9 - Lagrange's Planetary Equations

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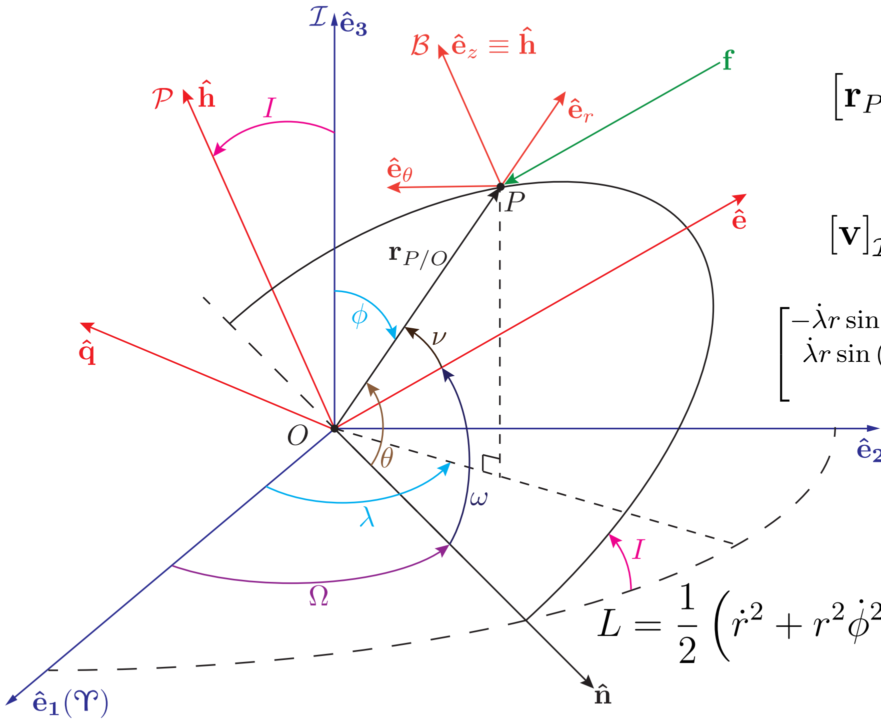
Lagrange's Planetary Equations

Now that we have applied Hamilton-Jacobi to central force motion in two dimensions, we must repeat the process in three dimensions in order to get at our desired variational equations for the orbital elements as functions of perturbing potentials. The 3D case is not qualitatively different from the 2D development, but does require a bit more careful bookkeeping. We will apply Hamilton-Jacobi to create a canonical set of orbital elements, and from there derive variational equations for the Keplerian elements (which are themselves not canonical).

Central Force Motion (3D)



$$\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}} \triangleq [\mathbf{r}]_{\mathcal{I}} = r \begin{bmatrix} \sin(\phi)\cos(\lambda) \\ \sin(\lambda)\sin(\phi) \\ \cos(\phi) \end{bmatrix}_{\mathcal{I}}$$

$$[\mathbf{v}]_{\mathcal{I}} \triangleq \left[\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r}\right]_{\mathcal{I}} =$$

$$\begin{bmatrix} -\dot{\lambda} r \sin(\lambda)\sin(\phi) + \dot{\phi} r \cos(\lambda)\cos(\phi) + \dot{r}\sin(\phi)\cos(\lambda) \\ \dot{\lambda} r \sin(\phi)\cos(\lambda) + \dot{\phi} r \sin(\lambda)\cos(\phi) + \dot{r}\sin(\lambda)\sin(\phi) \\ -\dot{\phi} r \sin(\phi) + \dot{r}\cos(\phi) \end{bmatrix}_{\mathcal{I}}$$

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + r^2\dot{\lambda}^2\sin^2\phi\right) + \frac{\mu}{r} + \underbrace{U^{(1)}(r,\phi,\lambda)}_{\text{Perturbing Potential}}$$

