# Fast Approximation of Continuous Thrust Optimal Relative Control in the Three Body Problem

Jackson Kulik, William Clark, and Dmitry Savransky
Cornell University

Astrodynamics Specialist Conference

August, 2022

# Dynamical Systems Preliminaries

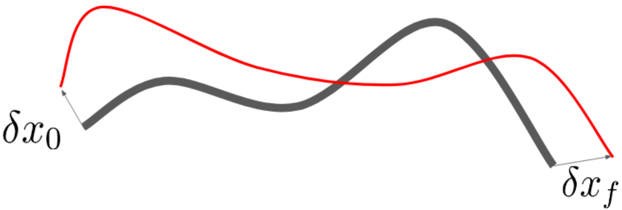
- State vector $\mathbf{x} \in \mathbb{R}^n$
- Dynamics/vector field $\mathbf{F}$:

$$\frac{d}{dt}\mathbf{x} = \mathbf{F}(\mathbf{x})$$

- Flow map $\varphi_t : \mathbf{x}_0 \to \mathbf{x}_t$

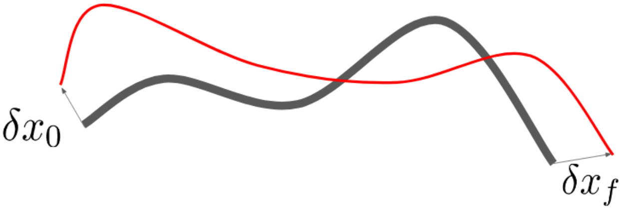
$$\frac{d}{dt}\varphi_t(\mathbf{x}) = \mathbf{F}(\varphi_t(\mathbf{x})), \quad \varphi_0(\mathbf{x}) = \mathbf{x}$$

# Taylor Expansion of the Flow Map




$$\mathbf{x}_f + \delta \mathbf{x}_f = \varphi_{t_f}(\mathbf{x}_0 + \delta \mathbf{x}_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \varphi_{t_f}}{\partial \mathbf{x}^n} \right|_{\mathbf{x}_0} \delta \mathbf{x}_0^n$$

# State Transition Matrix




The state transition matrix gives the linear approximation

$$\delta \mathbf{x}_f \approx \left. \frac{\partial \varphi_{t_f}}{\partial \mathbf{x}} \right|_{\mathbf{x}_0} \delta \mathbf{x}_{\mathbf{0}} = \Phi(t_f, t_0) \delta \mathbf{x}_0$$

# Calculating the State Transition Matrix

Assuming sufficient smoothness of $\varphi_t(\mathbf{x})$ (Clairaut's Theorem)

$$\frac{d\Phi(t,0)}{dt} = \frac{d}{dt}\frac{\partial\varphi_t}{\partial\mathbf{x}} = \frac{\partial}{\partial\mathbf{x}}\frac{d\varphi_t}{dt} = \frac{\partial}{\partial\mathbf{x}}\mathbf{F}(\varphi_t(\mathbf{x})) = \left.\frac{\partial\mathbf{F}(\mathbf{x})}{\partial\mathbf{x}}\right|_{\varphi_t(\mathbf{x})}\Phi(t,0)$$

$$\dot{\Phi} = D\mathbf{F}\cdot\Phi, \quad \Phi(0,0) = I_n$$

**First order variational equations** yield the state transition matrix.

- Integrate alongside original dynamical system.
- **Higher order variational equations** yield state transition tensors.

