7 - Towards General Perturbations

Dmitry Savransky

Cornell University

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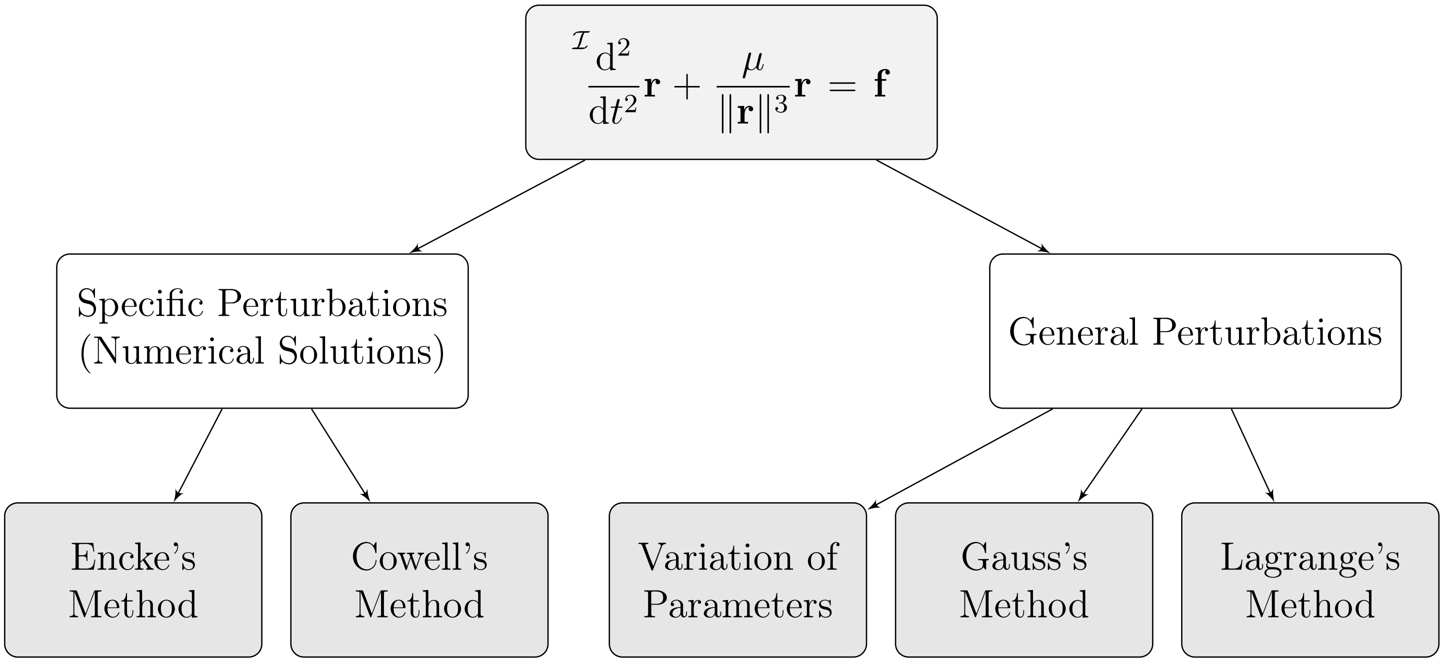
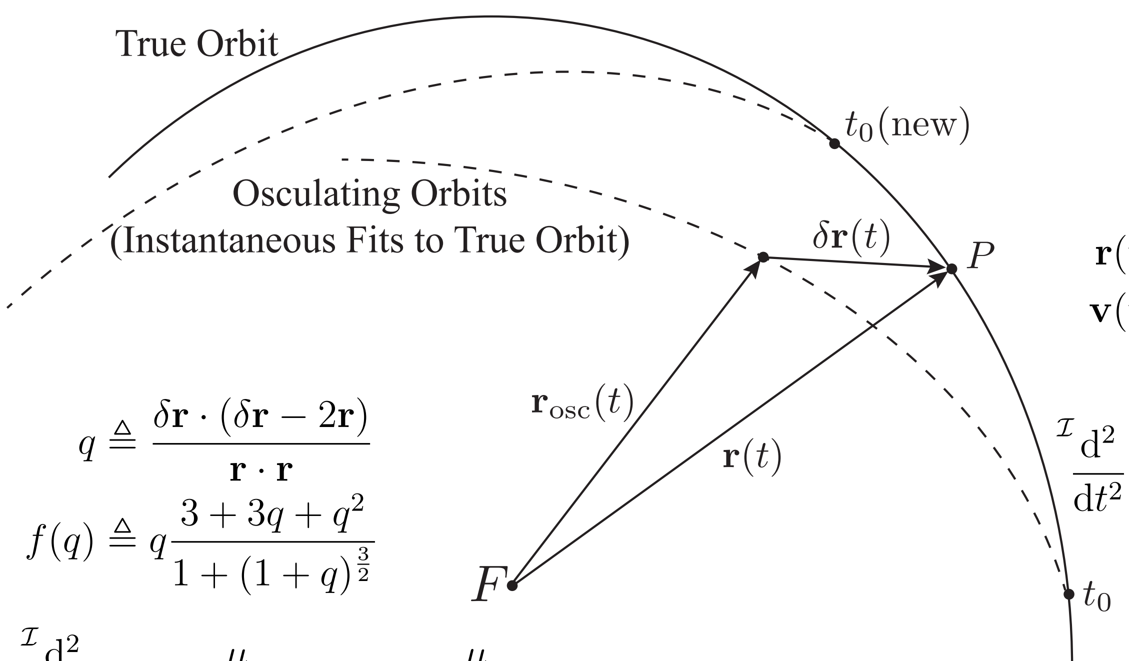
Towards General Perturbations

