}} U = -\frac{\mu^\star}{r_2}$$

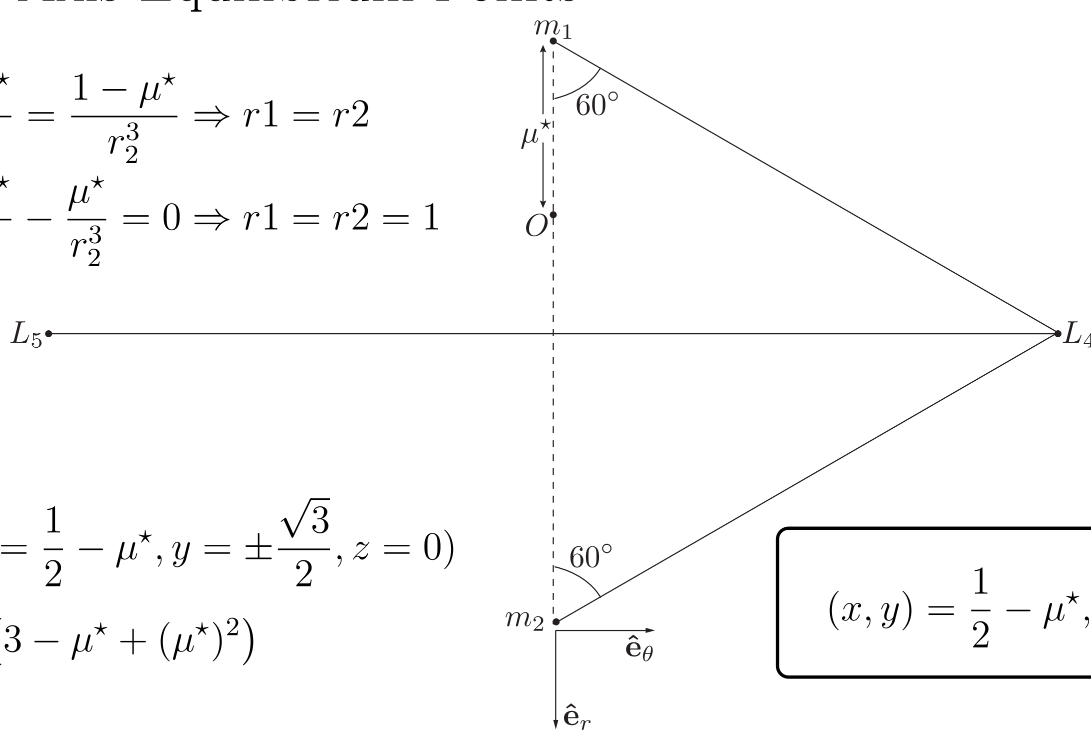
CR3BP Equilibrium Points

$$
\begin{aligned}
\frac{\partial U}{\partial x} &= -\frac{\mu^\star\left(-\mu^\star - x + 1\right)}{r_2^3} - x - \frac{\left(1-\mu^\star\right)\left(-\mu^\star - x\right)}{r_1^3} = 0 \\
\frac{\partial U}{\partial y} &= \frac{\mu^\star y}{r_2^3} - y + \frac{y\left(1-\mu^\star\right)}{r_1^3} = 0 \\
\frac{\partial U}{\partial z} &= \frac{\mu^\star z}{r_2^3} + \frac{z\left(1-\mu^\star\right)}{r_1^3} = 0
\end{aligned}
$$

$\frac{\partial U}{\partial z}$ is zero for $z = 0$ so we typically focus on in-plane solutions

$y \neq 0$: Off-Axis Equilibrium Points

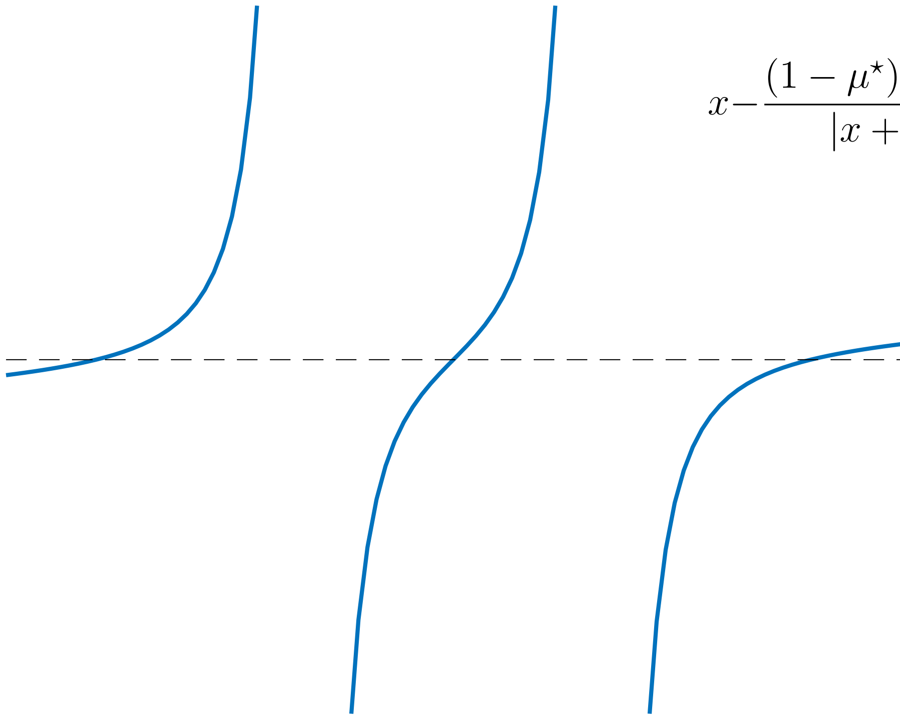
$$
\begin{aligned}
\frac{1-\mu^\star}{r_1^3} &= \frac{1-\mu^\star}{r_2^3} \Rightarrow r1 = r2 \\
1 - \frac{1-\mu^\star}{r_1^3} - \frac{\mu^\star}{r_2^3} &= 0 \Rightarrow r1 = r2 = 1
\end{aligned}
$$



