Relative Transfer Singularities and Multi-Revolution Lambert Uniqueness

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Roadmap

- Lambert's problem
  - Boundary value problem in two body motion
  - Existence/uniqueness characteristics well-known
- Implications to satellite relative motion
  - Debris creating events
  - Flight safety of satellite deployments
  - Relative motion boundary value problems
    - Existence and uniqueness properties
    - Linearized model of relative motion
    - Nonlinear model of relative motion

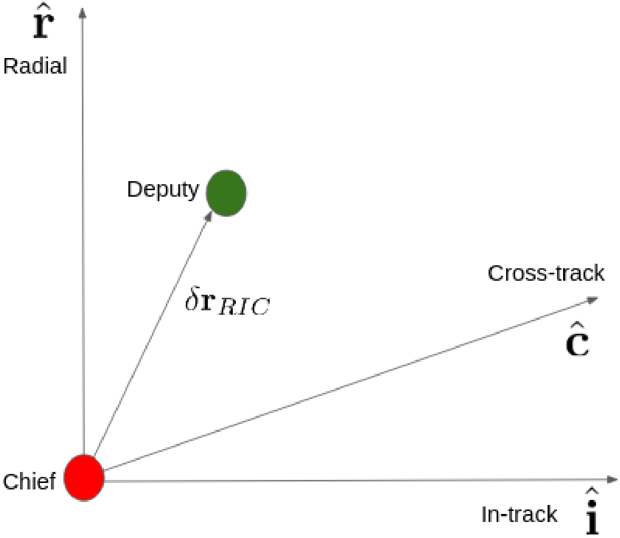
Lambert's Problem

Given $(\mathbf{r}_0, \mathbf{r}_f, \Delta t)$ find $\mathbf{v}_0$ such that $\mathbf{r}(0) = \mathbf{r}_0, \mathbf{r}(\Delta t) = \mathbf{r}_f$.

- There exists a unique solution such that $\mathbf{r}(t)$ does not complete any full revolutions around the central body.
- Adding the constraint that $\mathbf{r}(t)$ completes N revolutions about the central body, there exists Δt_N^* such that:
  - $\Delta t < \Delta t_N^* \implies$ No solutions
  - $\Delta t = \Delta t_N^* \implies$ A unique solution
  - $\Delta t > \Delta t_N^* \implies$ Two solutions

Relative Coordinate Frame

- Origin centered at the “chief” satellite
- Radial $\hat{\mathbf{r}}$ – directed from Earth to the satellite
- Cross–Track $\hat{\mathbf{c}}$ – direction of chief’s angular momentum
- In–Track $\hat{\mathbf{i}}$ – defined s.t. $\hat{\mathbf{r}} \times \hat{\mathbf{i}} = \hat{\mathbf{c}}$
- Relative position of “deputy” satellite as $\delta\mathbf{r}_{RIC} = x\hat{\mathbf{r}} + y\hat{\mathbf{i}} + z\hat{\mathbf{c}}$



Hill-Clohessy-Wiltshire (HCW) State Transition Matrix

$$
\begin{aligned}
\delta\mathbf{x} &= \left[x, y, z, \dot{x}, \dot{y}, \dot{z}\right] \\
\delta\mathbf{x}_t &= \Phi_0^t \delta\mathbf{x}_0 \\
\Phi_0^t &= \begin{bmatrix}
4-3c & 0 & 0 & s/n & 2/n-2c/n & 0 \\
-6nt+6s & 1 & 0 & -2/n+2c/n & 4s/n-3t & 0 \\
0 & 0 & c & 0 & 0 & s/n \\
3ns & 0 & 0 & c & 2s & 0 \\
-6n+6nc & 0 & 0 & -2s & -3+4c & 0 \\
0 & 0 & -ns & 0 & 0 & c
\end{bmatrix} \\
\Phi_0^t &= \begin{bmatrix}
\Phi_{\mathbf{rr}} & \Phi_{\mathbf{rv}} \\
\Phi_{\mathbf{vr}} & \Phi_{\mathbf{vv}}
\end{bmatrix}
\end{aligned}
$$

where $s = \sin(nt)$ and $c = \cos(nt)$ and n is the mean motion of the reference orbit

Relative Transfer

Initial relative velocity $\delta \mathbf{v}_0$ to change relative position $\delta \mathbf{r}_0 \to \delta \mathbf{r}_t$

- Can be found by solving Lambert's problem
  - Find corresponding inertial positions then solve BVP
  - Potentially 0, 1, or 2 solutions
- Linear approximation from HCW equations
  - 0, 1, or infinite solutions
  - Depending on null space of $\Phi_{\mathbf{rv}}^{-1}$ and boundary conditions

$$\delta \mathbf{v}_0 = \Phi_{\mathbf{rv}}^{-1}(\delta \mathbf{r}_t - \Phi_{\mathbf{rr}} \delta \mathbf{r}_0)$$

- What happens when $\Phi_{\mathbf{rv}}$ is singular/nearly singular?