We will continue our journey towards a completely general description of perturbations of two-body orbits by first considering specific numerical solutions to perturbed systems, and then developing a complete force-based description of perturbations in the form of Gauss's perturbation equations. Along the way, we will introduce an incredibly important piece of mathematical formalism (the Legendre polynomials, which will play a huge role throughout the rest of the course) as well as the concept of the sphere of influence, which is a key tool in preliminary interplanetary trajectory design.

Perturbations Roadmap



Encke's Method



$$\mathbf{r}(t_0) \equiv \mathbf{r}_{\mathrm{osc}}(t_0)$$
$$\mathbf{v}(t_0) \equiv \mathbf{v}_{\mathrm{osc}}(t_0)$$

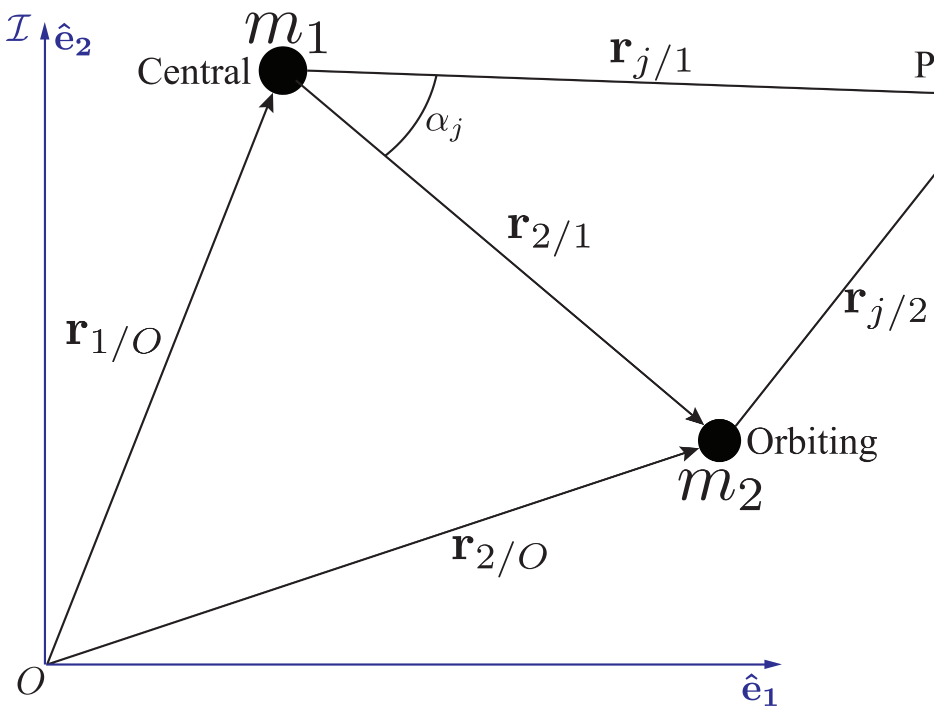
$$\mathbf{r}(t) = \mathbf{r}_{\mathrm{osc}}(t) + \delta\mathbf{r}(t)$$
$$\mathbf{v}(t) = \mathbf{v}_{\mathrm{osc}}(t) + \delta\mathbf{v}(t)$$

