The Putting Green:
The dynamics of a homogenous ball on an arbitrary surface

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Abstract

The dynamics of a golf ball on a putting green are studied as an instance of the general problem of a homogenous sphere on an arbitrary surface. Prior work in this area is examined and partially re-derived from first principles using alternate approaches. A series of simplified examples is solved to illustrate various properties of the physics of the system. The full system is modeled by defining the height of the ball in relation to the surface as an inequality constraint and utilizing a modified form of the Euler-Lagrange equation to derive the equations of motion. Finally, a method of numerical integration is presented to make use of the derived equations of motion.

1 Introduction

The problem discussed here was originally posed as an exercise in [2] and later developed as an optimization problem in [1]. The dynamics of a golf ball on a putting green can be generalized to the class of problems involving homogenous spheres sliding and rolling on an arbitrary surface of varying roughness. The purpose of this study will be to develop a group of general equations that describe the motion of a ball on a surface which can be described as a well-behaved function of one or more variables.

While there are numerous solutions to the problem of a ball sliding and rolling on a level plane, the majority of cases where the problem is redefined for an arbitrary surface include the assumption of pure rolling to simplify the dynamics. At the same time, generalizing to an arbitrary surface introduces a constraint that is implicit in the problem of motion on the level plane, but no longer holds true - that the ball remains attached to the surface. When solving this problem, it is very tempting to define the height of the ball as fully equivalent to the local height of the surface, which is clearly incorrect since there are times when the shape of the surface may cause the ball to fly into the air. This requires the inclusion of an inequality constraint and will be discussed in section 4.

The approach to finding the most general equations of motion will be to solve a series of simplified problems, each of which will motivate a different aspect of the dynamics and help us build an understanding of the effect of the surface shape on the motion of the ball.

2 Movement of a particle on an arbitrary surface

We begin by deriving the equations of motion for a particle on a surface given by a function of two coordinates $z = f(x, y)$ in the cartesian, dextral frame $I = (O, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$. This is simply an extension of the classical problem of a bead sliding on a wire of arbitrary shape into two dimensions, and is the simplest possible formulation of the problem of interest. While it neglects several important features of the full problem, it is useful for developing intuition as to how the equations depend on the functional description of the surface. We describe the position and velocity of the the particle as

$$\mathbf{r_p} = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3} \tag{1}$$

$$\mathbf{v_p} = \tfrac{dx}{dt}\mathbf{e_1} + \tfrac{dy}{dt}\mathbf{e_2} + \left(\tfrac{\partial f}{\partial x}\tfrac{dx}{dt} + \tfrac{\partial f}{\partial y}\tfrac{dy}{dt}\right)\mathbf{e_3} \tag{2}$$

where x and y are dependent on time only. Assuming a gravity field and adopting the notation $\dot{x} = \frac{dx}{dt}$ we can write the Lagrangian as

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \left(\dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y}\right)^2\right) - mgf \tag{3}$$

where m is the mass of the particle.

Applying the Euler-Lagrange equations to (3) with generalized coordinates x and y produces two second order differential equations of the form

$$\begin{aligned} \ddot{x} &= \frac{-\frac{\partial f}{\partial x}\left(\ddot{y}\frac{\partial f}{\partial y} + \dot{y}^2\frac{\partial^2 f}{\partial y^2} + \dot{x}\dot{y}\frac{\partial^2 f}{\partial x \partial y} + \dot{x}^2\frac{\partial^2 f}{\partial x^2} + g\right)}{1 + \left(\frac{\partial f}{\partial x}\right)^2} \\ \ddot{y} &= \frac{-\frac{\partial f}{\partial y}\left(\ddot{x}\frac{\partial f}{\partial x} + \dot{x}^2\frac{\partial^2 f}{\partial x^2} + \dot{x}\dot{y}\frac{\partial^2 f}{\partial x \partial y} + \dot{y}^2\frac{\partial^2 f}{\partial y^2} + g\right)}{1 + \left(\frac{\partial f}{\partial y}\right)^2} \end{aligned} \tag{4}$$

The mutual dependence on the second derivative of the other coordinate can be removed from these equations by setting $\ddot{y}$ equal to the value of the second equation in the first, and vice versa. After some algebraic simplification, this yields the equations

$$\begin{aligned} \ddot{x} &= \frac{-\frac{\partial f}{\partial x}\left(\dot{y}^2\frac{\partial^2 f}{\partial y^2} + \dot{x}\dot{y}\frac{\partial^2 f}{\partial x \partial y} + \dot{x}^2\frac{\partial^2 f}{\partial x^2} + g\right)}{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \\ \ddot{y} &= \frac{-\frac{\partial f}{\partial y}\left(\dot{x}^2\frac{\partial^2 f}{\partial x^2} + \dot{x}\dot{y}\frac{\partial^2 f}{\partial x \partial y} + \dot{y}^2\frac{\partial^2 f}{\partial y^2} + g\right)}{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \end{aligned} \tag{5}$$

As expected, the system is completely symmetric between the two coordinates. The motion of the particle will be determined wholly by the initial conditions and the gradients of the surface in the $\mathbf{e_1}$ and $\mathbf{e_2}$ directions. We can expand this formulation somewhat by adding the effects of friction between the ball and surface. If we assume that friction can be modeled as viscous damping with a single coefficient μ in each dimension then we can augment the Euler-Lagrange equations with the additional generalized forces of $-\mu\dot{x}$ and $-\mu\dot{y}$ [3]. Following the same procedure as before, we are left with the system described in (6). Once again, we see a high degree of symmetry between the equations of motion in the $\mathbf{e_1}$ and $\mathbf{e_2}$ dimensions.