The Unperturbed Hamiltonian

$$\text{Canonical Momenta: } p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \Longrightarrow \quad p_r = \dot{r} \qquad p_\phi = r^2\dot{\phi} \qquad p_\lambda = r^2\sin^2\phi\,\dot{\lambda}$$

$$H^{(0)} = \mathbf{p}\cdot\dot{\mathbf{q}} - L^{(0)} = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\phi^2 + \frac{1}{r^2\sin^2\phi}p_\lambda^2\right) - \frac{\mu}{r} \equiv \mathcal{E}$$

$$\alpha_1 \triangleq \sqrt{\mu a} \quad \Longrightarrow \quad \mathcal{E} = -\frac{\mu^2}{2\alpha_1^2}$$

$$W \triangleq W_r(r) + W_\phi(\phi) + \lambda \underbrace{\alpha_\lambda}_{\equiv P_\lambda}$$

$$p_r = \frac{\partial W_r}{\partial r} = \left(\frac{2\mu}{r} - \frac{\mu^2}{\alpha_1^2} - \frac{\alpha_2^2}{r^2}\right)^{1/2} \qquad\qquad p_\phi = \frac{\partial W_\phi}{\partial \phi} = \left(\alpha_2^2 - \frac{\alpha_\lambda^2}{\sin^2\phi}\right)^{1/2}$$

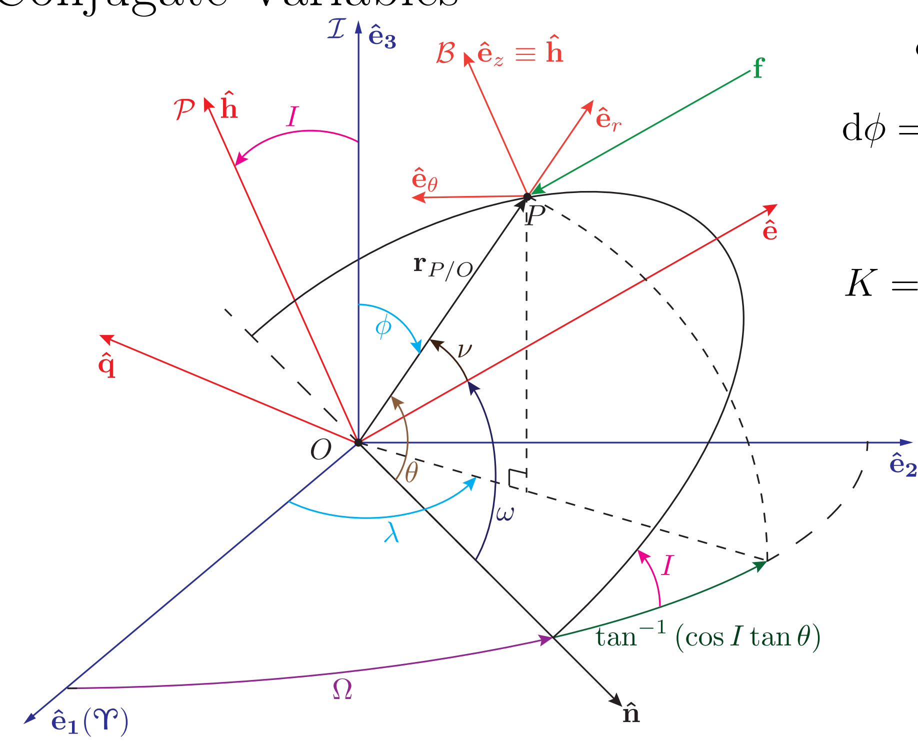
Constant Momenta

$$[\mathbf{h}]_{\mathcal{I}} = [\mathbf{r}\times\mathbf{v}]_{\mathcal{I}} = r^2\begin{bmatrix} \frac{\dot\lambda\sin(\lambda-2\phi)}{4} - \frac{\dot\lambda\sin(\lambda+2\phi)}{4} - \dot\phi\sin(\lambda) \\ -\frac{\dot\lambda\cos(\lambda-2\phi)}{4} + \frac{\dot\lambda\cos(\lambda+2\phi)}{4} + \dot\phi\cos(\lambda) \\ \dot\lambda\sin^2(\phi) \end{bmatrix}_{\mathcal{I}}$$

$$h^2 = \mathbf{h}\cdot\mathbf{h} = r^4\left(\dot\lambda^2\sin^2(\phi) + \dot\phi^2\right)$$

$$\alpha_1 \triangleq \sqrt{\mu a}$$
$$\alpha_2 \equiv h = \alpha_1\sqrt{1-e^2}$$
$$\alpha_\lambda \equiv \mathbf{h}\cdot\hat{\mathbf{e}}_3 = \alpha_2\cos I$$

