

9 - Lagrange's Planetary Equations

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Lagrange's Planetary Equations

Now that we have applied Hamilton-Jacobi to central force motion in two dimensions, we must repeat the process in three dimensions in order to get at our desired variational equations for the orbital elements as functions of perturbing potentials. The 3D case is not qualitatively different from the 2D development, but does require a bit more careful bookkeeping. We will apply Hamilton-Jacobi to create a canonical set of orbital elements, and from there derive variational equations for the Keplerian elements (which are themselves not canonical).

Central Force Motion (3D)

The diagram illustrates the geometry of 3D central force motion. It shows a central body at origin O and a particle at position P . The position vector is $\mathbf{r}_{P/O}$. The angular momentum vector is \mathbf{h} , and the Runge-Lenz vector is \mathbf{A} . The diagram shows the orientation of the orbital plane relative to the vertical axis $\hat{\mathbf{e}}_3$ and the horizontal axis $\hat{\mathbf{e}}_1(\mathcal{R})$. The angle between $\hat{\mathbf{e}}_3$ and \mathbf{h} is θ . The angle between \mathbf{h} and the orbital plane is ν . The angle between the orbital plane and the horizontal axis is λ . The angle between the position vector and the horizontal axis is ϕ . The angle between the position vector and the vertical axis is ψ . The angle between the position vector and the Runge-Lenz vector is ω . The angle between the Runge-Lenz vector and the horizontal axis is Ω . The angle between the Runge-Lenz vector and the vertical axis is \mathcal{I} . The angle between the Runge-Lenz vector and the position vector is \mathcal{I} . The angle between the Runge-Lenz vector and the horizontal axis is \mathcal{I} . The angle between the Runge-Lenz vector and the vertical axis is \mathcal{I} .

$$[\mathbf{r}_{P/O}]_{\mathcal{I}} \triangleq [\mathbf{r}]_{\mathcal{I}} = r \begin{bmatrix} \sin(\phi) \cos(\lambda) \\ \sin(\lambda) \sin(\phi) \\ \cos(\phi) \end{bmatrix}_{\mathcal{I}}$$

$$[\mathbf{v}]_{\mathcal{I}} \triangleq \left[\frac{\mathcal{I} d}{dt} \mathbf{r} \right]_{\mathcal{I}} = \begin{bmatrix} -\dot{\lambda} r \sin(\lambda) \sin(\phi) + \dot{\phi} r \cos(\lambda) \cos(\phi) + \dot{r} \sin(\phi) \cos(\lambda) \\ \dot{\lambda} r \sin(\phi) \cos(\lambda) + \dot{\phi} r \sin(\lambda) \cos(\phi) + \dot{r} \sin(\lambda) \sin(\phi) \\ -\dot{\phi} r \sin(\phi) + \dot{r} \cos(\phi) \end{bmatrix}_{\mathcal{I}}$$

$$L = \frac{1}{2} \left(\dot{r}^2 + r^2 \dot{\phi}^2 + r^2 \dot{\lambda}^2 \sin^2 \phi \right) + \frac{\mu}{r} + \underbrace{U^{(1)}(r, \phi, \lambda)}_{\text{Perturbing Potential}}$$

The Unperturbed Hamiltonian

Canonical Momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i} \implies p_r = \dot{r} \quad p_\phi = r^2 \dot{\phi} \quad p_\lambda = r^2 \sin^2 \phi \dot{\lambda}$

$$H^{(0)} = \mathbf{p} \cdot \dot{\mathbf{q}} - L^{(0)} = \frac{1}{2} \left(p_r^2 + \frac{1}{r^2} p_\phi^2 + \frac{1}{r^2 \sin^2 \phi} p_\lambda^2 \right) - \frac{\mu}{r} \equiv \mathcal{E}$$

$$\alpha_1 \triangleq \sqrt{\mu a} \implies \mathcal{E} = -\frac{\mu^2}{2\alpha_1^2}$$

$$W \triangleq W_r(r) + W_\phi(\phi) + \lambda \underbrace{\alpha_\lambda}_{\equiv P_\lambda}$$

$$p_r = \frac{\partial W_r}{\partial r} = \left(\frac{2\mu}{r} - \frac{\mu^2}{\alpha_1^2} - \frac{\alpha_2^2}{r^2} \right)^{1/2}$$

$$p_\phi = \frac{\partial W_\phi}{\partial \phi} = \left(\alpha_2^2 - \frac{\alpha_\lambda^2}{\sin^2 \phi} \right)^{1/2}$$