3 Rolling and Sliding on an Arbitrary Surface

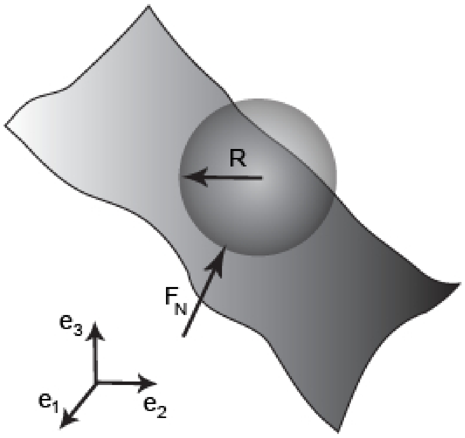

Figure 1: A ball sliding and rolling on a surface.

The next step in adding complexity to our model is to incorporate the concept of rolling. We shall use the Newtonian formulation presented in [1]. In order to use Newton's method, it is necessary to define the normal vector to the ball. Taking the cross product of the two surface tangent vectors $(1\mathbf{e_1} + \frac{\partial f}{\partial x}\mathbf{e_3})$ and $(1\mathbf{e_2} + \frac{\partial f}{\partial y}\mathbf{e_3})$ gives us the local surface normal vector: $(-\frac{\partial f}{\partial x}\mathbf{e_1} + -\frac{\partial f}{\partial y}\mathbf{e_2} + 1\mathbf{e_3})$. From this, we define the unit normal as

$$\mathbf{\hat{n}} \triangleq \frac{N}{\|N\|} = \frac{(-\frac{\partial f}{\partial x}\mathbf{e_1} + -\frac{\partial f}{\partial y}\mathbf{e_2} + 1\mathbf{e_3})}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} \tag{7}$$

We now describe the position of the center of mass of the ball as in (1) so that the function $f(x, y)$ now describes the height of the surface plus the radius of the ball multiplied by the normal vector: i.e. $x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3} = x_c\mathbf{e_1} + y_c\mathbf{e_2} + z_c\mathbf{e_3} + R\mathbf{\hat{n}}$ where (x_c, y_c, z_c) are the coordinates of the contact point between the ball and surface and R is the radius of

$$
\begin{aligned}
\ddot{x} &= \frac{\left(-\mu\dot{x} - gm\frac{\partial f}{\partial x} - m\dot{x}^2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2} - 2m\dot{x}\dot{y}\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x\partial y} + \mu\dot{y}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \mu\dot{x}\frac{\partial f}{\partial y}^2 - m\dot{y}^2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right)}{m\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)} \\
\ddot{y} &= \frac{\left(-\mu\dot{y} - gm\frac{\partial f}{\partial y} - m\dot{x}^2\frac{\partial^2 f}{\partial x^2}\frac{\partial f}{\partial y} - 2m\dot{x}\dot{y}\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y} + \mu\dot{x}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \mu\dot{y}\frac{\partial f}{\partial x}^2 - m\dot{y}^2\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2}\right)}{m\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)}
\end{aligned}
\tag{6}
$$

the ball. Differentiating twice to calculate the acceleration of the center of mass, we find that

$$
\begin{aligned}
\mathbf{a_p} &\triangleq \frac{d^2\mathbf{r_p}}{dt^2} = \ddot{x}\mathbf{e_1} + \ddot{y}\mathbf{e_2} + \ddot{z}\mathbf{e_3} \\
\ddot{z} &= \frac{\partial f}{\partial x}\ddot{x} + \frac{\partial f}{\partial y}\ddot{y} + \frac{\partial^2 f}{\partial x^2}\dot{x}^2 + \frac{\partial^2 f}{\partial y^2}\dot{y}^2 + 2\dot{x}\dot{y}\frac{\partial^2 f}{\partial x\partial y}
\end{aligned}
\tag{8}
$$