$$
\begin{aligned}
C_4 &= C_5 \\
&= U(x = \frac{1}{2} - \mu^\star, y = \pm\frac{\sqrt{3}}{2}, z = 0) \\
&= -\frac{1}{2}\left(3 - \mu^\star + (\mu^\star)^2\right)
\end{aligned}
$$

$$
(x, y) = \frac{1}{2} - \mu^\star, \pm\frac{\sqrt{3}}{2}
$$

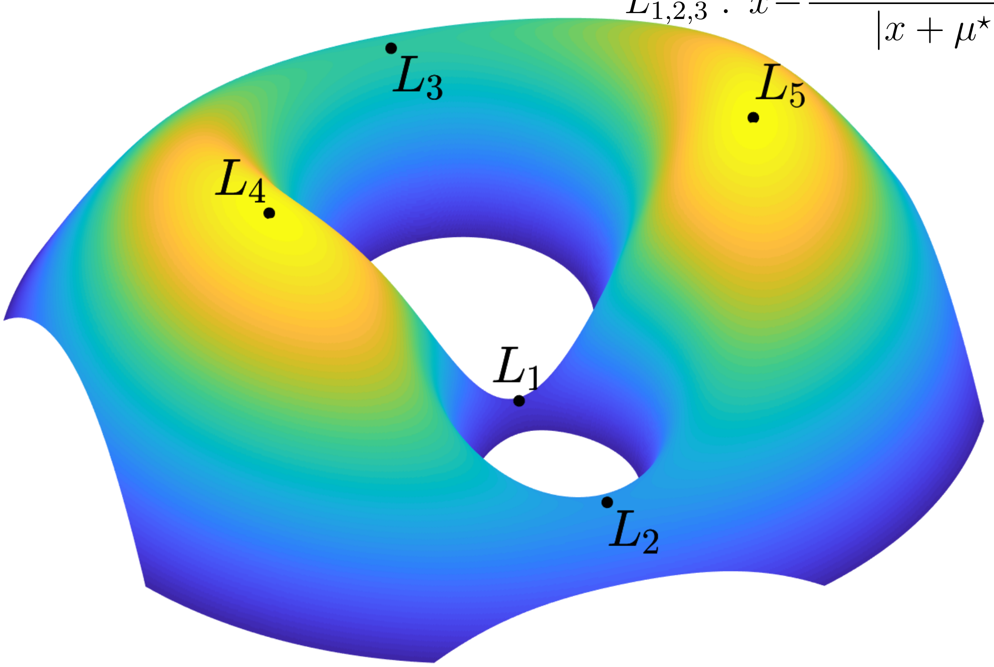
$y = 0$: On-Axis Equilibrium Points



$$x - \frac{(1-\mu^\star)(x+\mu^\star)}{|x+\mu^\star|^3} - \frac{\mu^\star(x-1+\mu^\star)}{|x-1+\mu^\star|^3} = 0$$

$$\begin{aligned} x &< -\mu^\star & (L_3) \\ -\mu^\star < x &< 1-\mu^\star & (L_1) \\ x &> 1-\mu^\star & (L_2) \end{aligned}$$

The Lagrange Points



$$L_{1,2,3}: \; x - \frac{(1-\mu^\star)(x+\mu^\star)}{|x+\mu^\star|^3} - \frac{\mu^\star(x-1+\mu^\star)}{|x-1+\mu^\star|^3} = 0$$

$$[L_{4,5}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} - \mu^\star \\ \pm\frac{\sqrt{3}}{2} \end{bmatrix}_{\mathcal{B}}$$