Constant Momenta

$$[\mathbf{h}]_{\mathcal{I}} = [\mathbf{r} \times \mathbf{v}]_{\mathcal{I}} = r^2 \begin{bmatrix} \frac{\dot{\lambda} \sin(\lambda - 2\phi)}{4} - \frac{\dot{\lambda} \sin(\lambda + 2\phi)}{4} - \dot{\phi} \sin(\lambda) \\ -\frac{\dot{\lambda} \cos(\lambda - 2\phi)}{4} + \frac{\dot{\lambda} \cos(\lambda + 2\phi)}{4} + \dot{\phi} \cos(\lambda) \\ \dot{\lambda} \sin^2(\phi) \end{bmatrix}_{\mathcal{I}}$$

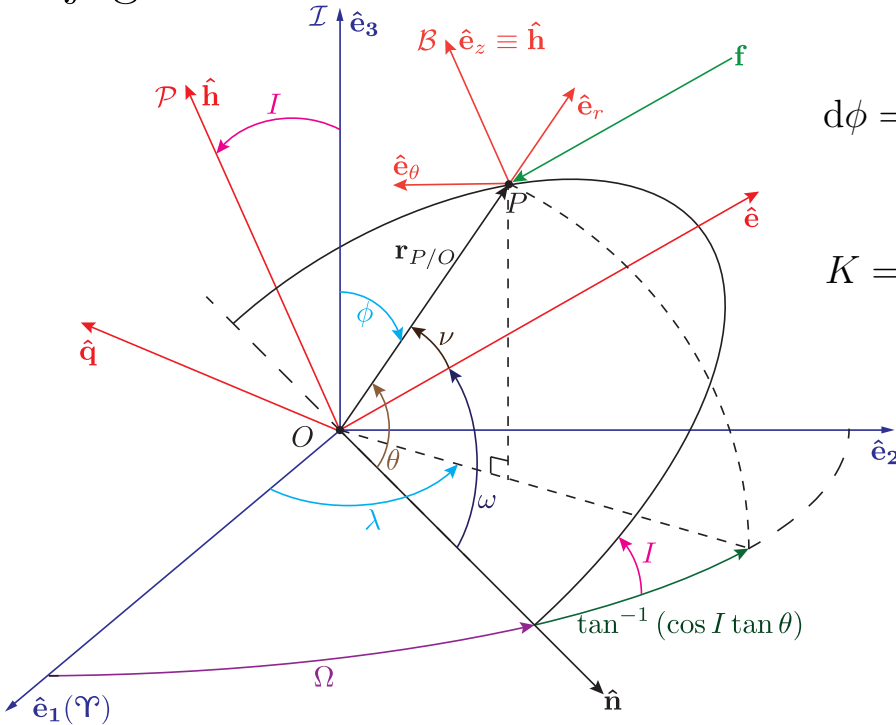
$$h^2 = \mathbf{h} \cdot \mathbf{h} = r^4 \left(\dot{\lambda}^2 \sin^2(\phi) + \dot{\phi}^2 \right)$$

$$\alpha_1 \triangleq \sqrt{\mu a}$$

$$\alpha_2 \equiv h = \alpha_1 \sqrt{1 - e^2}$$

$$\alpha_\lambda \equiv \mathbf{h} \cdot \hat{\mathbf{e}}_3 = \alpha_2 \cos I$$

Conjugate Variables



$$\phi = \cos^{-1}(\sin(I) \sin(\theta))$$

$$d\phi = -\frac{\sin(I) \cos(\theta) d\theta}{\sqrt{1 - \sin^2(I) \sin^2(\theta)}}$$