$$q \triangleq \frac{\delta\mathbf{r}\cdot(\delta\mathbf{r} - 2\mathbf{r})}{\mathbf{r}\cdot\mathbf{r}}$$
$$f(q) \triangleq q\frac{3 + 3q + q^2}{1 + (1+q)^{\frac{3}{2}}}$$

$${}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{\mathrm{osc}} + \frac{\mu}{\|\mathbf{r}_{\mathrm{osc}}\|^3}\mathbf{r}_{\mathrm{osc}} = 0$$
$${}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = \mathbf{f}$$

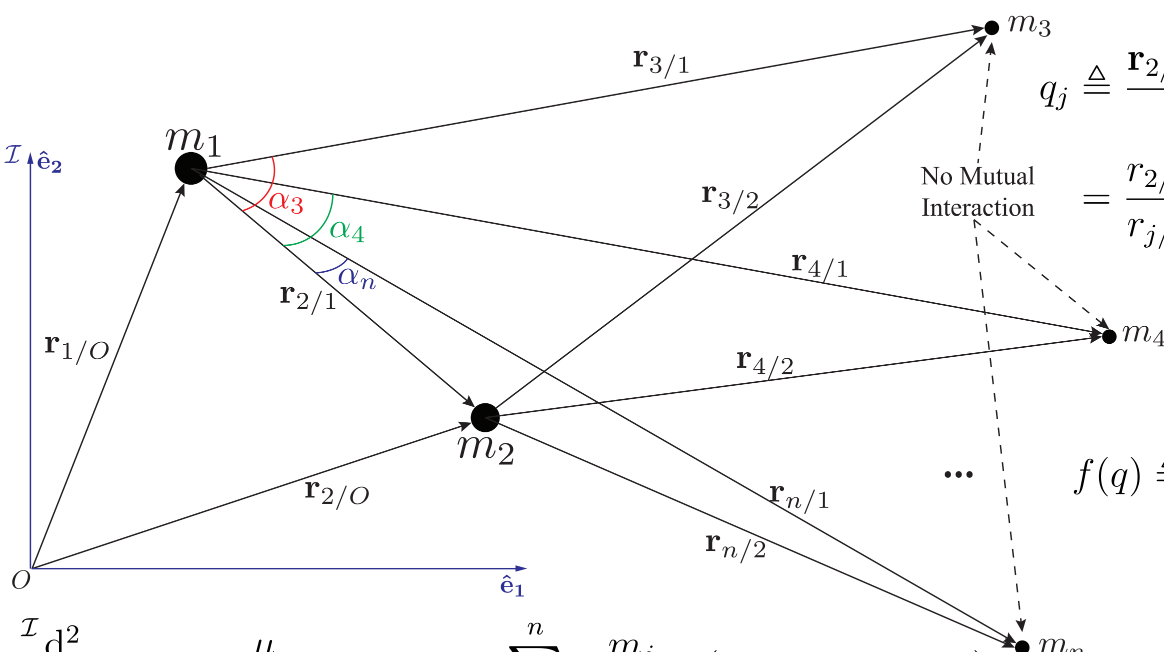
$${}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta\mathbf{r} + \frac{\mu}{\|\mathbf{r}_{\mathrm{osc}}\|^3}\delta\mathbf{r} = -\frac{\mu}{\|\mathbf{r}_{\mathrm{osc}}\|^3}f(q)\mathbf{r} + \mathbf{f}$$

