8 - Hamilton-Jacobi Perturbations

Dmitry Savransky

Cornell University

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Hamilton-Jacobi Perturbations

The development of Gauss's perturbation equations, along with the fact that gravity is a conservative force, leads us to consider whether we might be able to create an analogous description of perturbations based on perturbing potentials. We have already seen, via the introduction of the disturbing function, that a potential-based perturbation description is possible—now, we just need to variational equations for the orbital elements that can exploit such descriptions. However, the most straight-forward way of deriving these equations requires us to utilize multiple concepts from Hamiltonian mechanics, so we begin with a review of Hamiltonian formalism and perturbation methods.

Newton $\longrightarrow$ Euler-Lagrange

- Position of j^{th} (of n) particle with k constraints is a function of generalized coordinates q_i: $\mathbf{r}_j \triangleq \mathbf{r}_j(q_1, q_2, \ldots q_{3n-k}, t)$
- Virtual displacement (infinitesimal, instantaneous and consistent with all constraints): $\delta \mathbf{r}_j = \displaystyle\sum_{l=1}^{3n-k} \frac{\partial \mathbf{r}_j}{\partial q_l} \delta q_l$
- D'Alembert's Principle: $0 = \displaystyle\sum_{j=1}^{n} \left(\mathbf{F}_j^{(a)} - {}^{\mathcal{I}}\frac{\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\mathbf{p}_j \right) \cdot \delta \mathbf{r}_j$
- Generalized Forces: $Q_k \triangleq \displaystyle\sum_{j=1}^{n} \mathbf{F}_j^{(a)} \cdot \frac{\partial \mathbf{r}_j}{\partial q_k}$

Newton $\longrightarrow$ Euler-Lagrange (continued)

$$
\begin{aligned}
0 &= \sum_{j,k} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(m_j {}^{\mathcal{I}}\mathbf{v}_j \cdot \frac{\partial \mathbf{v}_j}{\partial \dot{q}_k} \right) - m_j \mathbf{v}_j \cdot \frac{\partial \mathbf{v}_j}{\partial q_k} \right) \delta q_k - \sum_k Q_k \delta q_k \\
&= \sum_{l=1}^{3n-k} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_l} \right) - \frac{\partial T}{\partial q_l} - Q_l \right) \delta q_l
\end{aligned}
$$

$$
Q_l^{(c)} = -\frac{\partial V}{\partial q_l}; \qquad L \triangleq T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, t)
$$

$$
\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_l} \right) - \frac{\partial L}{\partial q_l} = Q_l^{(nc)}}
$$

Euler-Lagrange $\longrightarrow$ Hamilton

- Generalized (canonical) momentum: $p_j \triangleq \dfrac{\partial L}{\partial \dot{q}_j}$
- Generalized Newton's 2nd law for conservative, holonomic systems: $\dot{p}_j = \dfrac{\partial L}{\partial q_j}$

$$h \triangleq \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \qquad\qquad \frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{\partial L}{\partial t} + \sum_j Q_j \dot{q}_j$$

$$H \triangleq \textstyle\sum_j \dot{q}_j p_j - L(\mathbf{q}, \dot{\mathbf{q}}, t) = \dot{\mathbf{q}} \cdot \mathbf{p} - L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad\qquad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i^{(nc)}$$

Canonical Transformations

$$\left.\begin{aligned} Q_j &\triangleq Q_j(\mathbf{q}, \mathbf{p}, t) \\ P_j &\triangleq P_j(\mathbf{q}, \mathbf{p}, t) \end{aligned}\right\} \quad \dot{Q}_j = \frac{\partial K}{\partial P_j} \qquad \dot{P}_j = -\frac{\partial K}{\partial Q_j}$$

NB: Q is a new coordinate, **not** a generalized force.

$$K(\mathbf{Q}, \mathbf{P}, t) = H(\mathbf{q}(\mathbf{Q}, \mathbf{P}, t), \mathbf{p}(\mathbf{Q}, \mathbf{P}, t), t)$$

$$\mathbf{p} \cdot \dot{\mathbf{q}} - H = \mathbf{P} \cdot \dot{\mathbf{Q}} - K + \frac{\mathrm{d}F}{\mathrm{d}t} \longrightarrow F : \int_{t_1}^{t_2} \frac{\mathrm{d}F}{\mathrm{d}t}\,\mathrm{d}t = F(t_2) - F(t_1) = 0$$