$$K = \mathcal{E} \quad \dot{Q}_i = \frac{\partial K}{\partial \alpha_i} \quad Q_i = \frac{\partial W}{\partial \alpha_i}$$

$$nt - \beta_1 = E - e \sin(E)$$

$$\Rightarrow \beta_1 = nt_0$$

$$\beta_2 = \theta - \nu = \omega$$

$$\beta_\lambda = \lambda - \tan^{-1}(\cos I \tan \theta)$$

$$\Rightarrow \beta_\lambda = \Omega$$

Canonical (Delaunay) Variables

| | Element | Kepler Mapping | Delaunay |
|---------------------|------------|--------------------------------|----------|
| Momenta (action) | α_1 | $\sqrt{\mu a}$ | L |
| | α_2 | $\sqrt{\mu a(1 - e^2)}$ | G |
| | α_3 | $\sqrt{\mu a(1 - e^2)} \cos I$ | H |
| Coordinates (angle) | β_1 | M | ℓ |
| | β_2 | ω | g |
| | β_3 | Ω | h |

$$H = H^{(0)} + H^{(1)} = -\frac{\mu^2}{2\alpha_1^2} - U^{(1)} \quad \dot{\alpha}_i = \frac{\partial U^{(1)}}{\partial \beta_i} \quad \dot{\beta}_i = -\frac{\partial U^{(1)}}{\partial \alpha_i} + \frac{\partial}{\partial \alpha_i} \left(\frac{\mu^2}{2\alpha_1^2} \right)$$

Symplectic Representation of Hamiltonians

$$\mathbf{q} \triangleq \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \quad \mathbf{p} \triangleq \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \quad \mathbf{z} \triangleq \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} \quad \Longrightarrow \quad \dot{\mathbf{z}} = \underbrace{\begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{bmatrix}}_{\triangleq J} \begin{bmatrix} \frac{\partial H}{\partial \mathbf{q}} \\ \frac{\partial H}{\partial \mathbf{p}} \end{bmatrix} = J \nabla_{\mathbf{z}} H$$

$$\text{Symplectic Matrix Properties:} \quad \begin{aligned} J J &\triangleq J^2 = -I & \det(J) &= 1 \\ J J^T &= J^T J = I & J^T &= -J = J^{-1} \end{aligned}$$

$$\mathbf{y} \triangleq \begin{bmatrix} \mathbf{Q} \\ \mathbf{P} \end{bmatrix} \quad \Longrightarrow \quad H(\mathbf{z}) = H(\mathbf{z}(\mathbf{y})) \triangleq K(\mathbf{y}) \quad M \triangleq \frac{\partial \mathbf{y}}{\partial \mathbf{z}} \quad \Longrightarrow \quad \dot{\mathbf{y}} = M J M^T \nabla_{\mathbf{y}} K$$

Symplectic M defines a canonical transformation!

Lagrange's Planetary Equations (setup)

Start with Delaunay Elements:

| | |
|------------|--------------------------------|
| α_1 | $\sqrt{\mu a}$ |
| α_2 | $\sqrt{\mu a(1 - e^2)}$ |
| α_3 | $\sqrt{\mu a(1 - e^2)} \cos I$ |
| β_1 | M |
| β_2 | ω |
| β_3 | Ω |

$$\mathbf{z} \triangleq \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

$$H = -\frac{\mu^2}{2\alpha_1^2} - U^{(1)}$$

$$\dot{\mathbf{z}} = J\nabla_{\mathbf{z}}H$$

Define New State and Hamiltonian: $\mathbf{y} \triangleq \begin{bmatrix} a \\ e \\ I \\ M \\ \omega \\ \Omega \end{bmatrix} \implies K = -\frac{\mu}{2a} - U^{(1)}(\mathbf{z}(\mathbf{y}))$