As in [1], we define the magnitude of the normal force to be equal to the sum of the force of gravity and the non-planar acceleration forces along the path of the ball, in the normal direction. Thus, $F_N = [(mg\mathbf{e_3} + m\mathbf{a_p})\cdot\hat{\mathbf{n}}]\mathbf{n}$. If we define

$$
F_{Nz} \triangleq \frac{m(g + \frac{\partial^2 f}{\partial x^2}\dot{x}^2 + \frac{\partial^2 f}{\partial y^2}\dot{y}^2 + 2\dot{x}\dot{y}\frac{\partial^2 f}{\partial x\partial y})}{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} \tag{9}
$$

then the normal force can be written as

$$
F_N = -\frac{\partial f}{\partial x}F_{Nz}\mathbf{e_1} - \frac{\partial f}{\partial y}F_{Nz}\mathbf{e_2} + F_{Nz}\mathbf{e_3} \tag{10}
$$

The force due to friction has a magnitude of $-\mu\|F_N\|$ in the direction of the velocity vector: $F_f = -\mu\|F_N\|\cdot\frac{\mathbf{v_p}}{\|\mathbf{v_p}\|}$. So

$$
\begin{aligned}
&F_f = \\
&-\mu\dot{x}C_f\mathbf{e_1} - \mu\dot{y}C_f\mathbf{e_2} - \mu C_f\left(\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y}\right)\mathbf{e_3} \\
&C_f \triangleq F_{Nz}\frac{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}{\sqrt{\dot{x}^2+\dot{y}^2+\left(\frac{\partial f}{\partial x}\dot{x}+\frac{\partial f}{\partial y}\dot{y}\right)^2}}
\end{aligned}
\tag{11}
$$

Newton's second law gives the relationship: $m\mathbf{a} = F_N + F_f - mg\mathbf{e_3}$ which yields the equations of motion

$$
\begin{aligned}
\ddot{x} &= -\frac{\partial f}{\partial x}\frac{F_{Nz}}{m} - \frac{\mu}{m}\dot{x}C_f \\
\ddot{y} &= -\frac{\partial f}{\partial y}\frac{F_{Nz}}{m} - \frac{\mu}{m}\dot{y}C_f \\
\ddot{z} &= -g + \frac{F_{Nz}}{m} - \frac{\mu}{m}\left(\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y}\right)C_f
\end{aligned}
\tag{12}
$$

Similarly, we can perform an angular momentum balance. Since the only force in this system which produces a moment about the center of mass of the ball is the friction force, we can write $I\dot{\omega} = \mathbf{r_{p/o}}\times F_f$ where I is the moment of inertia matrix of the ball, ω is the angular velocity and $\mathbf{r_{p/o}}$ is the vector between the center of mass and the contact point with the surface. This can be written simply as $-R\hat{\mathbf{n}}$. Since the moment of inertia for a uniform sphere is equal to $\frac{2}{5}mR^2$ in each of the principle axes we can write

$$
\begin{aligned}
\dot{\omega}_x &= -\frac{5\mu\left(\dot{x}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \dot{y}\left(1+\frac{\partial f}{\partial y}^2\right)\right)F_{N_z}}{2mR\sqrt{\dot{x}^2+\dot{y}^2+\left(\dot{x}\frac{\partial f}{\partial x}+\dot{y}\frac{\partial f}{\partial y}\right)^2}} \\
\dot{\omega}_y &= \frac{5\mu\left(\dot{x}\left(1+\frac{\partial f}{\partial x}^2\right) + \dot{y}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right)F_{N_z}}{2mR\sqrt{\dot{x}^2+\dot{y}^2+\left(\dot{x}\frac{\partial f}{\partial x}+\dot{y}\frac{\partial f}{\partial y}\right)^2}} \\
\dot{\omega}_z &= \frac{5\mu\left(-\dot{y}\frac{\partial f}{\partial x} + \dot{x}\frac{\partial f}{\partial y}\right)F_{N_z}}{2mR\sqrt{\dot{x}^2+\dot{y}^2+\left(\dot{x}\frac{\partial f}{\partial x}+\dot{y}\frac{\partial f}{\partial y}\right)^2}}
\end{aligned}
\tag{13}
$$

where $\omega_x, \omega_y, \omega_z$ are the components of the angular velocity about the three axes of the reference frame.

As expected, the equations of motion in (12) are very simmilar to those produced using Lagrange's method (6), the differences coming from the added complexity of rolling as well as sliding and the somewhat different modeling of friction. This does show that the argument presented in [1] for the necessity of augmenting the expression for the normal force acting on the ball with the forces of non-planar acceleration holds. When solving this problem using Newton's method, it is necessary to include this stipulation in order to agree with results produced by Lagrange's method.

As a simple check of these equations, we can use them to calculate the equations of motion for a surface for which this problem has been well characterized - the level plane. Let $f(x, y) = c$ where c is an arbitrary constant, independent of time. All of the partial derivatives of f become zero, and we are left with

$$\begin{aligned} \ddot{x} &= -\frac{g\mu\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ \ddot{y} &= -\frac{g\mu\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ \ddot{z} &= 0 \\ \dot{\omega} &= \frac{5g\mu}{2R} \end{aligned} \tag{14}$$

Thus, if we take $\mathbf{s}$ to be the straight path of the center of mass of the ball and $\hat{\mathbf{s}}$ to be the unit vector along this path, then, for a flat, level surface, the equations yield $\ddot{\mathbf{s}} = -\mu g\hat{\mathbf{s}}$ which is fully consistent with the standard solution to this problem [4].