# Relative Transfer

Partition the state into position and velocity

$$\delta \mathbf{x} = \begin{bmatrix} \delta \mathbf{r}^T & \delta \mathbf{v}^T \end{bmatrix}$$

$$\Phi(t,0) = \begin{bmatrix} \Phi_{\mathbf{r}}^{\mathbf{r}} & \Phi_{\mathbf{v}}^{\mathbf{r}} \\ \Phi_{\mathbf{r}}^{\mathbf{v}} & \Phi_{\mathbf{v}}^{\mathbf{v}} \end{bmatrix}$$

Initial relative velocity $\delta \mathbf{v}_0$ to change relative position $\delta \mathbf{r}_0 \to \delta \mathbf{r}_t$

$$\delta \mathbf{r}_t \approx \Phi_{\mathbf{r}}^{\mathbf{r}} \delta \mathbf{r}_0 + \Phi_{\mathbf{v}}^{\mathbf{r}} \delta \mathbf{v}_0 \implies$$

$$\delta \mathbf{v}_0 \approx (\Phi_{\mathbf{v}}^{\mathbf{r}})^{-1} (\delta \mathbf{r}_t - \Phi_{\mathbf{r}}^{\mathbf{r}} \delta \mathbf{r}_0)$$

# Continuous Thrust Control

Minimize cost $J = \int_{t_0}^{t_f} \frac{1}{2}\mathbf{u}^T\mathbf{u}\,\mathrm{d}t$ under boundary conditions $\mathbf{x}(t_0) = \mathbf{x_0}, \mathbf{x}(t_f) = \mathbf{x_f}$:

$$\frac{d}{dt}\mathbf{x} = \mathbf{F}(\mathbf{x}) + \mathbf{u}, \quad \frac{d}{dt}\boldsymbol{\lambda} = -\left(\frac{\partial \mathbf{F}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \boldsymbol{\lambda}$$

Two point boundary value problem with twelve equations for states and costates.
Optimal control given by $\mathbf{u} = -(0, 0, 0, \lambda_4, \lambda_5, \lambda_6)^T$

# Continuous Thrust Relative Transfers

Define the augmented state vector $\mathbf{z} = [\mathbf{x}^T \quad \boldsymbol{\lambda}^T \quad J]^T$ and its dynamics

$$\frac{d}{dt}\mathbf{z} = \mathbf{G}(\mathbf{z}) = \left[ \left( \mathbf{F}(\mathbf{x})^T + \begin{bmatrix} \mathbf{0} & \mathbf{u}^T \end{bmatrix} \right) \quad -\boldsymbol{\lambda}^T \left( \frac{\partial \mathbf{F}(\mathbf{x})}{\partial \mathbf{x}} \right) \quad \frac{1}{2}\mathbf{u}^T\mathbf{u} \right]^T$$

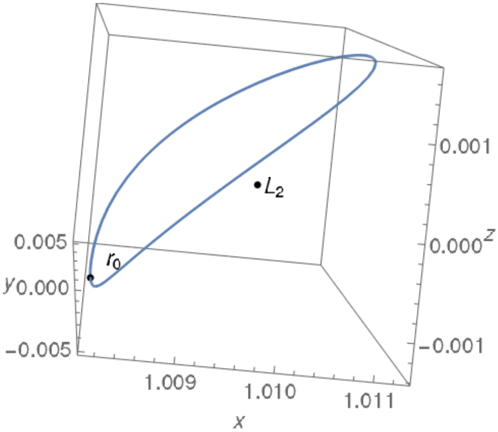
Initial costates are given by STM associated with $\mathbf{G}$

$$\delta\boldsymbol{\lambda}_0 \approx (\Phi^{\mathbf{x}}_{\boldsymbol{\lambda}})^{-1}(\delta\mathbf{x}_t - \Phi^{\mathbf{x}}_{\mathbf{x}}\delta\mathbf{x}_0)$$

Energy cost approximation from second order state transition tensor:

