Variational Equations for Control and Estimation of Satellite Relative Motion

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Why study satellite relative motion?

General applications

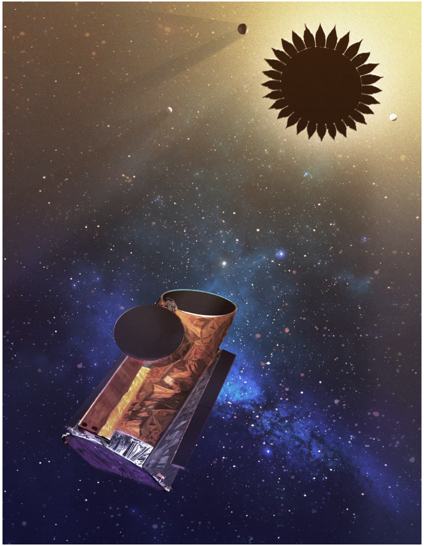
- Rendezvous/docking
- Inspection
- Servicing/refueling

Scientific applications

- Required large separation of instruments
  - Magnetosphere Multi-Scale mission (Roscoe et al. 2011)
  - **Starshade-based exoplanet imaging** (Gaudi et al. 2020)

Other benefits

- Redundancy
- Cost reductions



Outline

Dynamical Systems Background
- Calculating state transition tensors

Satellite Relative Motion Control
- State transition matrix for optimal control
- Composing STMs
- Interpolating STMs
- Precomputation and online algorithm

Angles-Only Relative Orbit Estimation
- Linear unobservable range
- Range ambiguous linear estimate
- Scale estimate
- Quadratic approximations

Dynamical Systems Preliminaries

- State vector $\mathbf{x} \in \mathbb{R}^n$
- Dynamics/vector field $\mathbf{F}$:

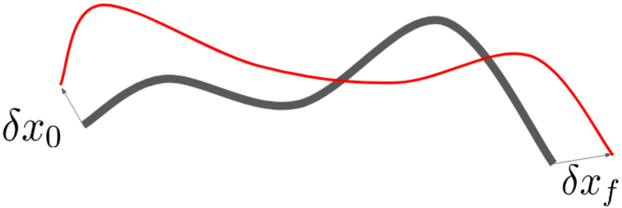
$$\frac{d}{dt}\mathbf{x} = \mathbf{F}(\mathbf{x})$$

- Flow map $\varphi_t : \mathbf{x}_0 \to \mathbf{x}_t$

$$\frac{d}{dt}\varphi_t(\mathbf{x}) = \mathbf{F}(\varphi_t(\mathbf{x})), \quad \varphi_0(\mathbf{x}) = \mathbf{x}$$

Taylor Expansion of the Flow Map

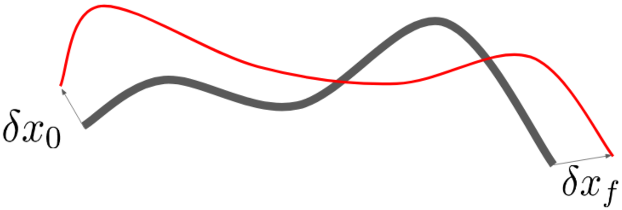
Application of high order expansion began in astrodynamics uncertainty propagation (Park and Scheeres 2006)



$$\mathbf{x}_f + \delta\mathbf{x}_f = \varphi_{t_f}(\mathbf{x}_0 + \delta\mathbf{x}_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \varphi_{t_f}}{\partial \mathbf{x}^n} \right|_{\mathbf{x}_0} \delta\mathbf{x}_0^n$$

State Transition Matrix

Applied to satellite relative motion since (Clohessy and Wiltshire 1960)



The state transition matrix gives the linear approximation

$$\delta \mathbf{x}_f \approx \left. \frac{\partial \varphi_{t_f}}{\partial \mathbf{x}} \right|_{\mathbf{x}_0} \delta \mathbf{x_0} = \Phi(t_f, t_0)\delta \mathbf{x}_0$$

Calculating the State Transition Matrix

Assuming sufficient smoothness of $\varphi_t(\mathbf{x})$ (Clairaut's Theorem)

$$\frac{d\Phi(t,0)}{dt} = \frac{d}{dt}\frac{\partial \varphi_t}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\frac{d\varphi_t}{dt} = \frac{\partial}{\partial \mathbf{x}}\mathbf{F}(\varphi_t(\mathbf{x})) = \left.\frac{\partial \mathbf{F}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\varphi_t(\mathbf{x})} \Phi(t,0)$$

$$\dot{\Phi} = D\mathbf{F} \cdot \Phi, \quad \Phi(0,0) = I_n$$

First order variational equations yield the state transition matrix.

- Integrate alongside original dynamical system.
- **Higher order variational equations** yield state transition tensors.