4 Removing the Constraint of Staying on the Surface

Although the system described in the previous section adds a description of rolling to the original model, it still includes a very important, unmodeled constraint. When writing the expression for the normal force acting on the ball (9), components of both the gravity force and acceleration forces were used. This has the same effect as setting z to be fully equivalent to the function describing the surface (as done in section 2). The ball is constrained to always stay on the surface. In reality this is, of course, incorrect. Even if the ball is given no initial vertical acceleration, the shape of the surface itself may lead the ball to go into the air. For example, a ball rolling down a steep incline and then back up a smaller one can be propelled into the air, just like a skier in a half-pipe. This leaves us with the inequality constraint: $z - f(x, y) \geq 0$.

A search through the relevant literature reveals that there are very few examples of inequality constraints applied to Lagrangian mechanics problems, so before proceeding with our derivation we will consider another greatly simplified case.

4.1 Particle Sliding Off a Hemisphere

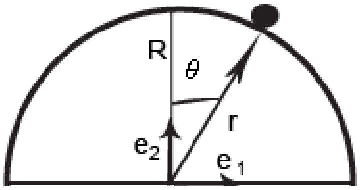

Figure 2: A particle sliding off a hemispherical surface.

The class of dynamics problems with inequality constraints will be motivated by the example of a particle sliding off of a hemispherical surface. A particle of mass m slides on the surface of a frictionless hemisphere of radius R in a fixed inertial frame $I = (O, \mathbf{e_1}, \mathbf{e_2})$ where the unit vectors correspond to the horizontal and vertical, respectively, and the origin is located in the center of the bottom of the hemisphere. The particle's position is described by the coordinates $(\mathbf{r}, \theta)$ where $\mathbf{r}$ is the vector from the coordinate frame origin and θ is the angle between r and $\mathbf{e_2}$. The particle starts from $\theta = 0$ with some initial velocity (θ_0) in the positive θ direction. This problem is generally posed to ask the question of when (and at what angular displacement) the particle will lose contact with the surface. The standard solution employs Newtonian mechanics to find the time at which the normal force becomes zero. We shall solve this problem using the Euler-Lagrange equations with the inequality constraint $r - R \geq 0$.

The position and velocity of the particle can be expressed as

$$\mathbf{r_p} = r\sin\theta\mathbf{e_1} + r\cos\theta\mathbf{e_2} \tag{15}$$

$$\begin{aligned} \mathbf{v_p} = {} & (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\mathbf{e_1} + \\ & (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\mathbf{e_2} \end{aligned} \tag{16}$$

where $\dot{r} = \frac{dr}{dt}$ and $\dot{\theta} = \frac{d\theta}{dt}$. Since we cannot assume that r remains constant in time, we must preserve all time derivatives of both θ and r. The Lagrangian is

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta \tag{17}$$

Applying the method of Lagrange multipliers produces the three equations

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 - g\cos\theta + \frac{\lambda}{m} \\ \ddot{\theta} &= \frac{g}{r}\sin\theta \\ r - R &\geq 0 \end{aligned} \tag{18}$$

Since the moment of contact loss is a stationary point, the Lagrange multiplier will go to zero, while the inequality constraint prior to that point in time will be a strict equality. Thus at the time of contact loss ($t^\star$) $\lambda \to 0$ and $r = R$, which implies that $\ddot{r} = \dot{r} = 0$. From this we get the relationship

$$\dot{\theta}^\star \triangleq \dot{\theta}(t^\star) = \sqrt{\frac{g}{R}\cos\theta^\star} \tag{19}$$

Integrating the differential equation for θ (18) with initial conditions $\theta(0) = 0$, $\dot{\theta}(0) = \dot{\theta}_0$ yields

$$\dot{\theta}(t) = \sqrt{\dot{\theta}_0^2 + \frac{2g}{R}(1 - cos\theta)} \tag{20}$$

Evaluating this at $t^\star$ and setting it equal to (19) allows us to solve for $\theta^\star$

$$\theta^\star = \cos^- 1\left(\frac{2}{3} + \frac{R}{3g}\dot{\theta}_0\right) \tag{21}$$

This matches the solution obtained from a Newtonian formulation [5] and gives us all of the equations necessary to numerically integrate the time history of an example of this system. By using the inequality constraint as a strict equality until $t^\star$ and a strict inequality afterwards an integrator can seamlessly go from one condition to the other. It is interesting to note that the Lagrange multiplier can be interpreted physically as the magnitude of the normal force. Solving (18) for λ gives the equation: $\lambda = m(\ddot{r} - r\dot{\theta} + g\cos\theta)$, which (when $\ddot{r} = 0$) is exactly the normal force that would be calculated by applying Newton's second law to the system in figure (2).

