

Week 1

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Dynamics

This is a class about dynamics, or classical mechanics (sometimes known as Newtonian mechanics). The entirety of this field is built upon laws first postulated by Isaac Newton (building upon the work of many others) and later expanded by Leonhard Euler. As with any study, the first step is to make sure that we have the appropriate tools and language to describe the phenomena under consideration. Since the early 20th century, thanks to the efforts of Josiah Willard Gibbs, the standard tools for studying classical mechanics are vector algebra and vector calculus, which we will review here. Remember that these handouts are not complete on their own. They are intended to accompany the recorded lectures, and to help in your note-taking and studying.

Newton's Laws of Motion

- ① *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare*

Every body preserves in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon

- ② *Mutationem motus proportionalem esse vi motrici impressae; et fieri secundum lineam rectam qua vis illa imprimitur*

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed

- ③ *Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi*

To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts

Laws vs. Theories

- A **law** is a descriptive generalization about how some aspect of the natural world behaves under stated circumstances.
 - Newton's laws are basically definitions or postulates. They describe but don't explain.
- A **theory** is a well-substantiated explanation of some aspect of the natural world that can incorporate facts, laws, inferences, and tested hypotheses.

-National Academy of Sciences (1999)

Circular Definition #1

Force

A **force** is something that changes the motion (momentum) of a body (mass).

NB: Forces are **not** required to do dynamics but are an enormously useful concept.

Circular Definition #2

Mass

Mass is the property of a body which determines how its motion (momentum) is modified by the application of force.

For our purposes: mass is how much matter is in the object. Mass and Force are the things that obey Newton's 2nd Law.

Newton's Laws: Restrictions and Corollaries

- Newton's laws **only** apply to point masses (infinitesimal particles)
- Change in momentum is due to the sum of all forces acting on a body. This is called the **resultant** force.
- We care about how much force there is (magnitude) and where it is coming from (direction).

Forces are **vector** quantities. Always! No exceptions!

Vectors

The vectors we use in this class are **Euclidean** (or **geometric** or **spatial**) vectors. These are:

- A geometric entity having magnitude and direction
- A connection between two points in space
- An element of a vector space (you don't need to care about this one)
- Absolutely **NOT** just a collection of 3 numbers

Notation

When I typeset vectors, they will always be bolded (e.g. $\mathbf{x}$). When I write vectors, they will be underlined (e.g. $\underline{x}$). *You* can write vectors however you wish, as long as I can always tell the difference between your vectors and your scalars.

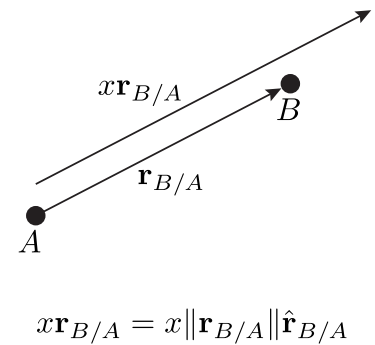
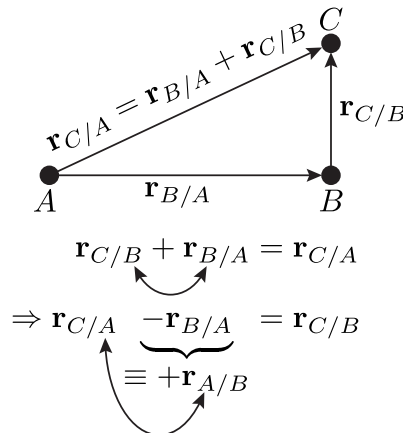
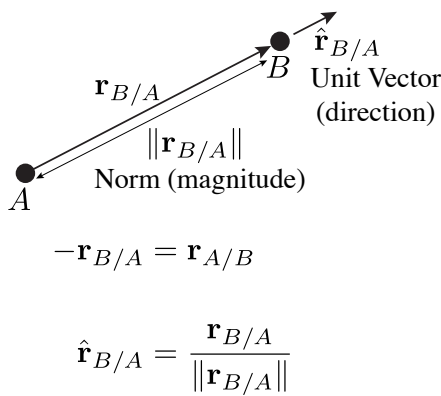
Vector Addition and Scalar Multiplication