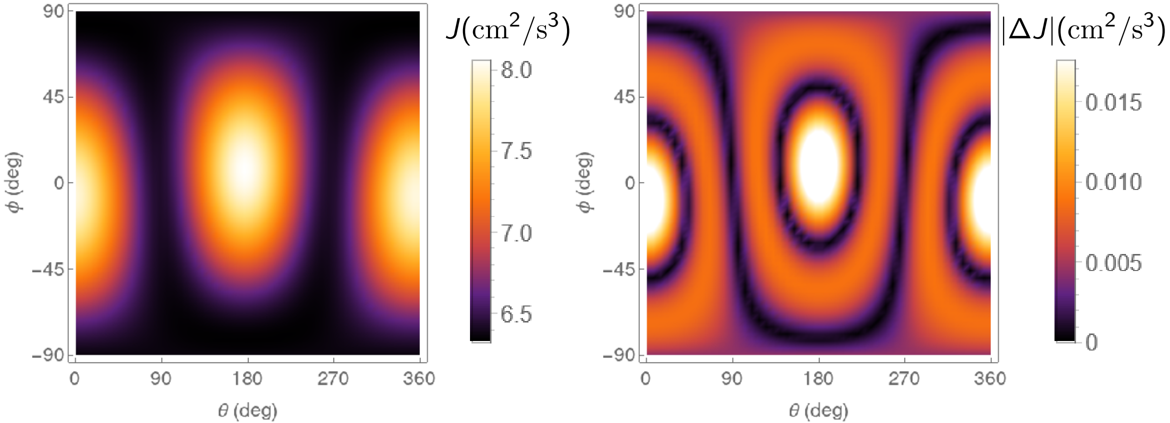
$$\delta J \approx \frac{\partial^2 J_f}{\partial \mathbf{x}_0^2}\delta\mathbf{x}_0^2 + \frac{\partial^2 J_f}{\partial \boldsymbol{\lambda}_0^2}\delta\boldsymbol{\lambda}_0^2$$

# Three Body Problem Example




The reference 6 month Sun-Earth L2 Halo orbit in canonical distance units.

# Rendezvous on SEL2 Halo: Two Weeks from 10,000km Sphere




The rendezvous control cost approximated by second order state transition tensor and the error in its approximation versus numerical integration.

# Problems with Practical Computation

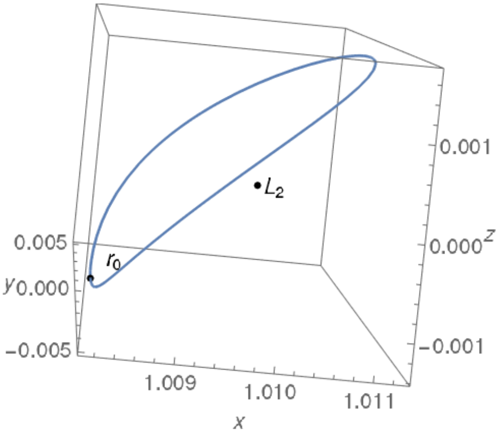
Fast calculations once we have $\Phi$ and potentially higher order state transition tensors.
What about calculating $\Phi$?

- Analytical solutions:
  - Clohessy-Wiltshire
  - Yamanaka-Ankersen
  - Their respective adjoint equations
- Numerical integration:
  - High fidelity two body motion with perturbations
  - Three body motion
  - Costate equations

Second order STT for optimal control requires integration of $13^3$ equations!
Likely **infeasible** for online computations

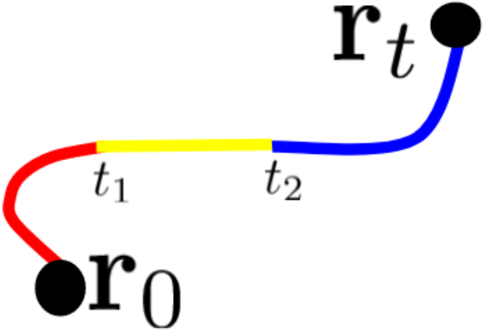
# Question:

Given prior knowledge of this reference trajectory,




how do we quickly compute the STTs along any arc?