Calculating the Second-Order State Transition Tensor

Ψ is a 1-2 tensor.

$$\delta \mathbf{x}_f \approx \Phi(t_f, 0)\delta \mathbf{x}_0 + \Psi(t_f, 0)\delta \mathbf{x}_0^2$$

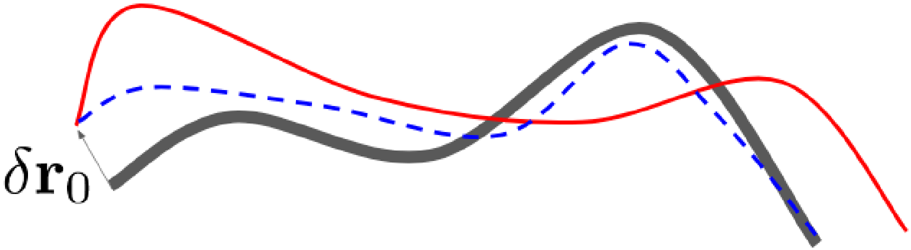
The second order variational equations:

$$\frac{d\Psi^i_{j,k}(t,0)}{dt} = \frac{\partial^2 F_i(\mathbf{x})}{\partial x_l \partial x_q}\Phi^l_j(t,0)\Phi^q_k(t,0) + \frac{\partial F_i(\mathbf{x})}{\partial x_l}\Psi^l_{j,k}(t,0), \quad \Psi^i_{j,k}(0,0) = 0$$

Outline

Rendezvous

What initial velocity leads the deputy satellite to meet the chief (reference) satellite?



Relative Transfer

Partition the state into position and velocity (Mullins 1992)

$$\delta \mathbf{x} = \begin{bmatrix} \delta \mathbf{r}^T & \delta \mathbf{v}^T \end{bmatrix}$$
$$\Phi(t, 0) = \begin{bmatrix} \Phi_{\mathbf{r}}^{\mathbf{r}} & \Phi_{\mathbf{v}}^{\mathbf{r}} \\ \Phi_{\mathbf{r}}^{\mathbf{v}} & \Phi_{\mathbf{v}}^{\mathbf{v}} \end{bmatrix}$$

Initial relative velocity $\delta \mathbf{v}_0$ to change relative position $\delta \mathbf{r}_0 \rightarrow \delta \mathbf{r}_t$

$$\delta \mathbf{r}_t \approx \boldsymbol{\Phi}_{\mathbf{r}}^{\mathbf{r}} \delta \mathbf{r}_0 + \boldsymbol{\Phi}_{\mathbf{v}}^{\mathbf{r}} \delta \mathbf{v}_0 \implies$$
$$\delta \mathbf{v}_0 \approx (\boldsymbol{\Phi}_{\mathbf{v}}^{\mathbf{r}})^{-1} (\delta \mathbf{r}_t - \boldsymbol{\Phi}_{\mathbf{r}}^{\mathbf{r}} \delta \mathbf{r}_0)$$

Continuous Thrust Control

Minimize cost $J = \int_{t_0}^{t_f} \frac{1}{2}\mathbf{u}^T\mathbf{u}\,\mathrm{d}t$ under boundary conditions $\mathbf{x}(t_0) = \mathbf{x_0}, \mathbf{x}(t_f) = \mathbf{x_f}$:

$$\frac{d}{dt}\mathbf{x} = \mathbf{F}(\mathbf{x}) + \mathbf{u}, \quad \frac{d}{dt}\boldsymbol{\lambda} = -\left(\frac{\partial \mathbf{F}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \boldsymbol{\lambda}$$

Two point boundary value problem with twelve equations for states and costates.
(Bryson and Ho 2018)
Optimal control given by $\mathbf{u} = -(0, 0, 0, \lambda_4, \lambda_5, \lambda_6)^T$

Continuous Thrust Relative Transfers

Define the augmented state vector $\mathbf{z} = [\mathbf{x}^T \quad \boldsymbol{\lambda}^T \quad J]^T$ and its dynamics

$$\frac{d}{dt}\mathbf{z} = \mathbf{G}(\mathbf{z}) = \left[\left(\mathbf{F}(\mathbf{x})^T + \begin{bmatrix}\mathbf{0} & \mathbf{u}^T\end{bmatrix}\right) \quad -\boldsymbol{\lambda}^T\left(\frac{\partial \mathbf{F}(\mathbf{x})}{\partial \mathbf{x}}\right) \quad \frac{1}{2}\mathbf{u}^T\mathbf{u}\right]^T$$

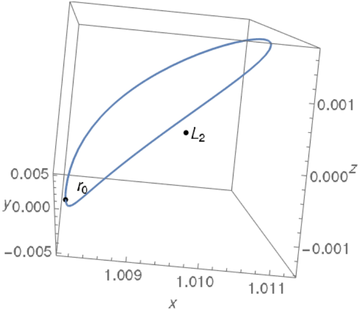
Initial costates are given by STM associated with $\mathbf{G}$