Generating Function	**Transformation**	**Example**
$F = F_1(\mathbf{q}, \mathbf{Q}, t)$	$p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = \mathbf{q} \cdot \mathbf{Q} \quad Q_i = p_i \quad P_i = -q_i$
$F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q} \cdot \mathbf{P}$	$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = \mathbf{q} \cdot \mathbf{P} \quad Q_i = q_i \quad P_i = p_i$
$F = F_3(\mathbf{p}, \mathbf{Q}, t) + \mathbf{q} \cdot \mathbf{p}$	$q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = \mathbf{p} \cdot \mathbf{Q} \quad Q_i = -q_i \quad P_i = -p_i$
$F = F_4(\mathbf{p}, \mathbf{P}, t) + \mathbf{q} \cdot \mathbf{p} - \mathbf{Q} \cdot \mathbf{P}$	$q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = \mathbf{p} \cdot \mathbf{P} \quad Q_i = p_i \quad P_i = -q_i$

Hamilton-Jacobi

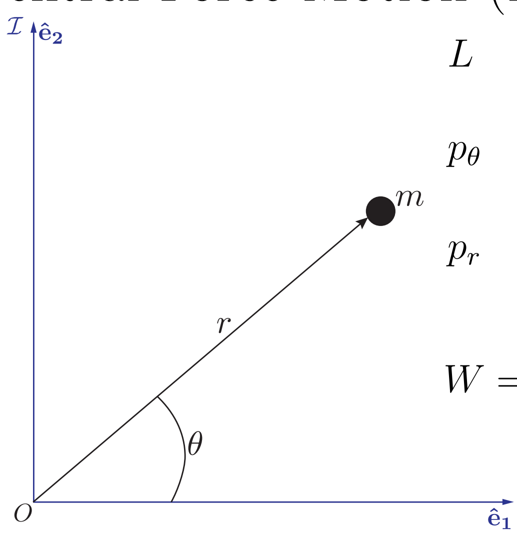
$$\text{Want:}\quad \left.\begin{aligned} \dot{Q}_j &= \frac{\partial K}{\partial P_j} \\ \dot{P}_j &= -\frac{\partial K}{\partial Q_j} \\ K &= H + \frac{\partial F}{\partial t} \end{aligned}\right\} \equiv 0 \quad \Longrightarrow \quad \begin{gathered} F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q} \cdot \mathbf{P} \\ p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \\ H\left(\mathbf{q}, \frac{\partial F_2}{\partial \mathbf{q}}, t\right) + \frac{\partial F_2}{\partial t} = 0 \end{gathered}$$

$$S \equiv F_2 = S(\mathbf{q}, \boldsymbol{\alpha}, t) \qquad p_i = \frac{\partial S}{\partial q_i} \qquad \beta_i \equiv Q_i = \frac{\partial S}{\partial \alpha_i}$$

$$P_i \equiv \alpha_i \longleftarrow \text{constants}$$

$$H = \alpha_1 : \quad S = W(\mathbf{q}, \boldsymbol{\alpha}) - \alpha_1 t \qquad p_i = \frac{\partial W}{\partial q_i} \qquad Q_i = \frac{\partial W}{\partial \alpha_i}$$

Central Force Motion (2D)



$$\left.\begin{aligned} L &= \tfrac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \\ p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \end{aligned}\right\} H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)$$

$$W = W_1(r) + \alpha_\theta \theta \Rightarrow \left(\frac{\partial W_1}{\partial r}\right)^2 + \left(\frac{\alpha_\theta}{r}\right)^2 + 2mV(r) = 2m\alpha_1$$

$$t + \beta_1 \triangleq \frac{\partial W}{\partial \alpha_1} = \int \frac{m\,\mathrm{d}r}{\sqrt{2m\left(\alpha_1 - V(r)\right) - \left(\frac{\alpha_\theta}{r}\right)^2}}$$

$$\beta_2 \triangleq \frac{\partial W}{\partial \alpha_\theta} = -\int \frac{\alpha_\theta\,\mathrm{d}r}{r^2\sqrt{2m\left(\alpha_1 - V(r)\right) - \left(\frac{\alpha_\theta}{r}\right)^2}} + \theta$$