# Cocycle Conditions

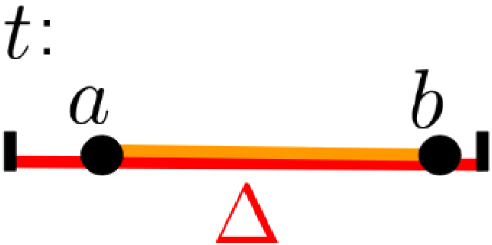



$$\Phi(t,0) = {\color{blue}\Phi}(t,t_2){\color{yellow}\Phi}(t_2,t_1){\color{red}\Phi}(t_1,0)$$

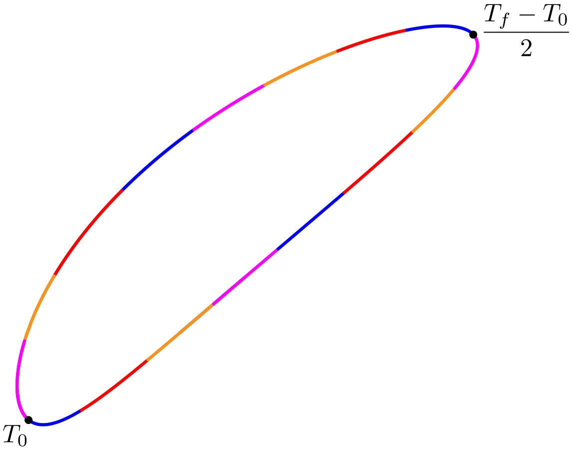
# Interpolation

Let $(a, b) \subseteq \Delta$, then an entrywise linear interpolant of $\Phi$ is given by

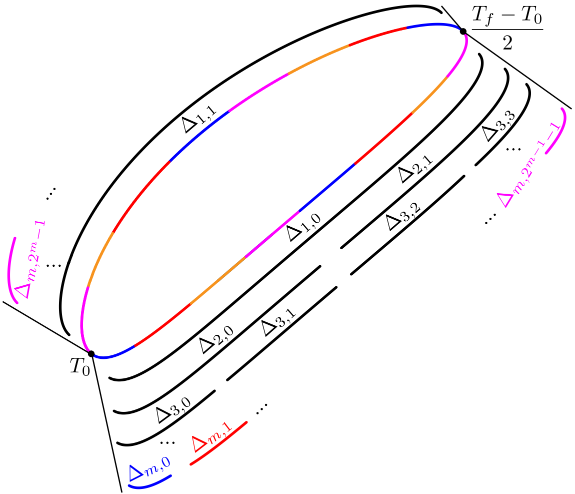
$$\Phi(b, a) \approx I + \frac{b - a}{|\Delta|}(\Phi(\Delta) - I)$$



# Building Up The State Transition Matrix



# Binary Search Construction




$$\Delta_{m,j} = \left(T_0 + \frac{T_f - T_0}{2^m} j,\ T_0 + \frac{T_f - T_0}{2^m}(j+1)\right)$$

# Conclusion

Benefits:

- 2-3 order of magnitude speedup in STM and STT calculation
- Order of megabytes of precomputed data
- Achieves order 0.1 to 1 percent error in energy estimates

Not Covered:

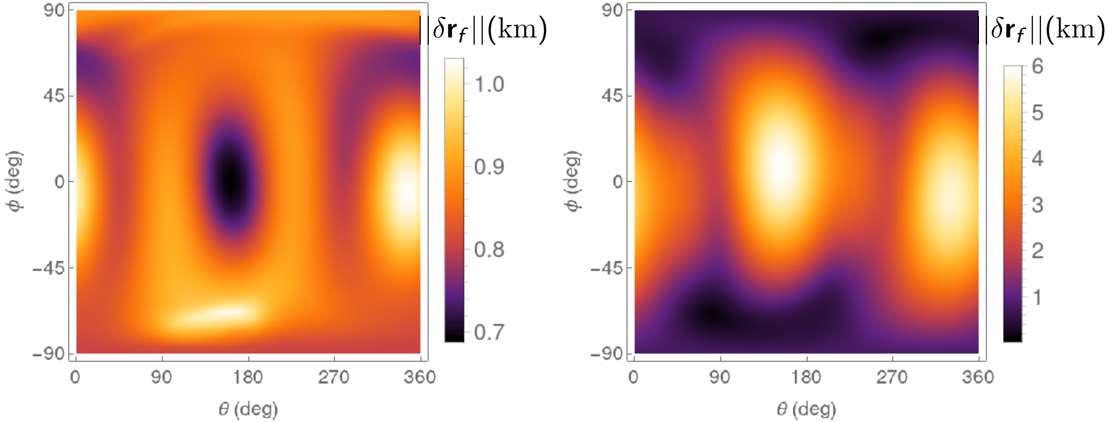
- Computation of state transition tensors (STTs)
- Cocycles and interpolation of STTs