Lagrange's Planetary Equations (transformation)

$$\mathbf{y} = \begin{bmatrix} \frac{\alpha_1^2}{\mu} \\ \sqrt{1 - \frac{\alpha_2^2}{\alpha_1^2}} \\ \cos^{-1}\left(\frac{\alpha_3}{\alpha_2}\right) \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$P \triangleq \frac{\partial \mathbf{y}}{\partial \mathbf{z}} = \begin{bmatrix} 0 & 0 & 0 & \frac{2\alpha_1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_2^2}{\alpha_1^2 \sqrt{\alpha_1^2 - \alpha_2^2}} & -\frac{\alpha_2}{\alpha_1 \sqrt{\alpha_1^2 - \alpha_2^2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha_3}{\alpha_2 \sqrt{\alpha_2^2 - \alpha_3^2}} & -\frac{1}{\sqrt{\alpha_2^2 - \alpha_3^2}} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\dot{\mathbf{y}} = PJP^T \nabla_{\mathbf{y}}K$$

Lagrange's Planetary Equations

$$\begin{aligned}
 \dot{a} &= 2\sqrt{\frac{a}{\mu}} \frac{\partial U^{(1)}}{\partial M} \\
 \dot{e} &= \frac{1}{e\sqrt{a\mu}} \left((1-e^2) \frac{\partial U^{(1)}}{\partial M} - \sqrt{1-e^2} \frac{\partial U^{(1)}}{\partial \omega} \right) \\
 \dot{i} &= \frac{\cos(I) \frac{\partial U^{(1)}}{\partial \omega} - \frac{\partial U^{(1)}}{\partial \Omega}}{\sqrt{a\mu} (1-e^2) \sin(I)} \\
 \dot{M} &= \sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}} \frac{\partial U^{(1)}}{\partial a} - \frac{1-e^2}{e\sqrt{a\mu}} \frac{\partial U^{(1)}}{\partial e} \\
 \dot{\omega} &= \frac{1}{\sqrt{a\mu}} \left(\frac{\sqrt{1-e^2}}{e} \frac{\partial U^{(1)}}{\partial e} - \frac{\cot(I)}{\sqrt{1-e^2}} \frac{\partial U^{(1)}}{\partial I} \right) \\
 \dot{\Omega} &= \frac{1}{\sqrt{a\mu} (1-e^2) \sin(I)} \frac{\partial U^{(1)}}{\partial I}
 \end{aligned}$$

Lagrange Planetary Equations (other versions)

$$\begin{aligned}
 \frac{d\Omega}{dt} &= \frac{1}{nab \sin i} \frac{\partial R}{\partial i} \\
 \frac{di}{dt} &= -\frac{1}{nab \sin i} \frac{\partial R}{\partial \Omega} + \frac{\cos i}{nab \sin i} \frac{\partial R}{\partial \omega} \\
 \frac{d\omega}{dt} &= -\frac{\cos i}{nab \sin i} \frac{\partial R}{\partial i} + \frac{b}{na^3 e} \frac{\partial R}{\partial e} \\
 \frac{da}{dt} &= \frac{2}{na} \frac{\partial R}{\partial \lambda} \\
 \frac{de}{dt} &= -\frac{b}{na^3 e} \frac{\partial R}{\partial \omega} + \frac{b^2}{na^4 e} \frac{\partial R}{\partial \lambda} \\
 \frac{d\lambda}{dt} &= -\frac{2}{na} \frac{\partial R}{\partial a} - \frac{b^2}{na^4 e} \frac{\partial R}{\partial e}
 \end{aligned}$$

Battin (1999) Eq. 10.31

$$\begin{aligned}
 \lambda &\triangleq nt_p \\
 b &= a\sqrt{1-e^2}
 \end{aligned}$$

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial M_o}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial M_o} - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial \omega}$$

$$\frac{di}{dt} = \frac{1}{na^2 \sqrt{1-e^2} \sin(i)} \left\{ \cos(i) \frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega} \right\}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial e} - \frac{\cot(i)}{na^2 \sqrt{1-e^2}} \frac{\partial R}{\partial i}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2 \sqrt{1-e^2} \sin(i)} \frac{\partial R}{\partial i}$$

$$\frac{dM_o}{dt} = -\frac{1-e^2}{na^2 e} \frac{\partial R}{\partial e} - \frac{2}{na} \frac{\partial R}{\partial a} + n$$

Vallado (2013) Eq. 9-12

$$M = M_0 + n(t - t_p)$$