Conjugate Variables



$$\phi = \cos^{-1}(\sin(I)\sin(\theta))$$

$$\mathrm{d}\phi = -\frac{\sin(I)\cos(\theta)\,\mathrm{d}\theta}{\sqrt{1-\sin^2(I)\sin^2(\theta)}}$$

$$K = \mathcal{E} \qquad \dot Q_i = \frac{\partial K}{\partial\alpha_i} \qquad Q_i = \frac{\partial W}{\partial\alpha_i}$$

$$nt - \beta_1 = E - e\sin(E)$$
$$\Rightarrow \beta_1 = nt_0$$
$$\beta_2 = \theta - \nu = \omega$$
$$\beta_\lambda = \lambda - \tan^{-1}(\cos I\tan\theta)$$
$$\Rightarrow \beta_\lambda = \Omega$$

Canonical (Delaunay) Variables

	Element	Kepler Mapping	Delaunay
Momenta (action)	α_1	$\sqrt{\mu a}$	L
	α_2	$\sqrt{\mu a(1-e^2)}$	G
	α_3	$\sqrt{\mu a(1-e^2)}\cos I$	H
Coordinates (angle)	β_1	M	ℓ
	β_2	ω	g
	β_3	Ω	h

$$H = H^{(0)} + H^{(1)} = -\frac{\mu^2}{2\alpha_1^2} - U^{(1)} \qquad \dot{\alpha}_i = \frac{\partial U^{(1)}}{\partial \beta_i} \qquad \dot{\beta}_i = -\frac{\partial U^{(1)}}{\partial \alpha_i} + \frac{\partial}{\partial \alpha_i}\left(\frac{\mu^2}{2\alpha_1^2}\right)$$

Symplectic Representation of Hamiltonians

$$\mathbf{q} \triangleq \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \qquad \mathbf{p} \triangleq \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \qquad \mathbf{z} \triangleq \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} \quad \Longrightarrow \quad \dot{\mathbf{z}} = \underbrace{\begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{bmatrix}}_{\triangleq J} \begin{bmatrix} \frac{\partial H}{\partial \mathbf{q}} \\ \frac{\partial H}{\partial \mathbf{p}} \end{bmatrix} = J\nabla_{\mathbf{z}} H$$

Symplectic Matrix Properties: $JJ \triangleq J^2 = -I \qquad \det(J) = 1$
$JJ^T = J^T J = I \qquad J^T = -J = J^{-1}$

$$\mathbf{y} \triangleq \begin{bmatrix} \mathbf{Q} \\ \mathbf{P} \end{bmatrix} \Longrightarrow H(\mathbf{z}) = H\left(\mathbf{z}(\mathbf{y})\right) \triangleq K(\mathbf{y}) \qquad M \triangleq \frac{\partial \mathbf{y}}{\partial \mathbf{z}} \Longrightarrow \dot{\mathbf{y}} = MJM^T \nabla_{\mathbf{y}} K$$

Symplectic M defines a canonical transformation!

Lagrange's Planetary Equations (setup)

Start with Delaunay Elements:

α_1	$\sqrt{\mu a}$
α_2	$\sqrt{\mu a(1-e^2)}$
α_3	$\sqrt{\mu a(1-e^2)}\cos I$
β_1	M
β_2	ω
β_3	Ω

$$\mathbf{z} \triangleq \begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{\alpha} \end{bmatrix}$$

$$H = -\frac{\mu^2}{2\alpha_1^2} - U^{(1)}$$

$$\dot{\mathbf{z}} = J\nabla_{\mathbf{z}} H$$

Define New State and Hamiltonian: $\mathbf{y} \triangleq \begin{bmatrix} a \\ e \\ I \\ M \\ \omega \\ \Omega \end{bmatrix} \implies K = -\frac{\mu}{2a} - U^{(1)}(\mathbf{z}(\mathbf{y}))$