Future directions:

- Explore Lunar Halo analogs
- Fast computation for path constrained optimal relative control
- Storage and interpolation with Chebyshev or Fourier methods
- Interpolation with Taylor series in time

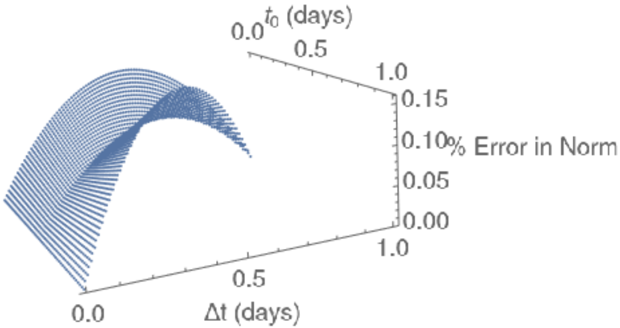
# Thank You – Questions?

# Final Position Error



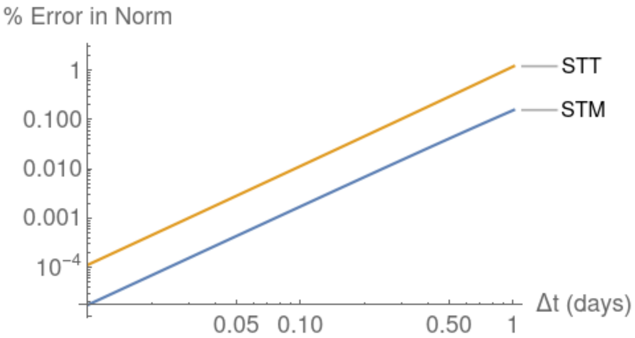

The error in final position due to the approximation of the rendezvous optimal control by second order and first order methods.

# Results of Interpolation Error



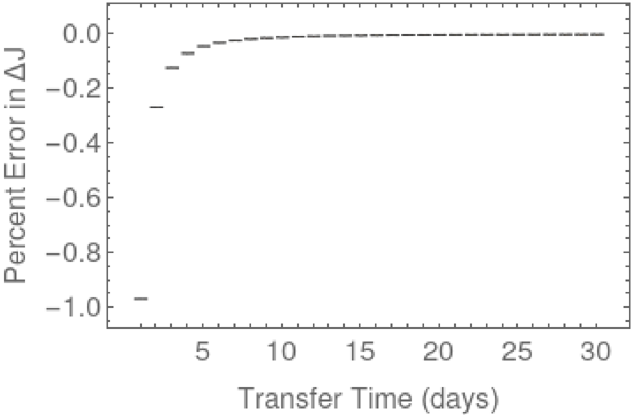

Error 2-norm for the state transition matrix $\Phi(t_0 + \Delta t, t_0)$ interpolated from $\Phi(1, 0)$ versus the exact value of $\Phi(t_0 + \Delta t, t_0)$.

# Results of Interpolation Error




Worst case interpolation error norm for the state transition matrix and second order state transition tensor

# Results of Interpolation Error




Worst case error in cost due to interpolation from 0.1 day intervals (66Mb data).