Having shown that the Lagrange method can produce good results with inequality constraints, we can move on to the problem of interest.

4.2 Ball Sliding and Rolling on an Arbitrary Surface

For this derivation, we will adopt a somewhat different formulation of the problem. To derive the equations of motion, we will use the modified Lagrange method presented by Rosen in the December 2000 issue of the Journal of Applied Mechanics [6]. Briefly, this method models friction losses as collisions between the rolling body and an infinite number of miniscule particles of equal mass and initial velocity that make up the surface. The energy losses due to sliding and pivoting are considered to be much greater than those from rolling so only the first two are considered. From these assumptions it is possible to define the change in kinetic energy of the particles per unit time ($K_f(t)$) as a function of the sliding velocity (v_s) and angular velocity normal to the tangent plane (ω_p). Following Rosen's example, for this system, we will take K_f to be a quadratic function of the form

$$K_f(t) \triangleq \frac{1}{2}\left(\mu v_s(t)^2 + \nu\omega_p(t)^2\right) \tag{22}$$

where μ and ν are coefficients of sliding and pivoting friction. Furthermore, we will make the assumption that friction due to pivoting is much smaller than friction due to sliding and neglect the ω_p term altogether. By introducing K_f into the equation for the generalized momentum of a particle, Rosen produces the modified Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_i}}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial K_f}{\partial \dot{q_i}} = Q_i \tag{23}$$

Essentialy, K_f serves the role of a dissipation function built into the Euler-Lagrange equation and based on physical values of sliding and pivoting velocities.

As before, we consider a homogenous ball of mass m, radius R and radius of gyration k on a surface characterized by the well-behaved function $z = f(x, y)$ which is defined for every point (x, y) within the region of interest. The coordinates (x, y, z) describe the motion of the center of mass, while the rotation of he ball is described by the coordinates (ψ, θ, ϕ). In Kane's notation, we will describe the rotation as a space-two 3-1-3 rotation [7]. The velocity of the center of mass and angular velocity are

$$\begin{aligned}\mathbf{v_p} &= \dot{x}\mathbf{e_1} + \dot{y}\mathbf{e_2} + \dot{z}\mathbf{e_3} && (24)\\ \bar{\omega} &= \omega_x\mathbf{e_1} + \omega_y\mathbf{e_2} + \omega_z\mathbf{e_3} && (25)\end{aligned}$$

where

$$\begin{aligned}\omega_x &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi && (26)\\ \omega_y &= \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi && (27)\\ \omega_z &= \dot{\phi}\cos\theta + \dot{\psi} && (28)\end{aligned}$$

Since the kinetic energy is equal to $\frac{m}{2}\mathbf{v_p}^2 + \frac{mk^2}{2}\bar{\omega}^2$ and the potential energy is due solely to the gravitational potential, the Lagrangian has the form

$$\begin{aligned}L &= \tfrac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\\ &+ \tfrac{mk^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta\right) - mgf\end{aligned} \quad (29)$$

The sliding is equal to the sum of the velocity of the center of mass and the cross product of the angular velocity and the vector between the center of mass and the contact point $\mathbf{v_s} = \mathbf{v_p} + \omega \times (-R\hat{\mathbf{n}})$ where $\hat{\mathbf{n}}$ was defined in (7). This is actually a restatement of the pure rolling condition - when the ball goes from rolling and sliding to just rolling, the value of this expression will be zero (and there will be no explicit friction force in the Newtonian formulation of the problem) [4] [8].

Because these equations can get very cumbersome, it helps at this point to define some new variables

$$N \triangleq \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1 \quad (30)$$

$$\gamma_x \triangleq \dot{x} - \frac{r}{\sqrt{N}}\left(\omega_y + \omega_z\frac{\partial f}{\partial y}\right) \quad (31)$$

$$\gamma_y \triangleq \dot{y} + \frac{r}{\sqrt{N}}\left(\omega_x + \omega_z\frac{\partial f}{\partial x}\right) \quad (32)$$

$$\gamma_z \triangleq \dot{z} + \frac{r}{\sqrt{N}}\left(\omega_x\frac{\partial f}{\partial y} - \omega_y\frac{\partial f}{\partial x}\right) \quad (33)$$

Using these, the definition in (22) and neglecting ω_p we can write K_f as

$$K_f = \frac{\mu_x}{2}\gamma_x^2 + \frac{\mu_y}{2}\gamma_y^2 + \frac{\mu_z}{2}\gamma_z^2 \quad (34)$$