Two vectors, $\mathbf{a}, \mathbf{b}$ can be added, creating a new vector $\mathbf{c}$: $\mathbf{a} + \mathbf{b} = \mathbf{c}$. Any vector $\mathbf{c}$ can be multiplied by any scalar x , creating a new vector $\mathbf{d}$: $x\mathbf{c} = \mathbf{d}$. These operations have the following properties:

- ① Commutativity of vector addition: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- ② Associativity of vector addition: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
- ③ Identity element of vector addition: $\exists \mathbf{0}$ s.t. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- ④ Inverse elements of vector addition: $\exists -\mathbf{a}$ s.t. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
- ⑤ Compatibility of scalar multiplication: $x(y\mathbf{a}) = (xy)\mathbf{a}$
- ⑥ Distributivity of scalar multiplication over vector addition: $x(\mathbf{a} + \mathbf{b}) = x\mathbf{a} + x\mathbf{b}$
- ⑦ Distributivity of scalar multiplication over scalar addition: $(x + y)\mathbf{a} = x\mathbf{a} + y\mathbf{a}$
- ⑧ Identity element of scalar multiplication: $\exists 1$ s.t. $1\mathbf{a} = \mathbf{a}$

Euclidean (Geometric) Vectors

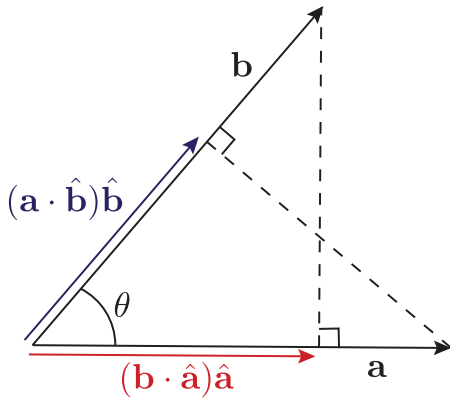
A Euclidean vector has a magnitude and a direction. A position vector $\mathbf{r}_{B/A}$ has a magnitude of the distance between points A and B and a direction pointing from A to B .



Vector Products

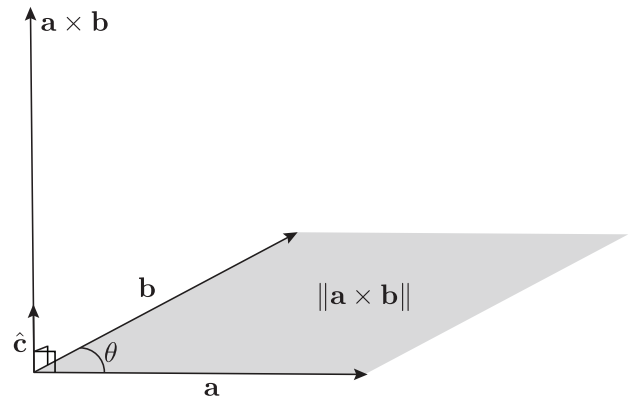
(Scalar) Dot Product

- $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $x\mathbf{a} \cdot y\mathbf{b} = xy(\mathbf{a} \cdot \mathbf{b})$



(Vector) Cross Product

- $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta \hat{\mathbf{c}}$
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $y\mathbf{a} \times \mathbf{b} = y(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times y\mathbf{b}$



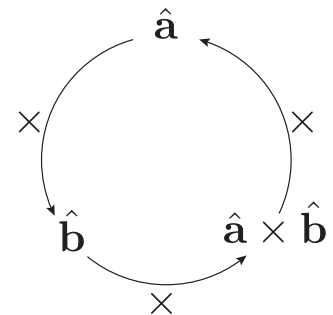
More Vector Products

- If vector $\mathbf{a}$ is perpendicular to vector $\mathbf{b}$ ($\mathbf{a} \perp \mathbf{b}$):