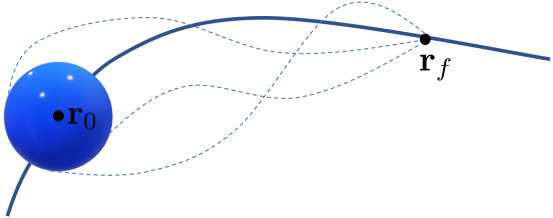
$$\delta\boldsymbol{\lambda}_0 \approx (\Phi^{\mathbf{x}}_{\boldsymbol{\lambda}})^{-1}(\delta\mathbf{x}_t - \Phi^{\mathbf{x}}_{\mathbf{x}}\delta\mathbf{x}_0)$$

Previous works have only applied to 2-body dynamics (Lembeck and Prussing 1993; Carter and Humi 1987)
Energy cost approximation from second order state transition tensor:

$$\delta J \approx \frac{\partial^2 J_f}{\partial \mathbf{x}_0^2}\delta\mathbf{x}_0^2 + \frac{\partial^2 J_f}{\partial \boldsymbol{\lambda}_0^2}\delta\boldsymbol{\lambda}_0^2$$

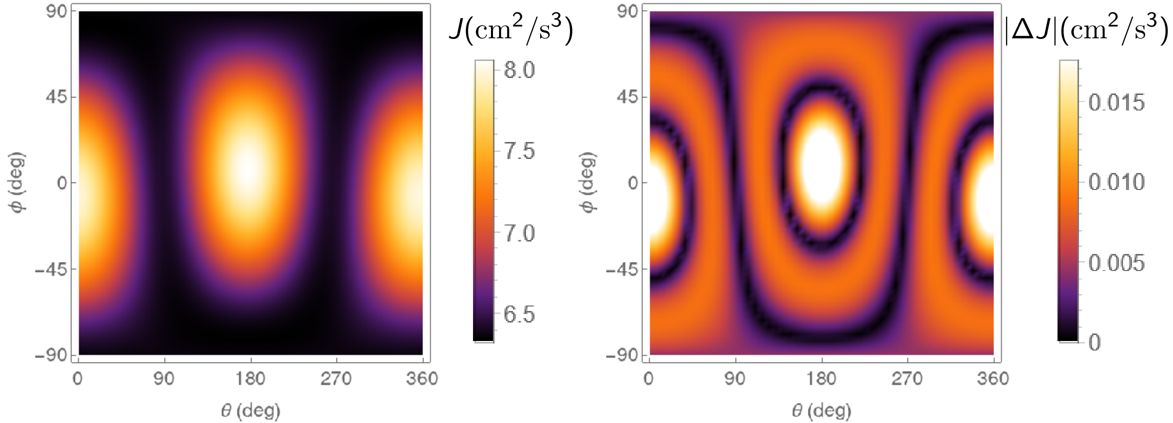
Three Body Problem Example





Rendezvous onto an Earth-Sun L2 Halo Orbit from a sphere of initial relative positions

Rendezvous on SEL2 Halo: Two Weeks from 10,000km Sphere



The rendezvous control cost approximated by second order state transition tensor and the error in its approximation versus numerical integration.

Problems with Practical Computation

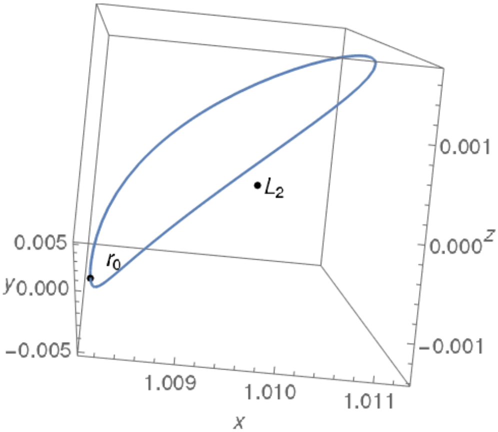
Fast calculations once we have Φ and potentially higher order state transition tensors.
What about calculating Φ?

- Analytical solutions:
  - Clohessy-Wiltshire
  - Yamanaka-Ankersen
  - Their respective adjoint equations
- Numerical integration:
  - High fidelity two body motion with perturbations
  - Three body motion
  - Costate equations

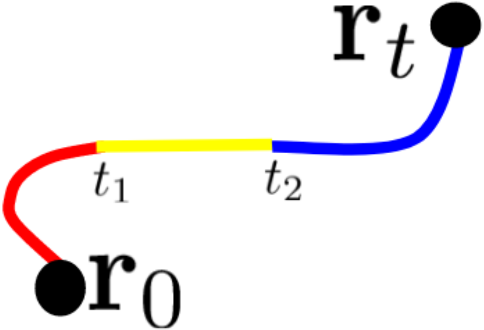
Second order STT for optimal control requires integration of 13^3 equations!
Likely **infeasible** for online computations

Question:

Given prior knowledge of this reference trajectory,



how do we quickly compute the STTs along any arc?