Applying the modified Euler-Lagrange equation (23) to (29) and (34) yields the equations of motion for the six generalized coordinates and one constraint equation

$$m\ddot{x} + \mu_x\gamma_x - \lambda_h\tfrac{\partial f}{\partial x} = 0 \quad (35)$$

$$m\ddot{y} + \mu_y\gamma_y - \lambda_h\tfrac{\partial f}{\partial y} = 0 \quad (36)$$

$$m\ddot{z} + \mu_z\gamma_z + gm + \lambda_h = 0 \quad (37)$$

$$\begin{gathered}k^2m\left(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta\right) + \\ \tfrac{r}{\sqrt{N}}\left(-\mu_x\gamma_x\tfrac{\partial f}{\partial y} + \mu_y\gamma_y\tfrac{\partial f}{\partial x}\right) = 0\end{gathered} \quad (38)$$

$$\begin{gathered}k^2m\left(\ddot{\theta} + \dot{\phi}\dot{\psi}\sin\theta\right) + \\ \tfrac{r}{\sqrt{N}}\left(\mu_x\gamma_x\sin\psi + \mu_y\gamma_y\cos\psi\right) + \\ \tfrac{r}{\sqrt{N}}\left(\mu_z\gamma_z\left(\cos\psi\tfrac{\partial f}{\partial y} + \sin\psi\tfrac{\partial f}{\partial x}\right)\right) = 0\end{gathered} \quad (39)$$

$$\begin{gathered}k^2m\left(\ddot{\phi} + \ddot{\psi}\cos\theta - \dot{\theta}\dot{\psi}\sin\theta\right) + \\ \tfrac{r}{\sqrt{N}}\left(-\mu_x\gamma_x\left(\cos\psi\sin\theta + \cos\theta\tfrac{\partial f}{\partial y}\right)\right) + \\ \tfrac{r}{\sqrt{N}}\left(\mu_y\gamma_y\left(\sin\psi\sin\theta + \cos\theta\tfrac{\partial f}{\partial x}\right)\right) + \\ \tfrac{r}{\sqrt{N}}\left(\mu_z\gamma_z\sin\theta\left(\sin\psi\tfrac{\partial f}{\partial y} - \cos\psi\tfrac{\partial f}{\partial x}\right)\right) = 0\end{gathered} \quad (40)$$

$$z - f(x, y) \geq 0 \quad (41)$$

We can now use (38) to eliminate $\ddot{\psi}$ from equation (40), producing

$$\begin{aligned}&\tfrac{\sin\theta}{\sqrt{N}}[k^2m\sqrt{N}\left(-\dot{\theta}\dot{\psi} + \cos\theta\dot{\theta}\dot{\phi} + \ddot{\phi}\sin\theta\right) + \\ &r\left(-\cos\psi\gamma_x\mu_x + \sin\psi\gamma_y\mu_y\right) + \\ &r\left(-\cos\psi\tfrac{\partial f}{\partial x} + \sin\psi\tfrac{\partial f}{\partial y}\right)\gamma_z\mu_z] = 0\end{aligned} \quad (42)$$

Now, multiplying (39) by $\sin\theta\sin\psi$ and subtracting (42) multiplied by $\cos\psi$, multiplying (39) by $\sin\theta\cos\psi$ and adding (42) multiplied by $\sin\psi$ and then replacing all relevant portions with the first derivatives of equations (26) through (28), equations (38) through (40) simplify to

$$-k^2m\dot{\omega}_y + \frac{r}{\sqrt{N}}\left(\gamma_x\mu_x + \frac{\partial f}{\partial x}\gamma_z\mu_z\right) = 0 \quad (43)$$

$$k^2 m\dot{\omega}_x + \frac{r}{\sqrt{N}}\left(\gamma_y\mu_y + \frac{\partial f}{\partial y}\gamma_z\mu_z\right) = 0 \tag{44}$$

$$k^2 m\dot{\omega}_z + \frac{r}{\sqrt{N}}\left(-\frac{\partial f}{\partial y}\gamma_x\mu_x + \frac{\partial f}{\partial x}\gamma_y\mu_y\right) = 0 \tag{45}$$

Note that here too, the Lagrange multiplier λ_h has the physical interpretation of the magnitude of the normal force. Equation (37) can be solved for: $\lambda_h = -\left(m\ddot{z} + \mu_z\gamma_z + gm\right)$, representing the magnitude of the normal force in the $\mathbf{e_3}$ direction. Thus, we have shown that the normal force contains components of gravity, friction, and the acceleration. This is the most general expression derived so far in this study that provides a rigorous proof for the argument regarding the magnitude of the normal force used in section 3.

As Rosen points out, the case of pure rolling can be modeled using this framework as the instance when the surface is infinitely rough (i.e. $\mu_x, \mu_y, \mu_z \to \infty$). When this occurs the following relationships form

$$\begin{aligned}
&\lim_{\mu_x\to\infty} \gamma_x = 0\\
&\lim_{\mu_y\to\infty} \gamma_y = 0\\
&\lim_{\mu_z\to\infty} \gamma_z = 0\\
&\lim_{\mu_x\to\infty} \mu_x\gamma_x = \lambda_x\\
&\lim_{\mu_y\to\infty} \mu_y\gamma_y = \lambda_y\\
&\lim_{\mu_z\to\infty} \mu_z\gamma_z = \lambda_z
\end{aligned} \tag{46}$$