Perturbation of $L_{4/5}$ Points

Consider a small displacement $\alpha\hat{\mathbf{e}}_r + \beta\hat{\mathbf{e}}_\theta$ from one of the equilibrium points L_i:

$$U \triangleq -\frac{1}{2}(x^2+y^2)-\left(\frac{1-\mu^\star}{r_1}+\frac{\mu^\star}{r_2}\right) \quad \Longleftrightarrow \quad \frac{\partial U}{\partial x} = \left.\frac{\partial U}{\partial x}\right|_{L_i} +\alpha \left.\frac{\partial^2 U}{\partial x^2}\right|_{L_i} +\beta \left.\frac{\partial^2 U}{\partial x \partial y}\right|_{L_i} +\cdots$$

$$\left.\begin{aligned} \left.\frac{\partial U}{\partial x}\right|_{L_{4/5}} &\approx -\left(\frac{3\alpha}{4}+\frac{3\sqrt{3}}{4}(1-2\mu^\star)\beta\right) \\ \left.\frac{\partial U}{\partial y}\right|_{L_{4/5}} &\approx -\left(\frac{9\beta}{4}+\frac{3\sqrt{3}}{4}(1-2\mu^\star)\alpha\right) \end{aligned}\right\} \quad \begin{aligned} \ddot{x}-2\dot{y} &= -\frac{\partial U}{\partial x} = \frac{3\alpha}{4}+\frac{3\sqrt{3}}{4}(1-2\mu^\star)\beta \\ \ddot{y}+2\dot{x} &= -\frac{\partial U}{\partial y} = \frac{9\beta}{4}+\frac{3\sqrt{3}}{4}(1-2\mu^\star)\alpha \end{aligned}$$

$$\left.\begin{aligned} \alpha &\triangleq Ae^{\lambda t} \\ \beta &\triangleq Be^{\lambda t} \end{aligned}\right\} \left.\begin{aligned} A\lambda^2-2B\lambda &= \frac{3A}{4}+\frac{3\sqrt{3}}{4}(1-2\mu^\star)B \\ B\lambda^2+2A\lambda &= \frac{9B}{4}+\frac{3\sqrt{3}}{4}(1-2\mu^\star)A \end{aligned}\right\} \begin{aligned} &\lambda^4+\lambda^2+\frac{27}{4}\mu^\star(1-\mu^\star)=0 \Rightarrow \\ &\lambda^2 = -\frac{1}{2}\pm\frac{1}{2}\sqrt{1-27\mu^\star(1-\mu^\star)} \end{aligned}$$

Stability of $L_{4/5}$ Points

- If λ^2 is complex, then at least one root will have a positive real part
- For $L_{4/5}$ to be stable, we therefore require λ^2 to be strictly real:

$$\lambda^2 = -\frac{1}{2}\pm\frac{1}{2}\sqrt{1-27\mu^\star(1-\mu^\star)}$$

- This imposes the condition: $1-27\mu^\star(1-\mu^\star)\geq 0$ which requires:

$$\mu^\star \leq \frac{1}{2}-\sqrt{\frac{23}{108}} \approx 0.0385$$

- $L_{4/5}$ are stable when $m_2 \lessapprox \frac{m_1}{25}$

Perturbation of $L_{1\ldots3}$ Points

Consider a small displacement $\alpha\hat{\mathbf{e}}_r + \beta\hat{\mathbf{e}}_\theta$ from one of the equilibrium points L_i:

$$\ddot{\alpha} - 2\dot{\beta} = -\left.\frac{\partial U}{\partial x}\right|_{L_{1\ldots3}} = \alpha(1+2D)$$

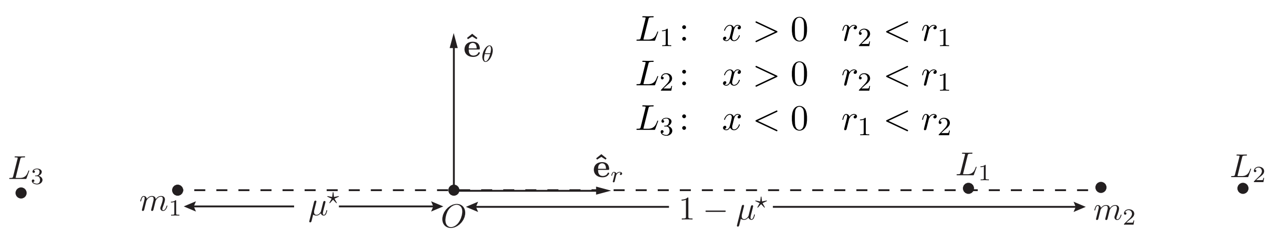
$$\ddot{\beta} + 2\dot{\alpha} = -\left.\frac{\partial U}{\partial y}\right|_{L_{1\ldots3}} = \beta(1-D)$$

$$D \triangleq \frac{1-\mu^\star}{r_1^3} + \frac{\mu^\star}{r_2^3}$$

$$\left.\begin{aligned} \alpha &\triangleq Ae^{\lambda t} \\ \beta &\triangleq Be^{\lambda t} \end{aligned}\right\} \quad \begin{aligned} \lambda^4 + (2-D)\lambda^2 + (1+2D)(1-D) = 0 \Longrightarrow \\ \lambda^2 = \left(\frac{D}{2} - 1\right) \pm \frac{1}{2}\sqrt{D(9D-8)} \end{aligned}$$