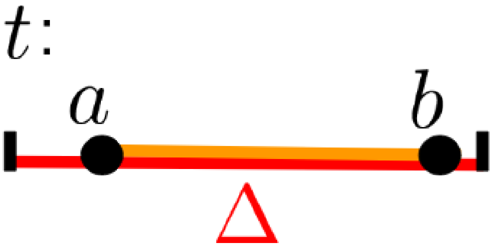


$$\Phi(t, 0) = \Phi(t, t_2)\Phi(t_2, t_1)\Phi(t_1, 0)$$

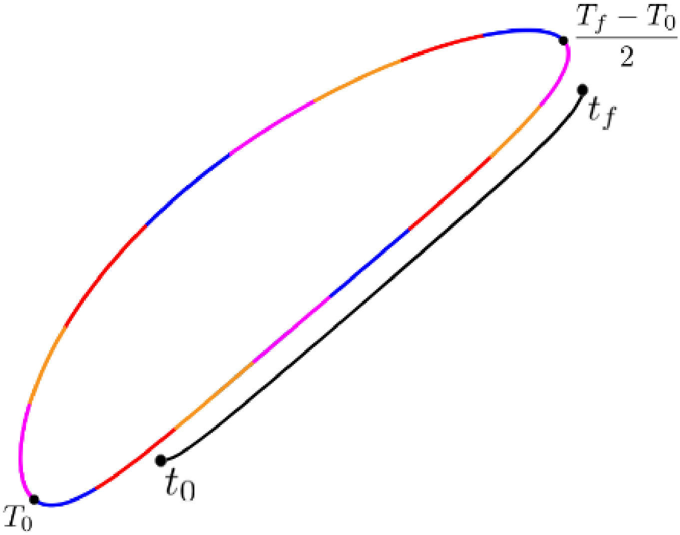
Interpolation

Let $(a, b) \subseteq \Delta$, then an entrywise linear interpolant of Φ is given by

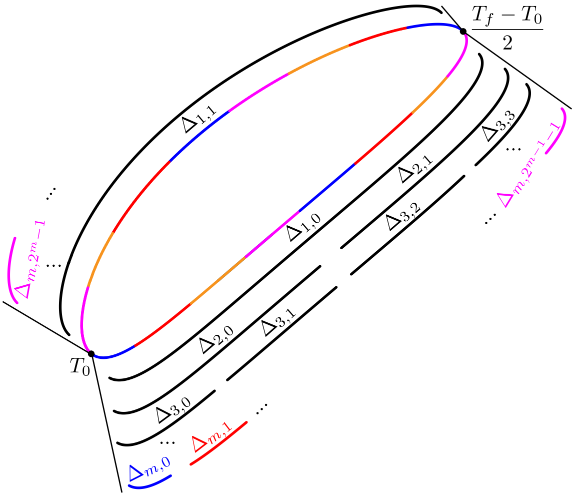
$$\Phi(b, a) \approx I + \frac{b - a}{|\Delta|}(\Phi(\Delta) - I)$$



Building Up The State Transition Matrix



Binary Search Construction



$$\Delta_{m,j} = \left(T_0 + \frac{T_f - T_0}{2^m} j,\ T_0 + \frac{T_f - T_0}{2^m}(j+1) \right)$$

Conclusion

Contributions:

- First linearized optimal control outside of Keplerian dynamics
- Novel use of STT to compute energy cost metric
- Precomputation and interpolation algorithm

Benefits:

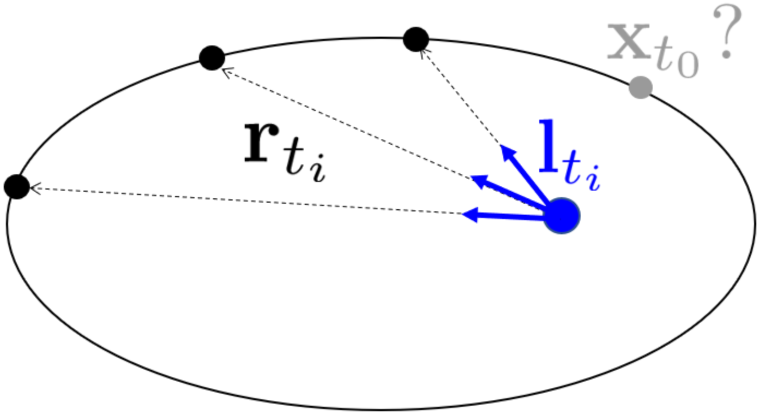
- 2-3 order of magnitude speedup in STM and STT calculation
- Order of megabytes of precomputed data
- Achieves order 0.1 to 1 percent error in energy estimates

Future directions:

- Explore Lunar Halo analogs
- Fast computation for path constrained optimal relative control
- Fuel optimal control

Outline

Angles-Only Space to Space Orbit Determination



Can a satellite determine another satellite's state with just a camera?