3rd (Nth) Body Perturbations



$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\underbrace{\mathbf{r}_{2/1}}_{\triangleq\, \mathbf{r}} + \frac{\overbrace{G(m_1+m_2)}^{\triangleq\, \mu}}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = \underbrace{-G\sum_j m_j\left(\frac{\mathbf{r}_{j/1}}{\|\mathbf{r}_{j/1}\|^3}+\frac{\mathbf{r}_{2/j}}{\|\mathbf{r}_{2/j}\|^3}\right)}_{\triangleq\, \mathbf{f}}$$

Cowell's Method



$$q_j \triangleq \frac{\mathbf{r}_{2/1}\cdot\left(\mathbf{r}_{2/1}-2\mathbf{r}_{j/1}\right)}{\mathbf{r}_{j/1}\cdot\mathbf{r}_{j/1}} = \frac{r_{2/1}}{r_{j/1}}\left(\frac{r_{2/1}}{r_{j/1}}-2\cos\alpha_j\right)$$

$$f(q)\triangleq q\left(\frac{3+3q+q^2}{1+(1+q)^{\frac{3}{2}}}\right)$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/1} + \frac{\mu}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = -G\sum_{j=3}^{n}\frac{m_j}{\|\mathbf{r}_{2/j}\|^3}\left(\mathbf{r}_{2/1}+f(q_j)\mathbf{r}_{j/1}\right)$$

The Disturbing Function

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/1} + \frac{\mu}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = -G\sum_j m_j \left(\frac{\mathbf{r}_{j/1}}{\|\mathbf{r}_{j/1}\|^3} + \frac{\mathbf{r}_{2/j}}{\|\mathbf{r}_{2/j}\|^3}\right) = \nabla \sum_{j=3}^{n} R_j$$

$$R_j \triangleq Gm_j \left(\frac{1}{r_{2/j}} - \frac{1}{r_{j/1}^3}\mathbf{r}_{2/1}\cdot\mathbf{r}_{j/1}\right) = \frac{Gm_j}{r_{j/1}}\left(\frac{r_{j/1}}{r_{2/j}} - \nu_j x_j\right)$$

$$x_j \triangleq \frac{r_{2/1}}{r_{j/1}} \qquad \nu_j \triangleq \cos\alpha_j = \frac{\mathbf{r}_{2/1}\cdot\mathbf{r}_{j/1}}{r_{2/1}r_{j/1}} \qquad q_j = x_j^2 - 2\nu_j x_j$$

$$\frac{r_{j/1}}{r_{2/j}} = (1+q_j)^{-\frac{1}{2}} = \sum_{k=0}^{\infty}\sum_{l=0}^{\lfloor\frac{k}{2}\rfloor} \frac{(-1)^l(2k-2l)!}{2^k l!(k-l)!(k-2l)!}\nu_j^{k-2l}x_j^k = \sum_{k=0}^{\infty} P_k(\nu_k)x_j^k$$

A Brief Review of Legendre Polynomials

- Legendre Polynomials are solutions to Legendre's Differential Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)\right) + \left(n^2+n\right)P_n(x) = 0$$

- Rodrigues Formula:

$$P_k(\nu) = \frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}\nu^k}\left(\nu^2-1\right)^k = \frac{1}{2^k}\sum_{n=0}^{k}\binom{k}{n}^2(\nu-1)^{k-n}(\nu+1)^n$$

- Generating Function: $L(x,\nu) \triangleq \left(1-2\nu x + x^2\right)^{-1/2} = \sum_{k=0}^{\infty} P_k(\nu)x^k$
- Properties: $P_k(-\nu) = (-1)^k P_k(\nu) \qquad P_k(1) = 1 \qquad P_k(-1) = (-1)^k$
- Recurrence Formulas: $kP_k(\nu) - (2k-1)\nu P_{k-1}(\nu) + (k-1)P_{k-2}(\nu) = 0$