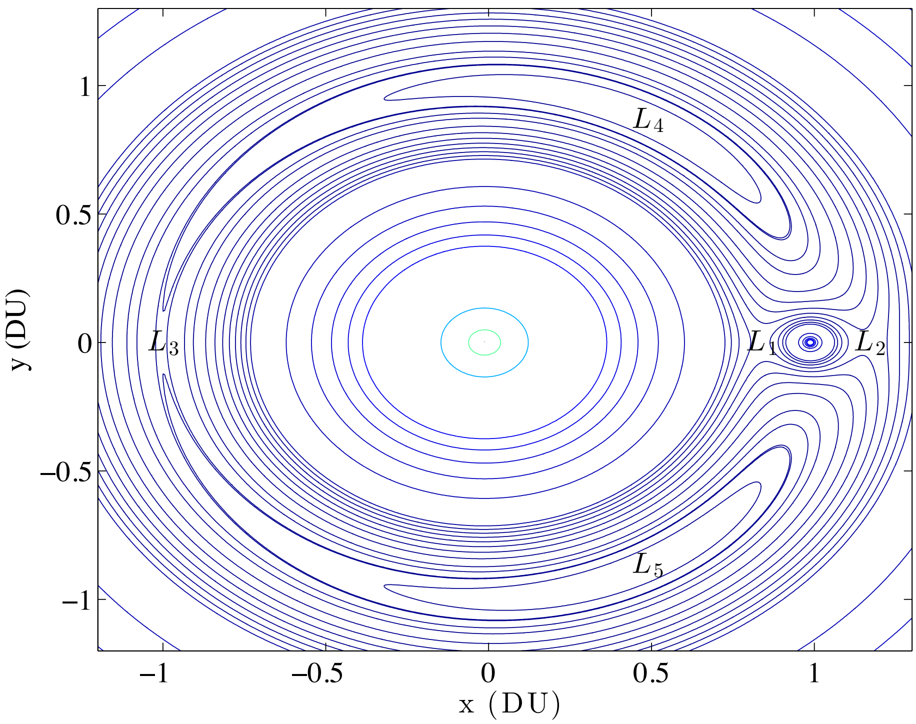
$$x - \frac{(1-\mu^\star)(x+\mu^\star)}{|x+\mu^\star|^3} - \frac{\mu^\star(x-1+\mu^\star)}{|x-1+\mu^\star|^3} = 0 \Longrightarrow 1 - D = \frac{\mu^\star(1-\mu^\star)}{x}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)$$

Stability of $L_{1\ldots3}$ Points

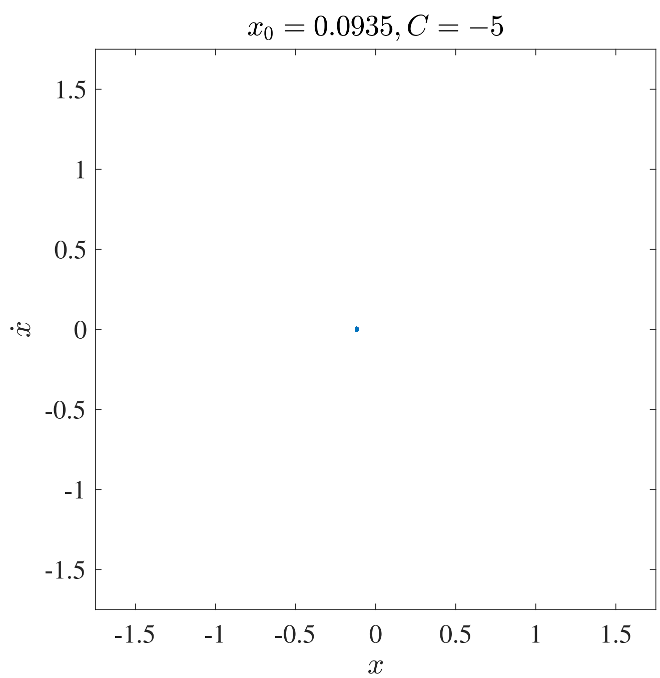


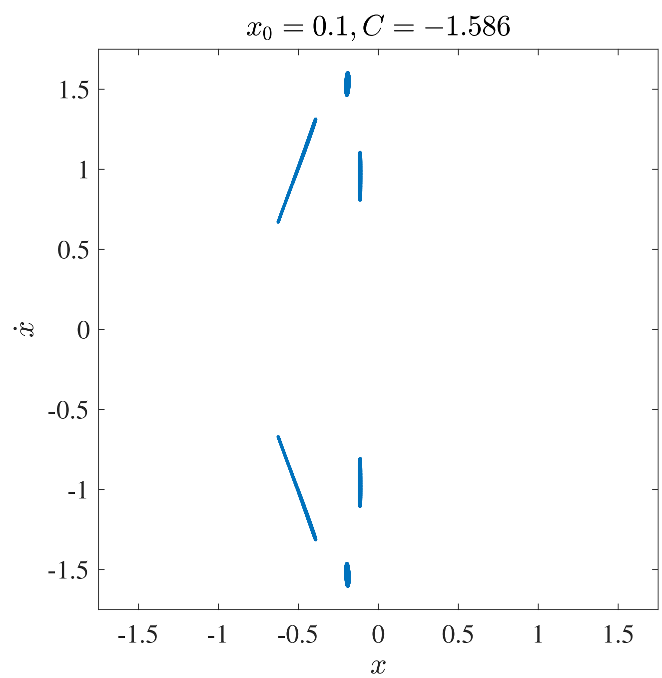
- We again require λ^2 to be strictly real and negative for stability
- D is positive by definition and we require $9D > 8$ and $D < 1$
- However, none of the three co-linear Lagrange points allows for $D < 1$
- $L_{1\ldots3}$ are inherently unstable