The Problem

Given relative position line of sight with QUEST measurement model (Shuster and Oh 1981)

$$\mathbf{l}_{t_i} = \frac{\mathbf{r}_{t_i}}{|\mathbf{r}_{t_i}|} + \mathbf{v}_{t_i}$$

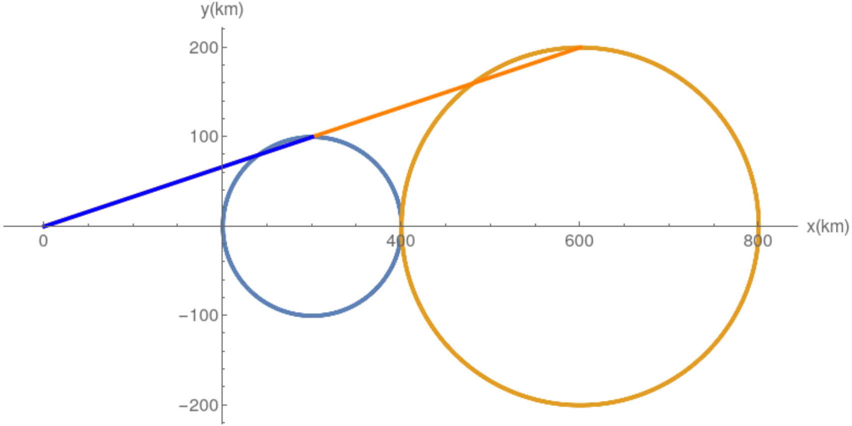
$$E(\mathbf{v}_{t_i}) = \mathbf{0}, \qquad E(\mathbf{v}_{t_i}\mathbf{v}_{t_i}^T) = \frac{\sigma^2}{2}\left(I_3 - \frac{\mathbf{r}_{t_i}\mathbf{r}_{t_i}^T}{||\mathbf{r}_{t_i}||^2}\right)$$

Find an initial relative state that fits

$$\begin{aligned}
&\operatorname{argmin}_{\delta\mathbf{x}_0\in\mathbb{R}^6}\left\{\sum_{i=1}^{n}\frac{||\mathbf{l}_{t_i}\times\mathbf{r}_{t_i}||^2}{||{\color{red}\mathbf{r}_{t_i}}||^2}\right\}\\
\approx\ &\operatorname{argmin}_{\delta\mathbf{x}_0\in\mathbb{R}^6}\left\{\frac{\sum_{i=1}^{n}||\mathbf{l}_{t_i}\times{\color{blue}\mathbf{r}_{t_i}}||^2}{||{\color{red}\delta\mathbf{x}_0}||^2}\right\}\\
=\ &\operatorname{argmin}_{\delta\mathbf{x}_0\in\mathbb{R}^6}\left\{\frac{\sum_{i=1}^{n}||\mathbf{l}_{t_i}\times{\color{blue}(\varphi^{\mathbf{r}}_{t_i}(\mathbf{x}_0+\delta\mathbf{x}_0)-\varphi^{\mathbf{r}}_{t_i}(\mathbf{x}_0))}||^2}{||\delta\mathbf{x}_0||^2}\right\}
\end{aligned}$$

Linear Unobservable Range

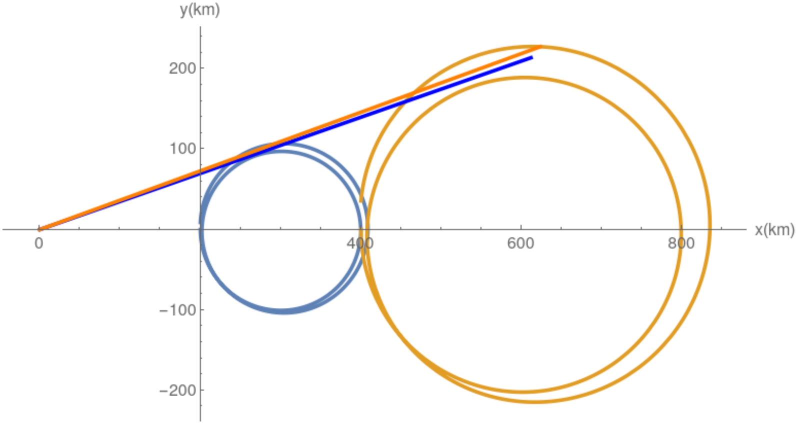
Linear dynamics of two satellites relative to a reference satellite at the origin



Relative orbit linear dynamics are unobservable with angles-only measurements (Woffinden and Geller 2007)

Nonlinear Observability

Nonlinear dynamics of the same satellites relative to a reference satellite at the origin



Nonlinear dynamics can enable observability (Ardaens, Gaias, et al. 2019; Jean-Sébastien Ardaens and Gaias 2019; Lovell, Sinclair, and Newman 2018). Current algorithms rely on bisection or homotopy continuation.