where λ_x, λ_y and λ_z are finite valued variables and correspond to the Lagrange multipliers that would enter the equations if pure rolling was modeled as a classical nonholonomic constraint [9]. The full system of equations in this case becomes

$$m\ddot{x} + \lambda_x - \lambda_h\tfrac{\partial f}{\partial x} = 0 \tag{47}$$

$$m\ddot{y} + \lambda_y - \lambda_h\tfrac{\partial f}{\partial y} = 0 \tag{48}$$

$$m\ddot{z} + \lambda_z + gm + \lambda_h = 0 \tag{49}$$

$$-k^2 m\dot{\omega}_y + \tfrac{r}{\sqrt{N}}\left(\lambda_x + \tfrac{\partial f}{\partial x}\lambda_z\right) = 0 \tag{50}$$

$$k^2 m\dot{\omega}_x + \tfrac{r}{\sqrt{N}}\left(\lambda_y + \tfrac{\partial f}{\partial y}\lambda_z\right) = 0 \tag{51}$$

$$k^2 m\dot{\omega}_z + \tfrac{r}{\sqrt{N}}\left(-\tfrac{\partial f}{\partial y}\lambda_x + \tfrac{\partial f}{\partial x}\lambda_y\right) = 0 \tag{52}$$

$$\gamma_x = 0 \tag{53}$$

$$\gamma_y = 0 \tag{54}$$

$$\gamma_z = 0 \tag{55}$$

$$z - f(x, y) \geq 0 \tag{56}$$

We are thus left with six numerically integrable equations, four constraint equations and ten unknowns, making this a well characterized system.

5 Strategies for numerical integration and optimization

While it is possible to solve analytically for the time when the ball will lose contact with surface (using the same approach taken in the sample problem of inequality constraints in section 4.1), it is not actually very useful to do so, from an optimization standpoint. For a given surface, the ball may leave the ground several times (for it shall invariably fall back down after every flight). This makes it simpler to numerically integrate the equations from some given starting conditions to find the path the ball will take. Since the optimization scheme described in [1] involves finding the position of the ball at discrete time steps, this approach is consistent with posing this as an optimization problem.

The ball starts at some initial position given by $(x_0, y_0, z_0 = f(x_0, y_0))$ with initial velocities $(\dot{x}_0, \dot{y}_0, \dot{z}_0, \dot{\psi}_0, \dot{\theta}_0, \dot{\phi}_0)$. At this point, the height constraint is a strict equality: $z = f(x, y)$ so equation (8) holds. The only way for the ball to immediately leave the surface is if it is imparted with a large initial velocity in the positive $\mathbf{e_3}$ direction, which is unlikely when considering the initial velocities given to a ball on a putting green, or if the surface is discontinuous, which would mean that its shape function is not well-behaved and make this analysis inapplicable. We then integrate the state space

$$\ddot{x} = -\tfrac{\mu_x}{m}\gamma_x + \tfrac{\lambda_h}{m}\tfrac{\partial f}{\partial x} \tag{57}$$

$$\ddot{y} = -\tfrac{\mu_y}{m}\gamma_y + \tfrac{\lambda_h}{m}\tfrac{\partial f}{\partial y} \tag{58}$$

$$\ddot{z} = -\tfrac{\mu_z}{m}\gamma_z - g - \tfrac{\lambda_h}{m} \tag{59}$$

$$\dot{\omega}_y = \tfrac{r}{k^2 m\sqrt{N}}\left(\gamma_x\mu_x + \tfrac{\partial f}{\partial x}\gamma_z\mu_z\right) \tag{60}$$

$$\dot{\omega}_x = -\tfrac{r}{k^2 m\sqrt{N}}\left(\gamma_y\mu_y + \tfrac{\partial f}{\partial y}\gamma_z\mu_z\right) \tag{61}$$

$$\dot{\omega}_z = -\frac{r}{k^2 m\sqrt{N}} \left(-\frac{\partial f}{\partial y}\gamma_x \mu_x + \frac{\partial f}{\partial x}\gamma_y \mu_y \right) \tag{62}$$

with the constraint $z = f(x, y)$ and calculate λ_h at each time step. When λ_h goes to zero, contact loss will have occured.

At this point the integrator function must change its behavior (or change to a different integrator). The ball will now follow a classic ballistic arc through the air, acted upon only by gravity (if air resistance is neglected) and taking the values of the last time step from the first integrator as the initial conditions. The equations of motion become simply: $\ddot{z} = -g$. Since air resistance is neglected, the other velocities do not change throughout the flight. When the second integrator reaches the point where the calculated z once again matches the surface function value, the ball has returned to the ground.