Earth/Moon ($\mu^\star = 1/82.3$)

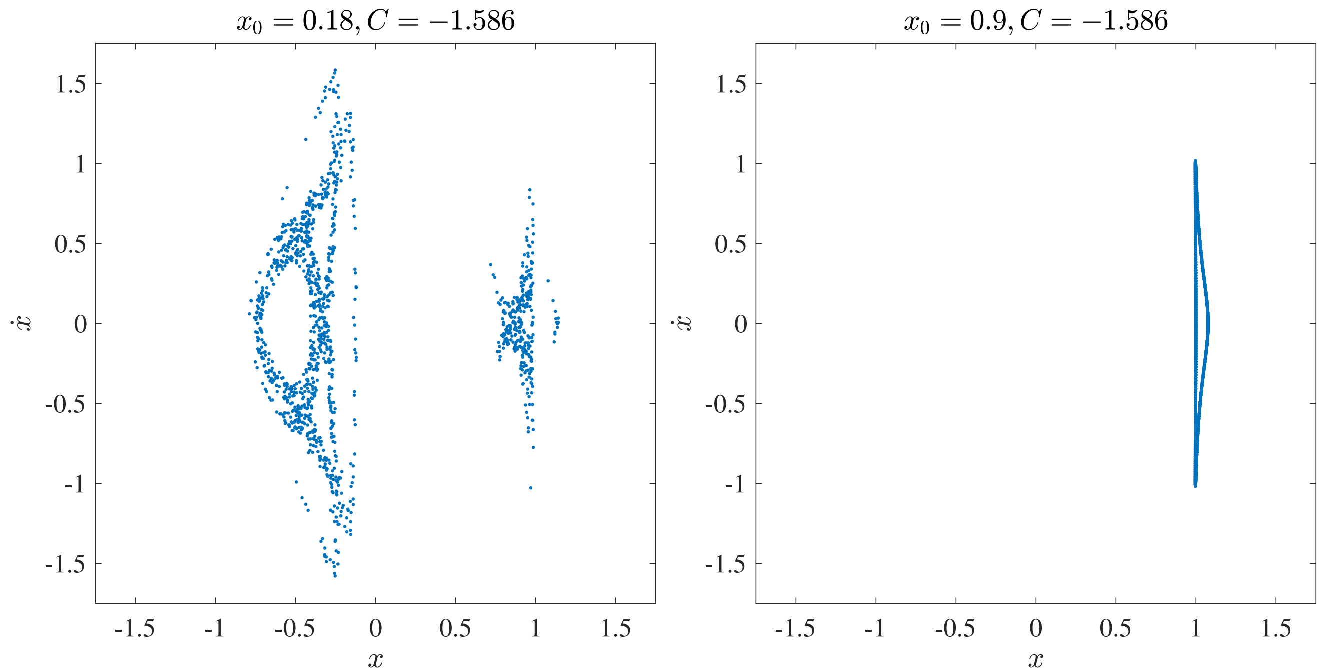


CR3BP Poincaré Maps

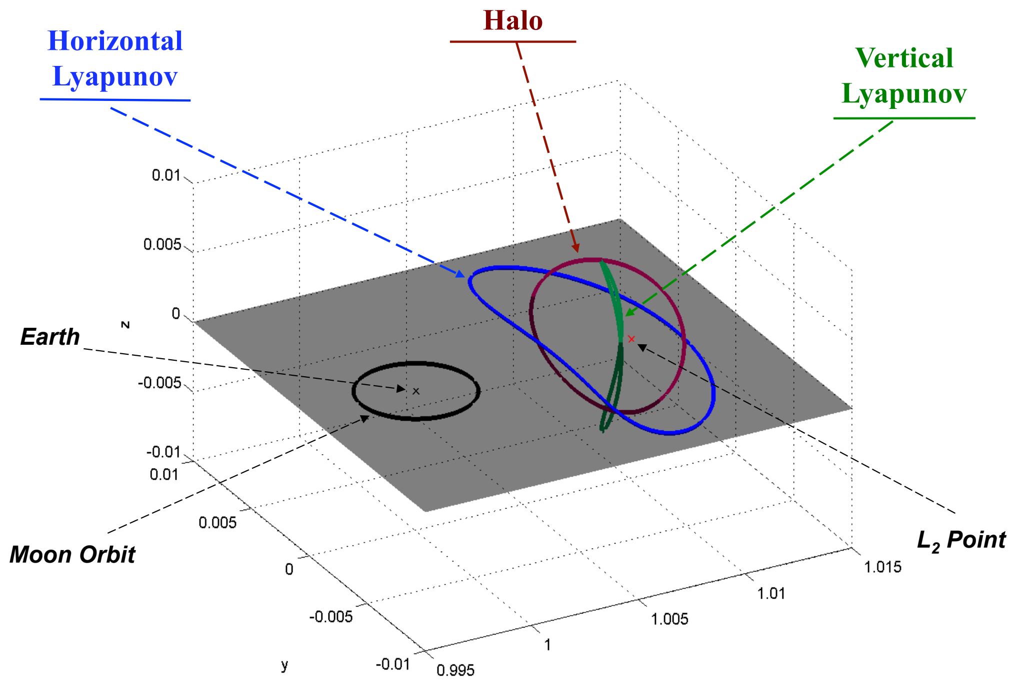




CR3BP Poincaré Maps

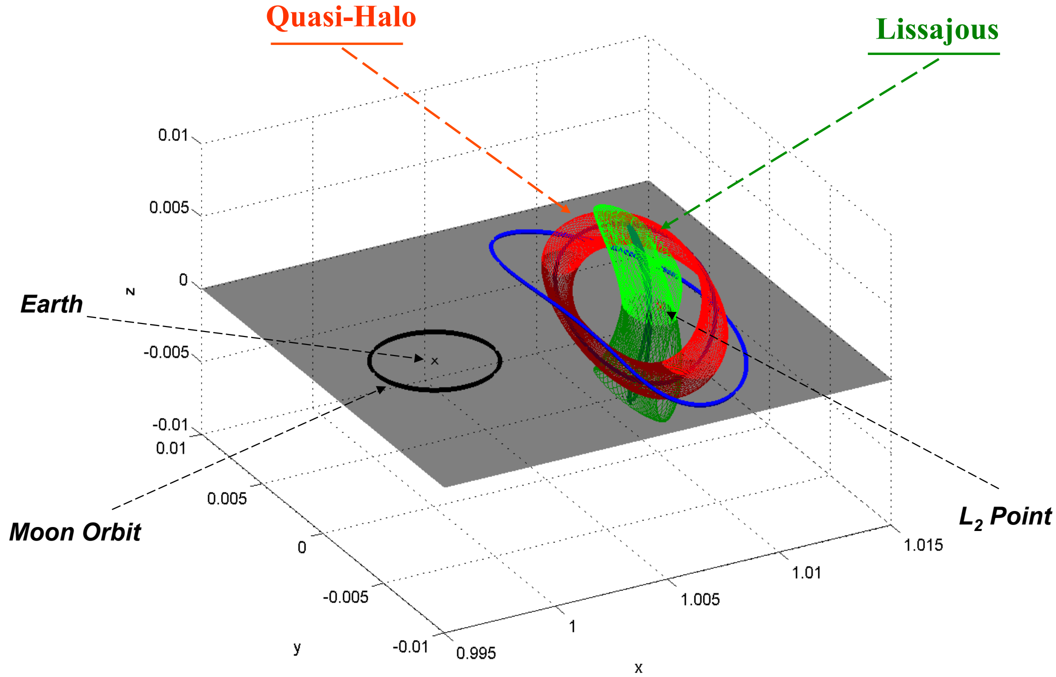