Relative Transfer Singularity and Uniqueness

- $\det(\Phi_{\mathbf{rv}}) = 0 \implies$ reference trajectory is unique solution to Lambert's problem between initial and terminal points of reference trajectory

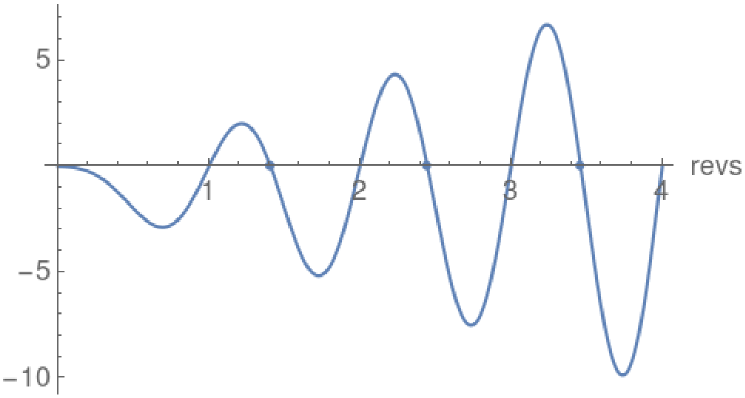


Figure 1: $\det(\Phi_{\mathbf{rv}})$ as a function of number of reference revolutions.

Debris Evolution

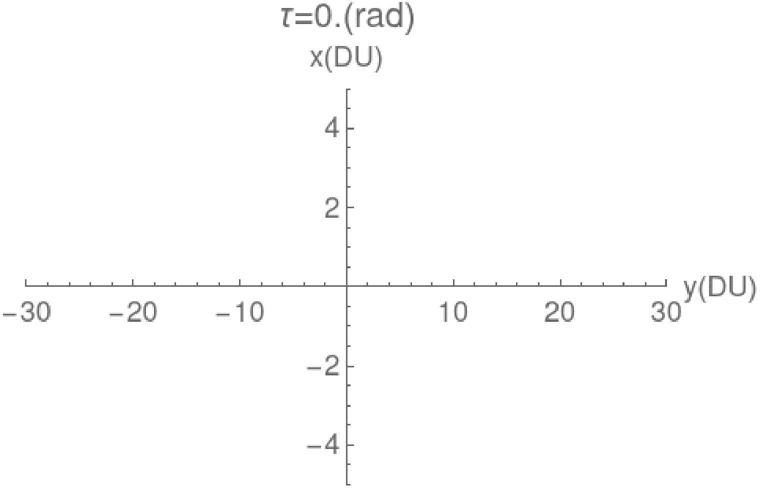


Figure 2: Objects relative to a circular reference orbit initialized with a uniform distribution of initial relative velocities in a square around zero.

Potential Collisions

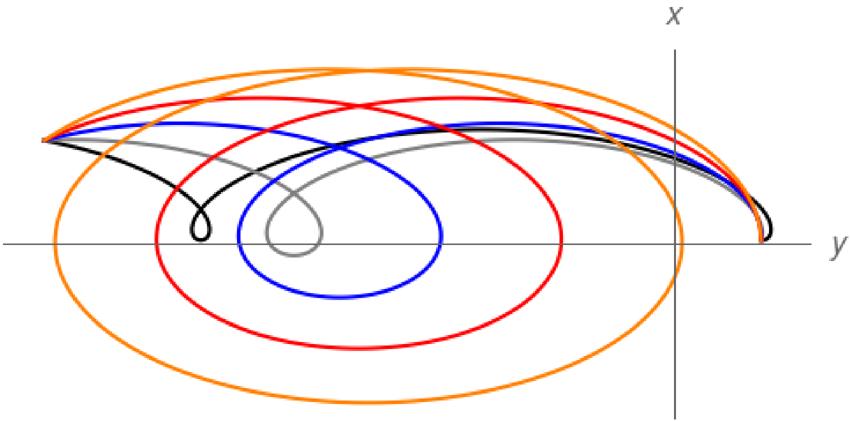


Figure 3: Objects with common initial position but velocities differing by a multiple of $(3\tau, -4)$.