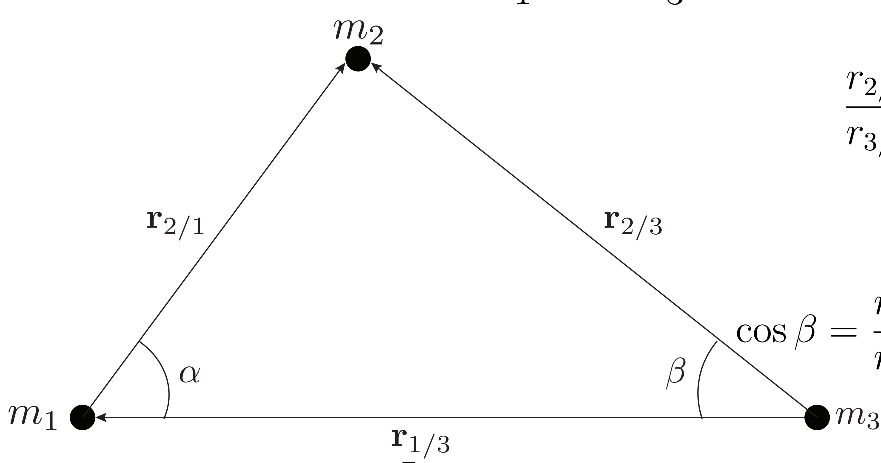
$$\left.\begin{aligned} kP_k(\nu) &= \nu P_k'(\nu) - P_{k-1}'(\nu) \\ kP_{k-1}(\nu) &= P_k'(\nu) - \nu P_{k-1}'(\nu)\end{aligned}\right\}\left(\nu^2-1\right)P_k'(\nu) = \nu k P_k(\nu) - kP_{k-1}(\nu)$$

Series Expansion of the Disturbing Function

$$R_j \triangleq Gm_j \left(\frac{1}{r_{2/j}} - \frac{1}{r_{j/1}^3} \mathbf{r}_{2/1} \cdot \mathbf{r}_{j/1} \right) = \frac{Gm_j}{r_{j/1}} \left[1 + \sum_{k=0}^{\infty} P_k(\cos\alpha_j) \left(\frac{r_{2/1}}{r_{j/1}} \right)^k \right]$$

$$\begin{aligned} {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/1} + \frac{\mu}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} &= \nabla \sum_{j=3}^{n} R_j \\ &= G \sum_{j=3}^{n} \frac{m_j}{r_{j/1}^2} \sum_{k=1}^{\infty} \left(\frac{r_{2/1}}{r_{j/1}} \right)^k \left[P'_{k+1}(\cos\alpha_j) \frac{\mathbf{r}_{j/1}}{r_{j/1}} - P'_k(\cos\alpha_j) \frac{\mathbf{r}_{2/1}}{r_{2/1}} \right] \end{aligned}$$

Which is Dominant: m_1 or m_3?



$$\frac{r_{2/3}}{r_{3/1}} = \left[1 - 2\frac{r_{2/1}}{r_{3/1}} \cos\alpha + \left(\frac{r_{2/1}}{r_{3/1}} \right)^2 \right]^{\frac{1}{2}}$$

$$\cos\beta = \frac{r_{3/1}}{r_{2/3}} - \frac{r_{2/1}}{r_{2/3}} \cos\alpha$$

$$m_1 \text{ dominant}: \quad {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/1} + \frac{G(m_1+m_2)}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = -Gm_3 \left(\frac{\mathbf{r}_{3/1}}{\|\mathbf{r}_{3/1}\|^3} + \frac{\mathbf{r}_{2/3}}{\|\mathbf{r}_{2/3}\|^3} \right)$$

$$m_3 \text{ dominant}: \quad {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/3} + \underbrace{\frac{G(m_2+m_3)}{\|\mathbf{r}_{2/3}\|^3}\mathbf{r}_{2/3}}_{\text{Central}} = \underbrace{-Gm_1 \left(\frac{\mathbf{r}_{2/1}}{\|\mathbf{r}_{2/1}\|^3} + \frac{\mathbf{r}_{1/3}}{\|\mathbf{r}_{1/3}\|^3} \right)}_{\text{Perturbing}}$$