At this point, one can model the ensuing momentum transfer in several ways. The most correct of these would be to take into account the possible inelasticity and bouncing which can occur [10], which is another very involved analysis. The simplest approach would be to consider that the net acceleration in the $\mathbf{e_3}$ direction instantly vanishes at the moment of impact, while the other quantities remain unaffected (the ball lands and continues to slide and roll exactly as it did immediately before leaving the ground). This would allow us to switch back to the first integrator, using the final state from the previous run as the initial conditions, with the exception of $\dot{z}$ and $\ddot{z}$, which would both be set to zero. In this way, one can simulate the full time history of the ball.

6 Conclusions

The problem of finding equations of motion of a ball on an arbitrary surface is greatly complicated by the number of nonholonomic constraints which must be taken into account to fully model the physics of the system. The added problem of including an inequality constraint makes the majority of formulations employ one simplification or another. While the equations cited here do stem from certain simplifications of the real physics in effect (such as the true nature of friction, the effects of air resistance and the nature of the collision between the ball and surface when returning from the air), they are more complete than any that could be found in existing literature.

The paper which motivated this exercise [1] presented an argument for the augmentation of the magnitude of the normal force with the out-of-tangent-plane acceleration (used in section 3). There was no rigorous argument presented, however, as to the correctness of this definition. Solving the problem of pure sliding using Lagrange's method (as in section 2) shows that the dynamics naturally produce this definition, with no need for any additional explanation. However, it is important to note that using this definition introduces the constraint of always keeping the ball on the surface, which could lead to unintentional errors if not recognized.

The final system of equations demonstrates the usefulness of Rosen's modified Lagrange method. While it is not radically different from introducing a dissipation function for friction, it makes the equations simpler to manipulate and defines friction completely in terms of velocity quantities naturally stemming from the system. While the derivation here employed a strictly quadratic, anisotropic function for K_f, there is no reason to consider this to be the best description of the friction effects of a putting green on a golf ball. It is highly likely that a lawn has the same friction characteristics in all directions (and is thus isotropic), which would require just one friction coefficient instead of the three employed in this derivation. Similarly, it may be that a linear K_f function is a better model than the quadratic one used. The quadratic, anisotropic model was selected solely to keep the derivation as general as possible. Employing an isotropic linear model of friction would certainly simplify the derived equations of motion. Alternatively, the equations could be further refined by including the effects of pivoting and rolling friction, which were neglected here.

Finally, it is important to point out that there are numerous other definitions that can be used to describe the shape of the surface. While all of the derivations presented here defined the height of the surface with a single equation in a cartesian coordinate frame, an equally correct approach would be to employ general curvilinear coordinates - to describe each point on the surface as the intersection of three two-dimensional functions. Another approach, used by Borisov et. al in their paper [8], has the surface described as a function of the vector between the origin of the inertial frame and the contact point: $F(r) = 0$. This implicit definition of the surface makes it very simple to define the surface normal ($\bigtriangledown F(r)$) and simplifies some of the equations produced by using Newton's method.

On the other hand, using cartesian coordinates makes it easier to define a variety of surfaces and is more intuitive than some of the other approaches. Also, having six initial conditions means that the initial velocity and spin imparted on the ball can be very easily characterized. While it is unlikely that the putter will apply a lateral spin to the ball, optimization routines often produce unexpected results. It may turn out that there are families of paths the ball can take which rely on specific initial spins. The only way for an optimizing routine to discover dynamics such as these is to give it the ability to fully control initial conditions in all dimensions.

References

[1] Vanderbei, R.J., *A case study in trajectory optimization: putting on an uneven green*, Technical Report ORFE-00-4, Dept. of Operations Research and Financial Engineering, Princeton University, Princeton NJ, 2000

[2] Alessandrini, S.M., *A motivational example for the numerical solution of two-point boundary-value problems*, SIAM Review, Vol. 37, No. 3, pp. 423-427, 1995

[3] Hand, L.N. and Finch, J.D., *Analytical Mechanics*, Cambridge University Press: Cambridge, 1998

[4] Hierrezuelo, J. and Carnero, C. *Sliding and Rolling: the physics of a rolling ball*, Phys. Educ., Vol. 30, pp. 177-182, 1995

[5] Kasdin, N.J. and Paley, D.A., *An Introduction to Engineering Dynamics*, in preparation, 2006

[6] Rosen, A., *Modified Lagrange method to analyze problems of sliding and rolling*, J. App. Mech. Vol. 67, pp. 697-704, 2000

[7] Kane, T.R., Likins, P.W., and Levnison, D.A., *Spacecraft Dynamics*, McGraw-Hill:New York, 1983

[8] Borisov, A.V., Mamaev, I.S., and Kilin, A.A., *The rolling motion of a ball on a surface. New integrals and hierarchy of dynamics*, Regular and Chaotic Dynamics, Vol. 7, No. 2, pp. 201-219, 2002

[9] Neimark, I. Jr., and Fufaev, N. A., *Dynamics of Nonholonomic Systems*, The American Mathematical Society: Providence, RI, 1982

[10] Burkhardt,T.W. and Kotsev, S.N., *Equilibrium statistics of an inelastically bouncing ball, subject to gravity and a random force*,Phys. Rev. E 73, 2006