Relative Transfers in the Vicinity of a Singularity

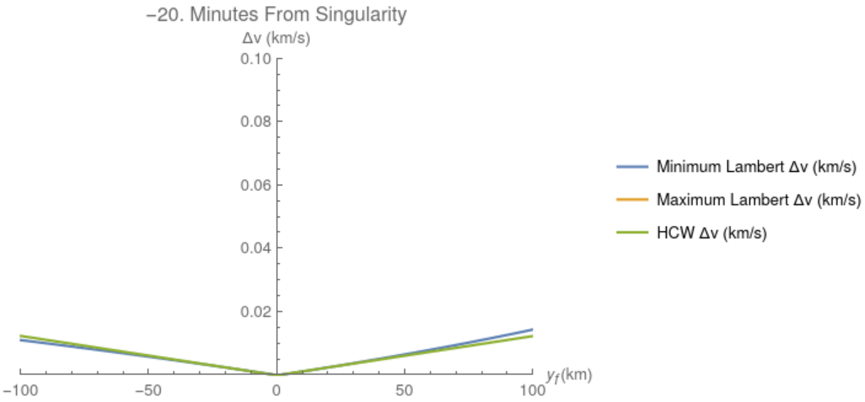


Figure 4: In geostationary orbit, relative transfer cost from origin to in-track offset in around 1.4 days.

Relative Transfers in the Vicinity of a Singularity

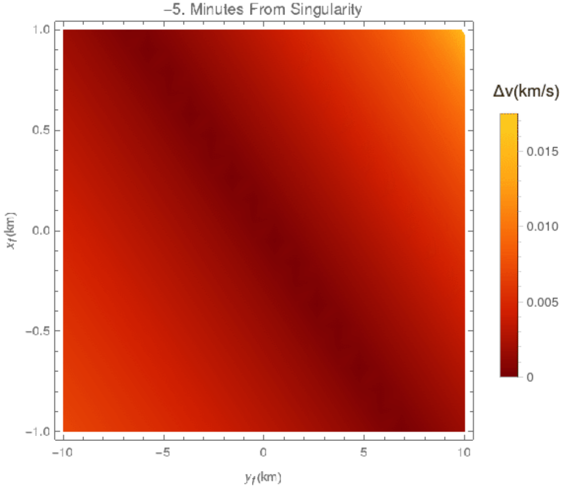


Figure 5: In geostationary orbit, relative transfer cost from origin to in-track, radial offset in around 1.4 days.

Generalization to Bifurcation in BVP Solutions

- Given $\ddot{\mathbf{r}} = \mathbf{f}(\mathbf{r}, t), \mathbf{r}(0) = \mathbf{r}_0, \mathbf{r}(t_f) = \mathbf{r}_f$
  - f twice differentiable with continuous second partials
  - Time change to $\mathbf{r}'' = t_f \mathbf{f}(\mathbf{r}, \tau), \mathbf{r}(0) = \mathbf{r}_0, \mathbf{r}(1) = \mathbf{r}_f$
  - Written as first order system $\mathbf{x}' = \mathbf{F}, \mathbf{x} = \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \mathbf{v} \\ \mathbf{f} \end{bmatrix}$
  - Variational equations $\Phi' = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \Phi, \Phi(0) = I$
- We proved a solution to the BVP such that $\det(\Phi_{rv}) = 0$ implies a bifurcation point at t_f.
- Applies to three body problem
- Continuation methods from bifurcation theory

Conclusion-New Results

- Previously connected:
  - Pinch points and relative transfer singularity (Fitzgerald 1998)
  - Relative transfer singularity and minimum time Lambert (Stern 1964)
  - We summarize all three connections
- Alternative proof connecting
  - relative transfer singularity
  - uniqueness of multi-revolution Lambert solution
- Bifurcation results generalize
  - General two point boundary value problems
  - Associated first order variational equations
- Relative transfer singularity effects
  - Debris and pinch points
  - Subsatellite deployment safety
  - Existence and cost of relative transfers
- Heuristics-formation reconfiguration (conference paper only)

References

- Fitzgerald, R. J., "Pinch Points of Debris from a Satellite Breakup," Journal of guidance, control, and dynamics, Vol. 21, No. 5, 1998, pp. 813–815.
- Stern, R., "Singularities in the analytic solution of the linearized variational equations of elliptical motion," 1st AIAA Annual Meeting, 1964, p. 398.

Thank You – Questions?