Sphere of Influence

$$\left(\frac{\|\mathbf{f}_{\text{perturbing}}\|}{\|\mathbf{f}_{\text{central}}\|}\right)_{m_1} = \frac{G m_3 \left[\left(\frac{\mathbf{r}_{2/3}}{r_{2/3}^3}+\frac{\mathbf{r}_{3/1}}{r_{3/1}^3}\right)\cdot\left(\frac{\mathbf{r}_{2/3}}{r_{2/3}^3}+\frac{\mathbf{r}_{3/1}}{r_{3/1}^3}\right)\right]^{\frac{1}{2}}}{G(m_1+m_2)\, r_{2/1}^{-2}}$$

$$= \frac{m_3}{m_2+m_1}\frac{\left(r_{2/1}/r_{3/1}\right)^2}{\left(r_{2/3}/r_{3/1}\right)^2}\left[1+\left(\frac{r_{2/3}}{r_{3/1}}\right)^4 - 2\left(\frac{r_{2/3}}{r_{3/1}}\right)\left(1-\frac{r_{2/1}}{r_{3/1}}\cos\alpha\right)\right]^{\frac{1}{2}}$$

$$\left(\frac{\|\mathbf{f}_{\text{perturbing}}\|}{\|\mathbf{f}_{\text{central}}\|}\right)_{m_3} = \frac{m_1}{m_2+m_3}\left(\frac{r_{2/1}}{r_{3/1}}\right)^{-2}\left(\frac{r_{2/3}}{r_{3/1}}\right)^{2}\left[1+\left(\frac{r_{2/1}}{r_{3/1}}\right)^4 - 2\left(\frac{r_{2/1}}{r_{3/1}}\right)^2\cos\alpha\right]^{\frac{1}{2}}$$

$$\text{Intersection :}\quad \left(\frac{r_{2/1}}{r_{3/1}}\right)^4 = \frac{m_1(m_1+m_2)}{m_3(m_2+m_3)}\left(\frac{r_{2/3}}{r_{3/1}}\right)^4\left[\frac{1+\left(\frac{r_{2/1}}{r_{3/1}}\right)^4 - 2\left(\frac{r_{2/1}}{r_{3/1}}\right)^2\cos\alpha}{1+\left(\frac{r_{2/3}}{r_{3/1}}\right)^4 - 2\left(\frac{r_{2/3}}{r_{3/1}}\right)\left(1-\frac{r_{2/1}}{r_{3/1}}\cos\alpha\right)}\right]^{\frac{1}{2}}$$

$$\boxed{\left(\frac{r_{2/1}}{r_{3/1}}\right)\approx\left(\frac{m_1}{m_3}\right)^{\frac{2}{5}} \quad\Rightarrow\quad r_{\text{SOI}}\approx a_{\text{planet}}\left(\frac{m_{\text{planet}}}{m_{\text{sun}}}\right)^{\frac{2}{5}}}$$

Variation of Parameters (Orbital Elements)

$$\mathbf{h} = \mathbf{r}\times\mathbf{v}$$

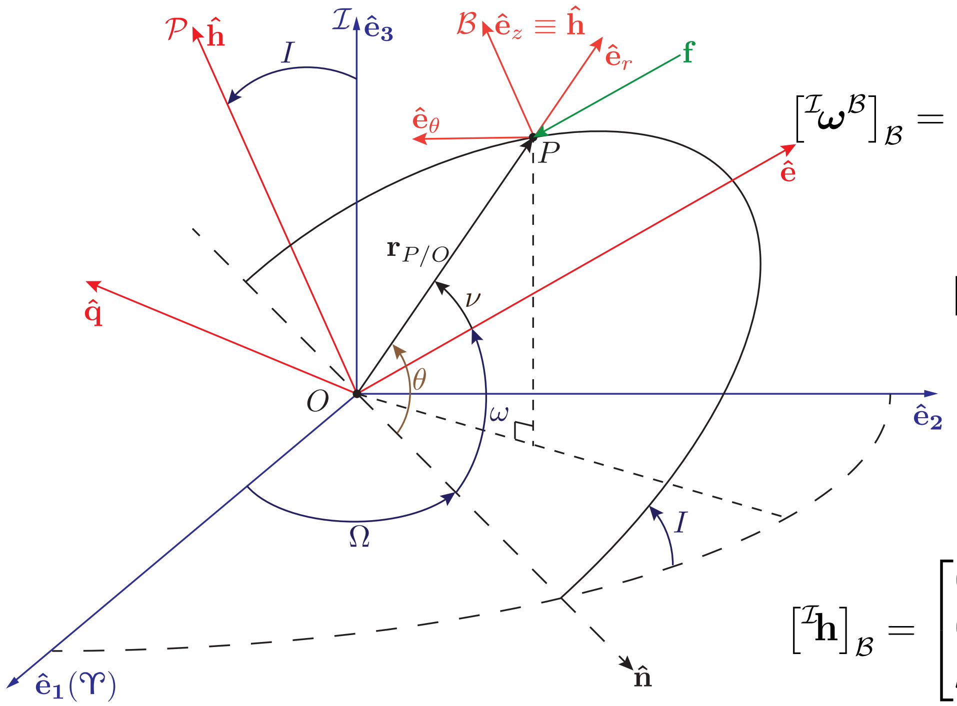
$$\mathbf{e} = \frac{\mathbf{v}\times\mathbf{h}}{\mu} - \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{v} = -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r}+\mathbf{f}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} = \mathbf{r}\times\mathbf{f}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{e} = \frac{1}{\mu}\left(\mathbf{f}\times\mathbf{h}+\mathbf{v}\times(\mathbf{r}\times\mathbf{f})\right)$$