Lagrange's Planetary Equations (transformation)

$$\mathbf{y} = \begin{bmatrix} \frac{\alpha_1^2}{\mu} \\ \sqrt{1-\frac{\alpha_2^2}{\alpha_1^2}} \\ \cos^{-1}\left(\frac{\alpha_3}{\alpha_2}\right) \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \qquad P \triangleq \frac{\partial \mathbf{y}}{\partial \mathbf{z}} = \begin{bmatrix} 0 & 0 & 0 & \frac{2\alpha_1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_2^2}{\alpha_1^2\sqrt{\alpha_1^2-\alpha_2^2}} & -\frac{\alpha_2}{\alpha_1\sqrt{\alpha_1^2-\alpha_2^2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha_3}{\alpha_2\sqrt{\alpha_2^2-\alpha_3^2}} & -\frac{1}{\sqrt{\alpha_2^2-\alpha_3^2}} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\dot{\mathbf{y}} = PJP^T\nabla_{\mathbf{y}} K}$$

Lagrange's Planetary Equations

$$
\begin{aligned}
\dot{a} &= 2\sqrt{\frac{a}{\mu}}\frac{\partial U^{(1)}}{\partial M} \\
\dot{e} &= \frac{1}{e\sqrt{a\mu}}\left(\left(1-e^2\right)\frac{\partial U^{(1)}}{\partial M} - \sqrt{1-e^2}\frac{\partial U^{(1)}}{\partial \omega}\right) \\
\dot{I} &= \frac{\cos\left(I\right)\frac{\partial U^{(1)}}{\partial \omega} - \frac{\partial U^{(1)}}{\partial \Omega}}{\sqrt{a\mu\left(1-e^2\right)}\sin\left(I\right)} \\
\dot{M} &= \sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}}\frac{\partial U^{(1)}}{\partial a} - \frac{1-e^2}{e\sqrt{a\mu}}\frac{\partial U^{(1)}}{\partial e} \\
\dot{\omega} &= \frac{1}{\sqrt{a\mu}}\left(\frac{\sqrt{1-e^2}}{e}\frac{\partial U^{(1)}}{\partial e} - \frac{\cot(I)}{\sqrt{1-e^2}}\frac{\partial U^{(1)}}{\partial I}\right) \\
\dot{\Omega} &= \frac{1}{\sqrt{a\mu\left(1-e^2\right)}\sin\left(I\right)}\frac{\partial U^{(1)}}{\partial I}
\end{aligned}
$$

Lagrange Planetary Equations (other versions)

$$
\begin{aligned}
\frac{d\Omega}{dt} &= \frac{1}{nab\sin i}\frac{\partial R}{\partial i} \\
\frac{di}{dt} &= -\frac{1}{nab\sin i}\frac{\partial R}{\partial \Omega} + \frac{\cos i}{nab\sin i}\frac{\partial R}{\partial \omega} \\
\frac{d\omega}{dt} &= -\frac{\cos i}{nab\sin i}\frac{\partial R}{\partial i} + \frac{b}{na^3 e}\frac{\partial R}{\partial e} \\
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \lambda} \\
\frac{de}{dt} &= -\frac{b}{na^3 e}\frac{\partial R}{\partial \omega} + \frac{b^2}{na^4 e}\frac{\partial R}{\partial \lambda} \\
\frac{d\lambda}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{b^2}{na^4 e}\frac{\partial R}{\partial e}
\end{aligned}
$$

Battin (1999) Eq. 10.31

$$\lambda \triangleq nt_p$$

$$b = a\sqrt{1-e^2}$$

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial M_o} \\
\frac{de}{dt} &= \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M_o} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega} \\
\frac{di}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\,\mathrm{SIN}(i)}\left\{\mathrm{COS}(i)\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right\} \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{\mathrm{COT}(i)}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\,\mathrm{SIN}(i)}\frac{\partial R}{\partial i} \\
\frac{dM_o}{dt} &= -\frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} \quad +n
\end{aligned}
$$

Vallado (2013) Eq. 9-12

$$M = M_0 + n(t - t_p)$$