Stable Structures About L_2



See: Kolemen et al. (2012)

Stable Structures About L_2



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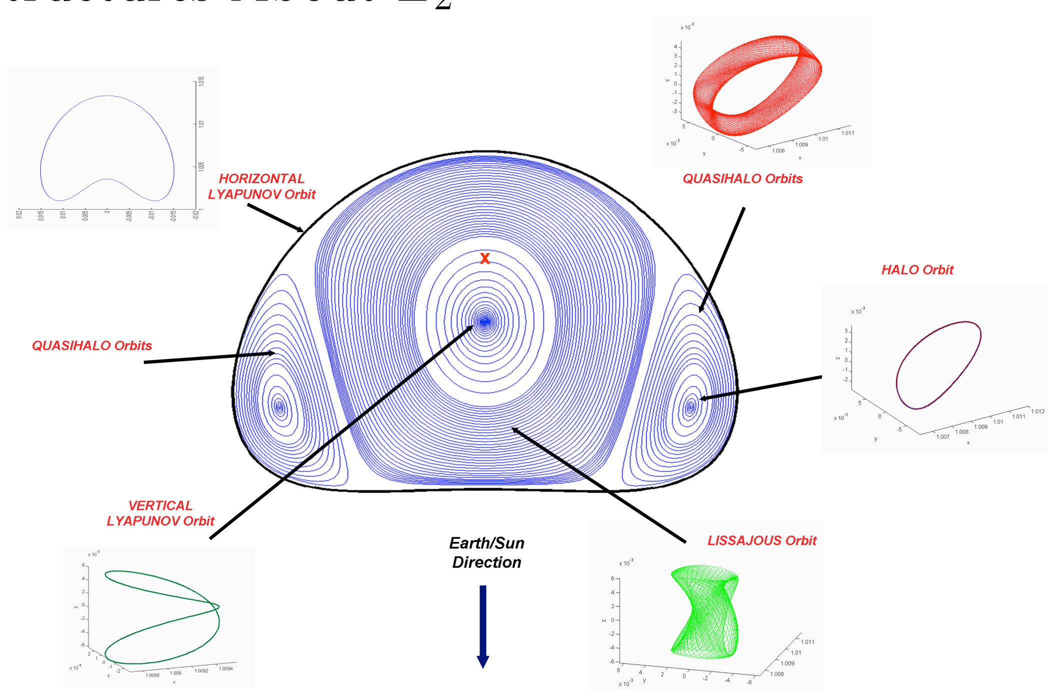
Stable Structures About L_2