Variation of Parameters Reference Frames



$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\Omega}\hat{\mathbf{e}}_3 + \dot{I}\hat{\mathbf{n}} + \dot{\theta}\hat{\mathbf{h}}$$

$$\left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} = \begin{bmatrix} \dot{I}\cos(\theta) + \dot{\Omega}\sin(I)\sin(\theta) \\ -\dot{I}\sin(\theta) + \dot{\Omega}\sin(I)\cos(\theta) \\ \dot{\Omega}\cos(I) + \dot{\theta} \end{bmatrix}_{\mathcal{B}}$$

$$[\mathbf{r}]_{\mathcal{B}} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \qquad \left[{}^{\mathcal{I}}\mathbf{v}\right]_{\mathcal{B}} = \begin{bmatrix} v_r \\ v_\theta \\ 0 \end{bmatrix}_{\mathcal{B}}$$

$$[\mathbf{f}]_{\mathcal{B}} = \begin{bmatrix} f_r \\ f_\theta \\ f_h \end{bmatrix}_{\mathcal{B}}$$

$$\left[{}^{\mathcal{I}}\mathbf{h}\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}_{\mathcal{B}} \qquad [\mathbf{e}]_{\mathcal{B}} = \begin{bmatrix} e\cos\nu \\ -e\sin\nu \\ 0 \end{bmatrix}_{\mathcal{B}}$$

Gauss's Perturbation Equations (the setup)

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{h} = \mathbf{r} \times \mathbf{f} \quad \Rightarrow$$

$$\begin{bmatrix} 0 \\ 0 \\ \dot{h} \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} h\left(-\dot{I}\sin(\theta) + \dot{\Omega}\sin(I)\cos(\theta)\right) \\ -h\left(\dot{I}\cos(\theta) + \dot{\Omega}\sin(I)\sin(\theta)\right) \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 \\ -f_h r \\ f_\theta r \end{bmatrix}_{\mathcal{B}}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{e} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{e} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{e} = \frac{1}{\mu}\left(\mathbf{f} \times \mathbf{h} + \mathbf{v} \times (\mathbf{r} \times \mathbf{f})\right) \quad \Rightarrow$$

$$\begin{bmatrix} -e\left(\dot{\omega} - \dot{\theta}\right)\sin(\omega - \theta) + \dot{e}\cos(\omega - \theta) \\ e\left(\dot{\omega} - \dot{\theta}\right)\cos(\omega - \theta) + \dot{e}\sin(\omega - \theta) \\ 0 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} -e\left(\dot{\Omega}\cos(I) + \dot{\theta}\right)\sin(\omega - \theta) \\ e\left(\dot{\Omega}\cos(I) + \dot{\theta}\right)\cos(\omega - \theta) \\ e\left(\dot{I}\sin(\omega) - \dot{\Omega}\sin(I)\cos(\omega)\right) \end{bmatrix}_{\mathcal{B}} = \frac{1}{\mu}\left(\begin{bmatrix} f_\theta h \\ -f_r h \\ 0 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} f_\theta r v_\theta \\ -f_\theta r v_r \\ -f_h r v_r \end{bmatrix}_{\mathcal{B}}\right)$$