① $\mathbf{a} \cdot \mathbf{b} = 0$

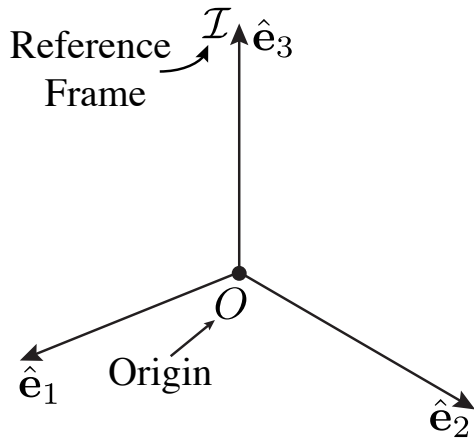
② $\hat{\mathbf{a}} \times \hat{\mathbf{b}}$ is perpendicular to both $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$

- Scalar Triple Product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
- Vector Triple Product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$
- $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\| \|\mathbf{a}\| \cos(0) = \|\mathbf{a}\|^2$



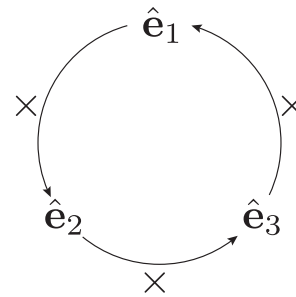
Reference Frames

A reference frame defines the space with respect to which you make measurements:



$$\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \text{ or } \mathcal{I} \triangleq (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$$

- A reference frame is a collection of at least 3 non-collinear points that are rigidly connected (the distance between the points does not change).
- A reference frame is defined by three mutually orthogonal unit vectors and (optionally) by a point (the origin)



Inertial Reference Frames

- Newton defined one special reference frame: absolute space, defined with respect to the ‘fixed stars’
- Today, we know that there are no truly fixed stars, and now call this special frame an **inertial reference frame** - one not undergoing acceleration
- Newton’s second law only works for accelerations calculated in an inertial reference frame
- But how do we find a reference frame that is not accelerating?
 - The Earth spins on its axis at ~ 1000 mph
 - The Earth orbits the sun at ~ 66000 mph
 - The Sun orbits the galactic center at ~ 483000 mph
 - The whole galaxy moves at $\sim 1.3 \times 10^6$ mph
 - And so on...

Circular Definition #3

Inertial Frame

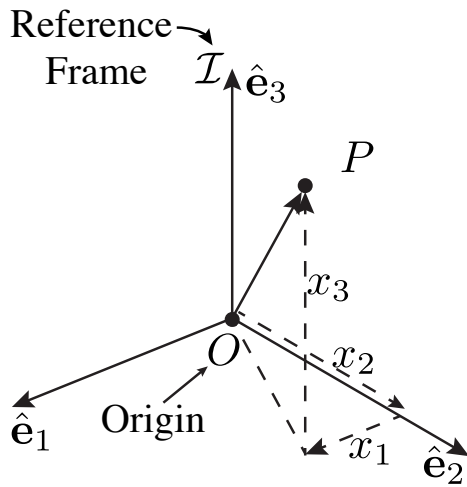
An **inertial reference frame** is a frame of reference in which Newton's laws apply

- An unforced body travels in straight lines at constant speeds
- Accelerations calculated with respect to this frame are proportional to applied forces
- Any reference frame traveling at constant velocity (not rotating) with respect to an inertial frame is also inertial
- For our purposes, a reference frame can be treated as inertial if it has negligible accelerations as compared with the system we are considering
 - For objects and systems on the surface of the Earth, the Earth's surface can be approximated as an inertial reference frame

Coordinate Systems

- A **coordinate system** is a set of scalars that locate the position of one point with respect to another within a reference frame
- Coordinate systems are **not** the same as reference frames
- Any reference frame can have an infinite number of coordinate systems
- One special coordinate system (Cartesian) measures along the unit vectors of the frame
- Remember: a position vector is just a geometric object connecting two points. It exists independently of any reference frame or coordinate system.
- However: *any* vector can be written as the weighted sum of the unit vectors defining a reference frame