Second Order Position Approximation

Consider the 3 by 6 matrix

$$\Phi^{\mathbf{r}}(t,0) = \frac{\partial \mathbf{r}_t}{\partial \mathbf{x}_0}$$

and the 3 by 6 by 6 rank (1,2) tensor

$$\Psi^{\mathbf{r}}(t,0) = \frac{\partial^2 \mathbf{r}_t}{\partial \mathbf{x}_0^2}$$

$$\delta \mathbf{r}_t \approx \Phi^{\mathbf{r}}(t,0)\delta \mathbf{x}_0 + \frac{1}{2}\Psi^{\mathbf{r}}\delta \mathbf{x}_0^2$$

Optimal Linear Orbit Determination

Linear approximation of initial orbit direction vector: 6th right singular vector of A (Sinclair and Alan Lovell 2020)

$$A = \begin{bmatrix} [\mathbf{l}_{t_1}]_\times \Phi^{\mathbf{r}}(t_1, t_0) \\ \vdots \\ [\mathbf{l}_{t_n}]_\times \Phi^{\mathbf{r}}(t_n, t_0) \end{bmatrix}$$

$$\hat{\delta \mathbf{x}}_0 \approx \bar{\mathbf{x}}_0 = \mathrm{argmin}_{||\delta \mathbf{x}_0||=1} \left\{ ||A \delta \mathbf{x}_0||^2 \right\}$$

Scale Estimate

$$0 = ||\mathbf{l}_t \times \delta \mathbf{r}_t|| \approx ||\mathbf{l}_t \times \Phi^{\mathbf{r}}(t,0)\delta \mathbf{x}_0 + \mathbf{l}_t \times \frac{1}{2}\Psi^{\mathbf{r}}(t,0)\delta \mathbf{x}_0^2|| \implies$$

$$||\delta \mathbf{x}_0|| \approx \frac{||\mathbf{l}_t \times \Phi^{\mathbf{r}}(t,0)\hat{\delta \mathbf{x}_0}||}{||\mathbf{l}_t \times \frac{1}{2}\Psi^{\mathbf{r}}(t,0)\hat{\delta \mathbf{x}_0}^2||}$$

10.30	0.22	103.79	0.83	1.02	2.37	0.07	5.37	0.29

Relative orbit semi-minor axis sizes predicted from this estimate.
Mean of 13.81km order of magnitude approximation for actual 20km size.

Improved Scale Estimate

By solving a generalized eigenvalue problem, we can improve initial state approximation

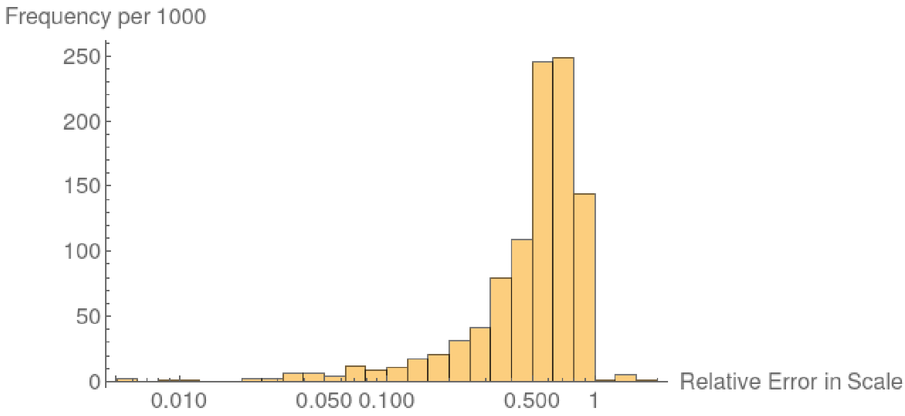
$$0 = ||\mathbf{l}_t \times \delta \mathbf{r}_t|| \approx ||\mathbf{l}_t \times \Phi^{\mathbf{r}}(t,0)\delta \mathbf{x}_0 + \mathbf{l}_t \times \frac{1}{2}\Psi^{\mathbf{r}}(t,0)\delta \mathbf{x}_0^2|| \implies$$

$$||\delta \mathbf{x}_0|| \approx \frac{||\mathbf{l}_t \times \Phi^{\mathbf{r}}(t,0)\hat{\delta \mathbf{x}_0}||}{||\mathbf{l}_t \times \frac{1}{2}\Psi^{\mathbf{r}}(t,0)\hat{\delta \mathbf{x}_0}^2||}$$