Gauss's Perturbation Equations (the solution)

$$
\begin{aligned}
\dot{I} &= \frac{f_h r}{h}\cos(\theta) & \dot{e} &= \frac{e f_\theta}{h} r\sin^2(\nu) + \frac{f_r h}{\mu}\sin(\nu) + \frac{2 f_\theta}{\mu} h\cos(\nu) \\
\dot{\Omega} &= \frac{f_h r\sin(\theta)}{h\sin(I)} & \dot{\omega} &= -\frac{f_h r\sin(\theta)}{h\tan(I)} - \frac{f_\theta r}{2h}\sin(2\nu) - \frac{f_r h}{e\mu}\cos(\nu) + \frac{2 f_\theta h}{e\mu}\sin(\nu) \\
\dot{h} &= f_\theta r & \frac{h}{r^2} &= \dot{\Omega}\cos(I) + \dot{\theta}
\end{aligned}
$$

$$
\dot{a} = \frac{2a^2}{h}\left[e\sin\nu f_r + (1 + e\cos(\nu)) f_\theta\right]
$$

Gauss's Perturbation Equations (other versions)

$$
\begin{aligned}
\frac{d\Omega}{dt} &= \frac{r\sin\theta}{h\sin i} a_{dh} \\
\frac{di}{dt} &= \frac{r\cos\theta}{h} a_{dh} \\
\frac{d\omega}{dt} &= \frac{1}{he}[-p\cos f\, a_{dr} + (p+r)\sin f\, a_{d\theta}] - \frac{r\sin\theta\cos i}{h\sin i} a_{dh} \\
\frac{da}{dt} &= \frac{2a^2}{h}\left(e\sin f\, a_{dr} + \frac{p}{r} a_{d\theta}\right) \\
\frac{de}{dt} &= \frac{1}{h}\{p\sin f\, a_{dr} + [(p+r)\cos f + re] a_{d\theta}\} \\
\frac{dM}{dt} &= n + \frac{b}{ahe}[(p\cos f - 2re) a_{dr} - (p+r)\sin f\, a_{d\theta}]
\end{aligned}
$$

Battin (1999) Eq. 10.41
NB: $f \equiv \nu$, $p \equiv \ell$
$(a_{dr}, a_{d\theta}, a_{dh}) \equiv (f_r, f_\theta, f_h)$

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{n\sqrt{1-e^2}}\left\{e\,\mathrm{SIN}(\nu) F_R + \frac{p}{r} F_S\right\} \\
\frac{de}{dt} &= \frac{\sqrt{1-e^2}}{na}\left\{\mathrm{SIN}(\nu) F_R + \left(\mathrm{COS}(\nu) + \frac{e + \mathrm{COS}(\nu)}{1 + e\,\mathrm{COS}(\nu)}\right) F_S\right\} \\
\frac{di}{dt} &= \frac{r\,\mathrm{COS}(u)}{na^2\sqrt{1-e^2}} F_W \\
\frac{d\Omega}{dt} &= \frac{r\,\mathrm{SIN}(u)}{na^2\sqrt{1-e^2}\,\mathrm{SIN}(i)} F_W \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{nae}\left\{-\mathrm{COS}(\nu) F_R + \mathrm{SIN}(\nu)\left(1 + \frac{r}{p}\right) F_S\right\} - \frac{r\,\mathrm{COT}(i)\,\mathrm{SIN}(u)}{h} F_W \\
\frac{dM_o}{dt} &= \frac{1}{na^2 e}\left\{(p\,\mathrm{COS}(\nu) - 2er) F_R - (p+r)\,\mathrm{SIN}(\nu) F_S\right\} - \frac{dn}{dt}(t - t_o)
\end{aligned}
$$

Vallado (2013) Eq. 9-24
NB: $p \equiv \ell$
$(F_R, F_S, F_W) \equiv (f_r, f_\theta, f_h)$