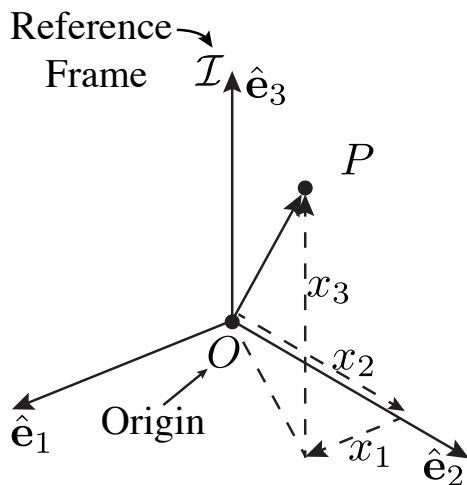
Vector Components



- $\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ is the **reference frame**
- (x_1, x_2, x_3) are the **Cartesian coordinates** defining the position of point P with respect to point O in reference frame $\mathcal{I}$
- The position vector of P with respect to O can be written as:

$$\mathbf{r}_{P/O} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3$$

Vector Components



- The position vector of P with respect to O can be written as:

$$\mathbf{r}_{P/O} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3$$

- We can also write:

$$[\mathbf{r}_{P/O}]_{\mathcal{I}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\mathcal{I}}$$

A vector is **not** a list of 3 numbers ...

...but the vector's components with respect to a reference frame is

Vector Addition and Scalar Multiplication

$$\text{Given: } \begin{cases} \mathbf{x} \triangleq x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 \\ \mathbf{y} \triangleq y_1\hat{\mathbf{e}}_1 + y_2\hat{\mathbf{e}}_2 + y_3\hat{\mathbf{e}}_3 \end{cases}$$

- $\mathbf{x} + \mathbf{y} = (x_1 + y_1)\hat{\mathbf{e}}_1 + (x_2 + y_2)\hat{\mathbf{e}}_2 + (x_3 + y_3)\hat{\mathbf{e}}_3$
- $\mathbf{x} - \mathbf{y} = (x_1 - y_1)\hat{\mathbf{e}}_1 + (x_2 - y_2)\hat{\mathbf{e}}_2 + (x_3 - y_3)\hat{\mathbf{e}}_3$
- $z\mathbf{x} = (zx_1)\hat{\mathbf{e}}_1 + (zx_2)\hat{\mathbf{e}}_2 + (zx_3)\hat{\mathbf{e}}_3$

Dot Product

$$\text{Given: } \begin{cases} \mathbf{x} \triangleq x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 \\ \mathbf{y} \triangleq y_1\hat{\mathbf{e}}_1 + y_2\hat{\mathbf{e}}_2 + y_3\hat{\mathbf{e}}_3 \end{cases}$$

- $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$
- Recall: $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$
- $\mathbf{x} \cdot \mathbf{x} \equiv \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2$
- Therefore:

$$\bullet \quad \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2} \qquad \cos\theta = \frac{x_1y_1 + x_2y_2 + x_3y_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}\sqrt{y_1^2 + y_2^2 + y_3^2}}$$

Cross Product

$$\text{Given: } \begin{cases} \mathbf{x} \triangleq x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3 \\ \mathbf{y} \triangleq y_1 \hat{\mathbf{e}}_1 + y_2 \hat{\mathbf{e}}_2 + y_3 \hat{\mathbf{e}}_3 \end{cases}$$

$$\bullet \mathbf{x} \times \mathbf{y} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2) \hat{\mathbf{e}}_1 + (x_3 y_1 - x_1 y_3) \hat{\mathbf{e}}_2 + (x_1 y_2 - x_2 y_1) \hat{\mathbf{e}}_3$$

$$\bullet \text{ Recall: } \|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \|\mathbf{y}\| \sin \theta$$