See: Kolemen et al. (2012)

The Tisserand Criterion

$$\frac{1}{2}\left({}^{\mathcal{B}}\mathbf{v}_{P/O}\cdot\underbrace{{}^{\mathcal{B}}\mathbf{v}_{P/O}}\right)-\frac{x^2+y^2}{2}-\left(\frac{1-\mu^\star}{r_1}+\frac{\mu^\star}{r_2}\right)=C$$

$${}^{\mathcal{B}}\mathbf{v}_{P/O}={}^{\mathcal{I}}\mathbf{v}_{P/O}-\hat{\mathbf{e}}_3\times\mathbf{r}_{P/O}$$

$$C=\frac{1}{2}\left({}^{\mathcal{I}}\mathbf{v}_{P/O}\cdot{}^{\mathcal{I}}\mathbf{v}_{P/O}\right)-\hat{\mathbf{e}}_3\cdot{}^{\mathcal{I}}\mathbf{h}_{P/O}-\left(\frac{1-\mu^\star}{r_1}+\frac{\mu^\star}{r_2}\right)$$

$$\frac{1}{2}\left({}^{\mathcal{I}}\mathbf{v}_{P/1}\cdot{}^{\mathcal{I}}\mathbf{v}_{P/1}\right)=\frac{1}{r_1}-\frac{1}{2a}\qquad\qquad\hat{\mathbf{e}}_3\cdot{}^{\mathcal{I}}\mathbf{h}_{P/O}=\sqrt{a(1-e^2)}\cos(I)$$

$$\frac{1}{a}+2\sqrt{a(1-e^2)}\cos(I)+\underbrace{2\mu^\star\left(\frac{1}{r_2}-\frac{1}{r_1}\right)}_{\text{small}}=-2C$$

$$\boxed{T\triangleq\frac{1}{a}+2\sqrt{a(1-e^2)}\cos(I)\approx-2C}$$

The Tisserand Criterion and Trajectory Design

Tisserand's Criterion can be used as a trajectory design tool:

- The conditions of the approximation (small $\mu^\star$ and $r_2^{-1}-r_1^{-1}$) apply when planning deep-space flybys.
- Can therefore match a, e, I pre- and post- flyby allowing for rapid iteration on flyby trajectories
- The Tisserand criterion was explicitly used in initial TESS orbit design when modeling lunar flybys. See Gangestad et al. (2013) and Dichmann et al. (2016) for details

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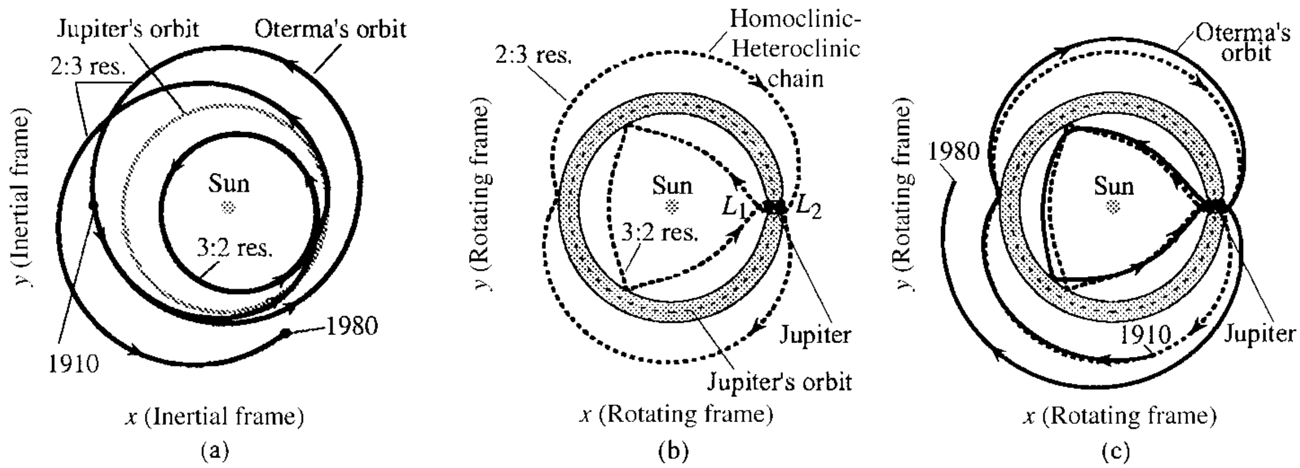
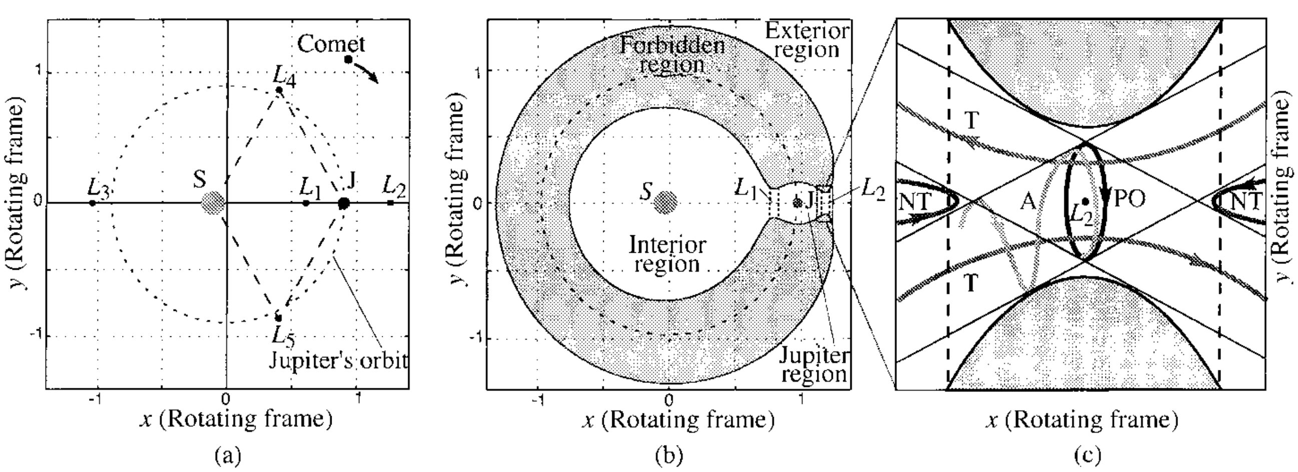