20.002	20.010	19.631	20.011	20.013	19.999	20.014	20.041	20.003

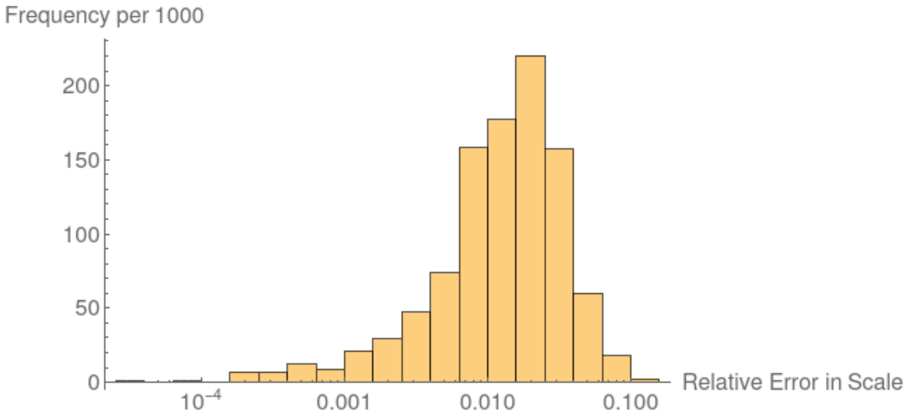
Relative orbit semi-minor axis sizes predicted from this estimate.
Mean of 19.97km is within 1 percent of the actual 20km size.

Simple estimate $\sigma = 10^{-4}$ error (3σ bound $\approx 1'$)



Relative error in scale for 1000 initial relative orbits in a 200km cube centered about the reference orbit.

Improved estimate $\sigma = 10^{-4}$ error



Relative error in scale for 1000 initial relative orbits in a 200km cube centered about the reference orbit.

Conclusion

Contributions:

- Approximate solution of passive angles-only relative orbit determination
- Quadratic model of dynamics resolves linear unobservability
- Only linear methods employed

Future directions:

- Iterative methods to improve estimate
- Convergence studies from starting estimates
- Showcase algorithm in cislunar space
- Extension to line of sight velocity problem
- Inertial orbit determination with similar methods

Course Work

- Math
  - Dynamical Systems
  - Applied Dynamical Systems
  - Perturbation Theory and Asymptotics
  - Probability Theory
  - Optimal Control and Differential Games (Expected Spring 2023)
- MAE
  - Model Based Estimation
  - Advanced Astrodynamics
  - Attitude Dynamics and Control (Expected Fall 2022)
- Other
  - Parallel Computing
  - Inverse Problems

Publications

Journal papers:

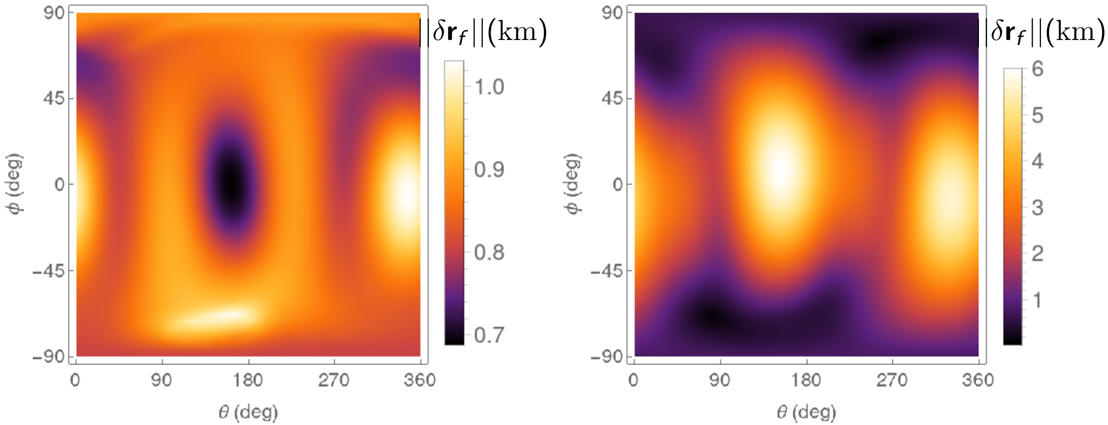
- J. Kulik, G.J. Soto, and D. Savransky. "Minimal differential lateral acceleration configurations for starshade stationkeeping in exoplanet direct imaging." Journal of Astronomical Telescopes, Instruments, and Systems 8.1 (2022): 017003.
- J. Kulik "An in-plane J2-invariance condition and control algorithm for highly elliptical satellite formations." Celestial Mechanics and Dynamical Astronomy 133.2 (2021): 1-25.

Conference papers:

- J. Kulik, W. Clark and D. Savransky, "Fast approximation of continuous thrust optimal relative control in the three body problem," Astrodynamics Specialist Conference 2022, AAS, 2022.
- J. Kulik and D. Savransky, "Precomputation and interpolation of the matrizant for starshade slewing," Space Telescopes and Instrumentation 2022, SPIE, 2022.
- J. Kulik, and D. Savransky. "Relative Transfer Singularities and Multi-Revolution Lambert Uniqueness." AIAA SCITECH 2022 Forum. 2022.