• Therefore:

$$\sin \theta = \frac{\sqrt{(x_2 y_3 - x_3 y_2)^2 + (x_3 y_1 - x_1 y_3)^2 + (x_1 y_2 - x_2 y_1)^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2} \sqrt{y_1^2 + y_2^2 + y_3^2}}$$

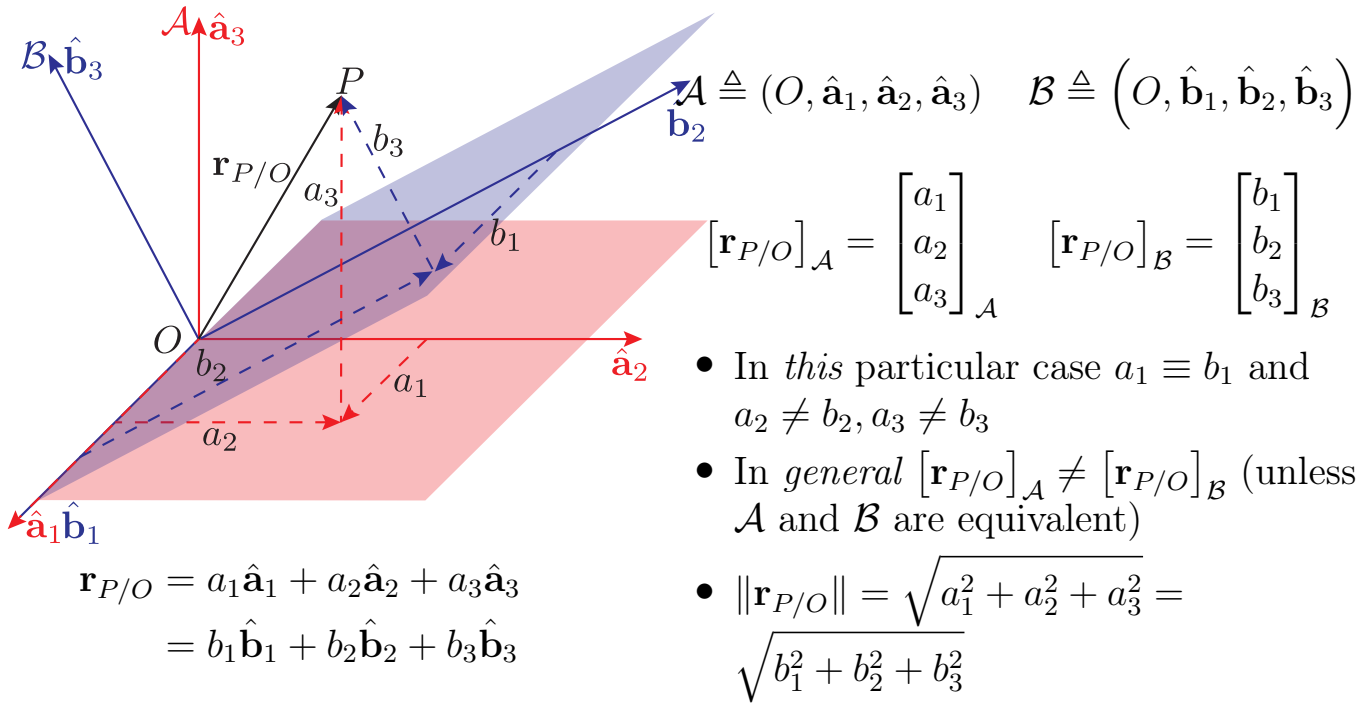
Dynamics with Multiple Reference Frames

Remember:

You only ever need one reference frame to do dynamics (an inertial frame)

However, for many problems, it is frequently convenient to define multiple inertial and non-inertial frames.