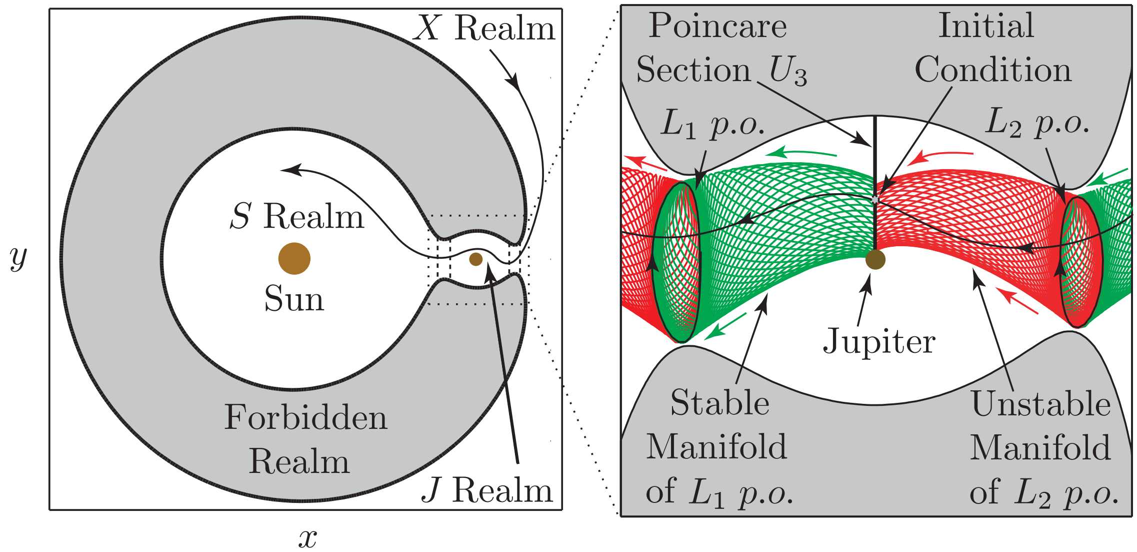
Koon et al. (2001) Fig. 1

Flows about Equilibrium Points



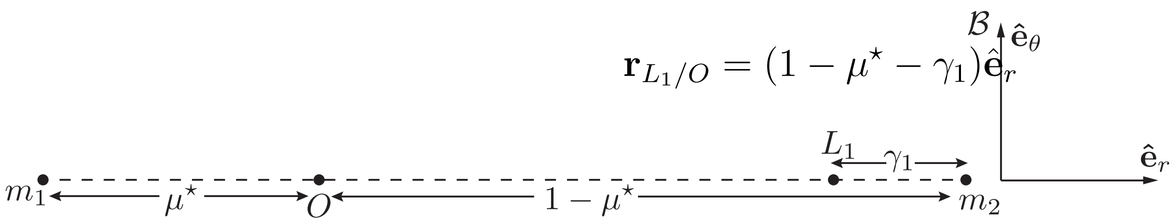
Koon et al. (2001) Fig. 2

Invariant Manifolds



Koon et al. (2011) Fig. 4.4.1

The Hill Sphere



$$\text{For } L_{1-3}\text{: } x - \frac{(1-\mu^\star)(x+\mu^\star)}{|x+\mu^\star|^3} - \frac{\mu^\star(x-1+\mu^\star)}{|x-1+\mu^\star|^3} = 0$$

$$\implies -(1-\mu^\star-\gamma_1) = -\frac{(1-\mu^\star)(1-\gamma_1)}{(1-\gamma_1)^2} + \frac{\mu^\star}{\gamma_1^2}$$

$$\mu^\star \ll \gamma_1 \implies \gamma_1^3 \approx \frac{\mu^\star}{3}$$

$$\boxed{R_H \triangleq \left(\frac{\mu^\star}{3}\right)^{1/3} r_{1/2}}$$