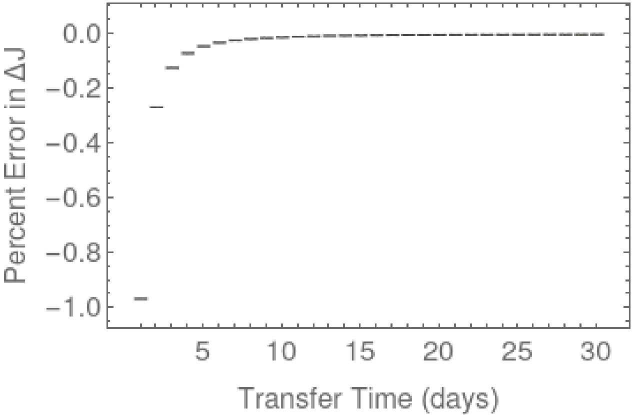
Thank You – Questions?

Final Position Error



The error in final position due to the approximation of the rendezvous optimal control by second order and first order methods.

Results of Interpolation Error



Worst case error in cost due to interpolation from 0.1 day intervals (66Mb data).

Leading Order Interpolation

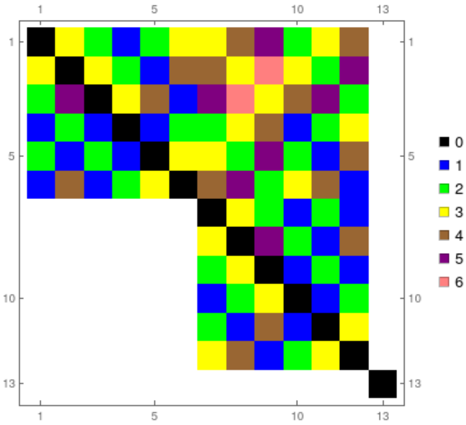
$$\text{Given } J = \frac{\partial \mathbf{F}(\mathbf{x_0})}{\partial \mathbf{x}}$$

$$P^i_j = \min\{p \in \mathbb{N} \quad | \quad (J^p)^i_j \neq 0\}$$

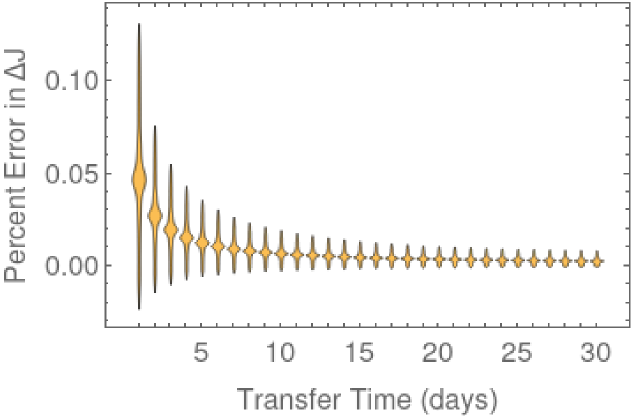
$$\Phi(t,0) \approx e^{Jt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} J^k$$

$$\Phi(\alpha t,0) \approx e^{J\alpha t} = \sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} J^k$$

$$\Phi^i_j(\alpha t,0) \approx I_n + \alpha^{P^i_j}(\Phi^i_j(t,0) - I_n)$$



Results of Interpolation Error



Worst case error in cost due to interpolation from 1 day intervals (6Mb data).

Improving the Estimate

$$\mathbf{a} = \begin{bmatrix} \mathbf{l}_{t_1} \times \Phi^{\mathbf{r}}(t_1, 0)\hat{\delta \mathbf{x}}_0 \\ \vdots \\ \mathbf{l}_{t_n} \times \Phi^{\mathbf{r}}(t_n, 0)\hat{\delta \mathbf{x}}_0 \end{bmatrix} = A\hat{\delta \mathbf{x}}_0 \qquad || \qquad \mathbf{b} = \begin{bmatrix} \mathbf{l}_{t_1} \times \Psi^{\mathbf{r}}(t_1, 0)\hat{\delta \mathbf{x}}_0^2 \\ \vdots \\ \mathbf{l}_{t_n} \times \Psi^{\mathbf{r}}(t_n, 0)\hat{\delta \mathbf{x}}_0^2 \end{bmatrix} \approx B\hat{\delta \mathbf{x}}_0$$

$$A = \begin{bmatrix} [\mathbf{l}_{t_1}]_\times \Phi^{\mathbf{r}}(t_1, 0) \\ \vdots \\ [\mathbf{l}_{t_n}]_\times \Phi^{\mathbf{r}}(t_n, 0) \end{bmatrix} \qquad B = \begin{bmatrix} [\mathbf{l}_{t_1}]_\times \Psi^{\mathbf{r}}(t_1, 0)\bar{\mathbf{x}}_0 \\ \vdots \\ [\mathbf{l}_{t_n}]_\times \Psi^{\mathbf{r}}(t_n, 0)\bar{\mathbf{x}}_0 \end{bmatrix}$$

$$B\hat{\delta \mathbf{x}}_0 = \lambda A\hat{\delta \mathbf{x}}_0$$

$$A^\dagger B\hat{\delta \mathbf{x}}_0 = \lambda\hat{\delta \mathbf{x}}_0$$

References I

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