Vector Components in Multiple Frames



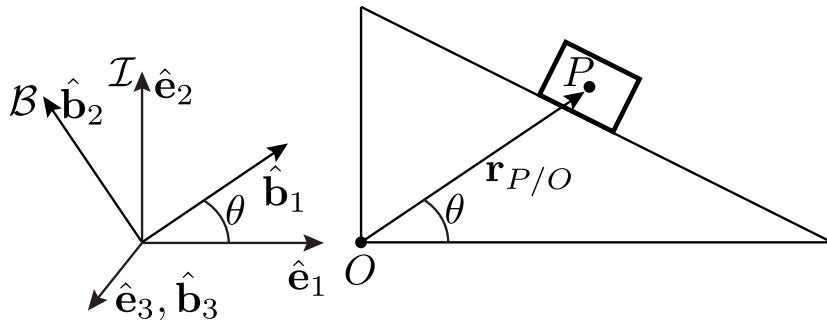
Extracting Vector Components

We can always extract the vector components in a given direction by using the dot product. For example, given $\mathbf{r}_{P/O} = a_1 \hat{\mathbf{a}}_1 + a_2 \hat{\mathbf{a}}_2 + a_3 \hat{\mathbf{a}}_3$:

$$\mathbf{r}_{P/O} \cdot \hat{\mathbf{a}}_1 = (a_1 \hat{\mathbf{a}}_1 + a_2 \hat{\mathbf{a}}_2 + a_3 \hat{\mathbf{a}}_3) \cdot \hat{\mathbf{a}}_1 = a_1 \underbrace{(\hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_1)}_{=1} + a_2 \underbrace{(\hat{\mathbf{a}}_2 \cdot \hat{\mathbf{a}}_1)}_{=0} + a_3 \underbrace{(\hat{\mathbf{a}}_3 \cdot \hat{\mathbf{a}}_1)}_{=0} = a_1$$

Similarly, $\mathbf{r}_{P/O} \cdot \hat{\mathbf{a}}_2 = a_2$ and $\mathbf{r}_{P/O} \cdot \hat{\mathbf{a}}_3 = a_3$.

A Motivating Example



$$\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \quad \mathcal{B} \triangleq (O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$$

$$[\mathbf{r}_{P/O}]_{\mathcal{I}} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}_{\mathcal{I}}$$

$$[\mathbf{r}_{P/O}]_{\mathcal{B}} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}$$

$$\hat{\mathbf{b}}_1 \equiv \hat{\mathbf{r}}_{P/O}$$

Vector Derivatives in Time

- The derivative of a **position** vector with respect to time is a **velocity** vector.
- The derivative of a **velocity** vector with respect to time is an **acceleration** vector. This is also the second derivative of the position vector with respect to time, and is the quantity needed to write the mathematical form of Newton's second law.
- Unlike regular Euclidean vectors, which exist independently of any reference frame, vector derivatives in time are specific to a frame. We denote this via a superscript to indicate the frame in which we are differentiating. For example:

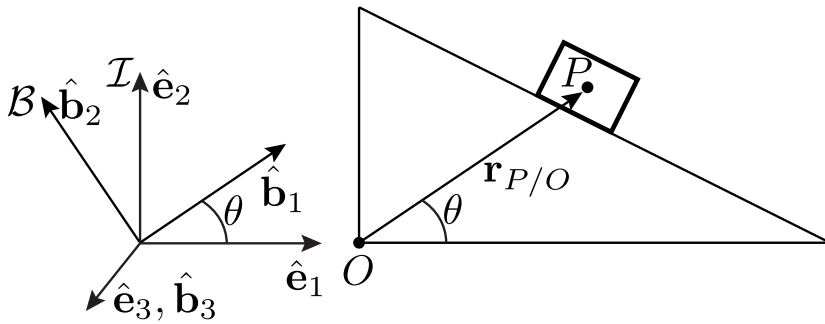
$${}^{\mathcal{I}}\mathbf{v}_{P/O} \triangleq \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} \quad \text{and} \quad {}^{\mathcal{I}}\mathbf{a}_{P/O} \triangleq \frac{{}^{\mathcal{I}}d}{dt}\mathbf{v}_{P/O} = \frac{{}^{\mathcal{I}}d^2}{dt^2}\mathbf{r}_{P/O}$$

Vector Derivatives in Time

- The unit vectors defining a reference frame are fixed in that frame for all time, and therefore **always** have zero time derivatives in that frame (they may, however, have non-zero time derivatives in other frames that are moving with respect to their own frame).
- $\mathbf{r}_{P/O} = a_1\hat{\mathbf{a}}_1 + a_2\hat{\mathbf{a}}_2 + a_3\hat{\mathbf{a}}_3$ is differentiable in time at a time t_1 with respect to frame $\mathcal{A} = (O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)$ if $a_1(t), a_2(t), a_3(t)$ are differentiable at $t = t_1$:

$$\begin{aligned} \left. \frac{{}^{\mathcal{A}}d}{dt} \mathbf{r}_{P/O} \right|_{t=t_1} &= \left. \frac{da_1}{dt} \right|_{t=t_1} \hat{\mathbf{a}}_1 + a_1 \underbrace{\left. \frac{{}^{\mathcal{A}}d}{dt} \hat{\mathbf{a}}_1 \right|_{t=t_1}}_{=0} + \left. \frac{da_2}{dt} \right|_{t=t_1} \hat{\mathbf{a}}_2 + a_2 \underbrace{\left. \frac{{}^{\mathcal{A}}d}{dt} \hat{\mathbf{a}}_2 \right|_{t=t_1}}_{=0} \\ &\quad + \left. \frac{da_3}{dt} \right|_{t=t_1} \hat{\mathbf{a}}_3 + a_3 \underbrace{\left. \frac{{}^{\mathcal{A}}d}{dt} \hat{\mathbf{a}}_3 \right|_{t=t_1}}_{=0} \end{aligned}$$

Back to Our Motivating Example



$$\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$$

$$\mathcal{B} \triangleq (O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$$

$$\hat{\mathbf{b}}_1 \equiv \hat{\mathbf{r}}_{P/O}$$

$$\frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O} \neq \frac{{}^{\mathcal{B}}d}{dt} \mathbf{r}_{P/O}$$

$$\mathbf{r}_{P/O} = x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2 = r\hat{\mathbf{b}}_1$$

$$\frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O} = \frac{{}^{\mathcal{I}}d}{dt} (x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2) = \dot{x}\hat{\mathbf{e}}_1 + \dot{y}\hat{\mathbf{e}}_2$$

$$\frac{{}^{\mathcal{B}}d}{dt} \mathbf{r}_{P/O} = \frac{{}^{\mathcal{B}}d}{dt} (r\hat{\mathbf{b}}_1) = \dot{r}\hat{\mathbf{b}}_1$$

$$\begin{aligned} \frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O} &= \frac{{}^{\mathcal{I}}d}{dt} (r\hat{\mathbf{b}}_1) \\ &= \dot{r}\hat{\mathbf{b}}_1 + r \frac{{}^{\mathcal{I}}d}{dt} (\hat{\mathbf{b}}_1) \end{aligned}$$

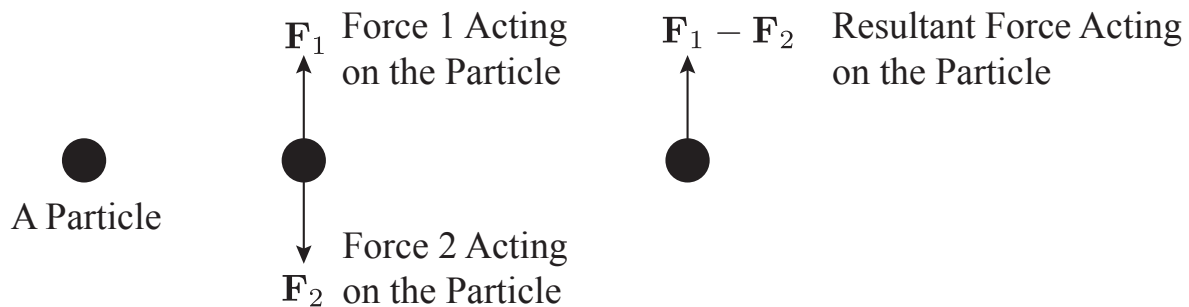
Newton's Second Law (for real)

$$\mathbf{F}_P = \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \left({}^{\mathcal{I}}\mathbf{p}_{P/O} \right) = \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \left(m_P {}^{\mathcal{I}}\mathbf{v}_{P/O} \right) = m_P {}^{\mathcal{I}}\mathbf{a}_{P/O}$$

Inertial Frame Derivative Inertially Fixed Point Mass (Assumed Constant)
 Resultant Force on P Linear Momentum Inertial Velocity and Acceleration

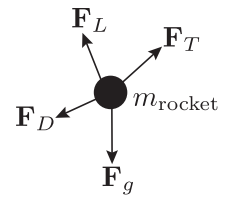
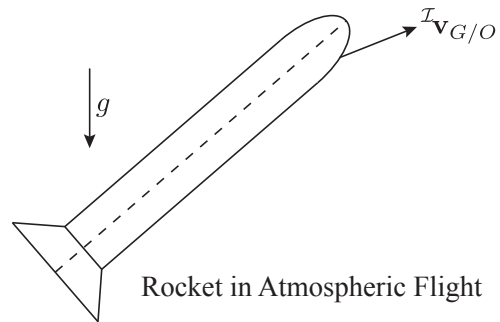
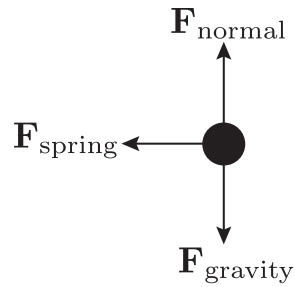
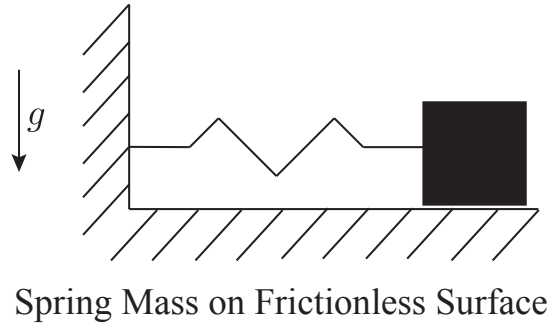
- Force has units of mass $\times$ length $\times$ time $^{-2}$
- The SI unit of force is the Newton: $1 \text{ N} = 1 \text{ kg} \cdot \text{m} \cdot \text{s}^{-2}$

Free Body Diagrams (FBDs)

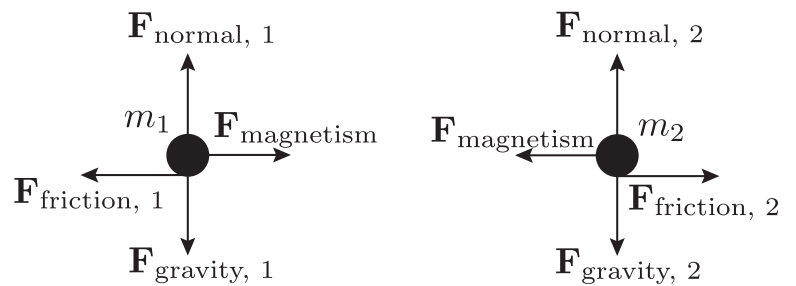
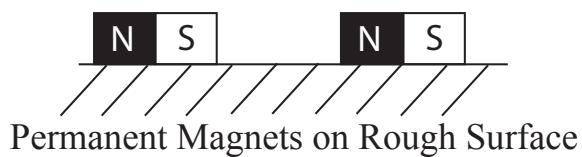
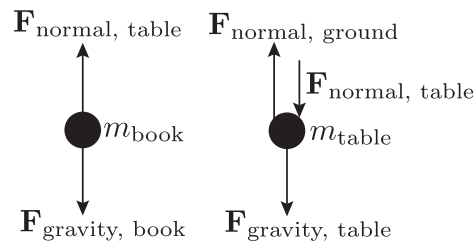
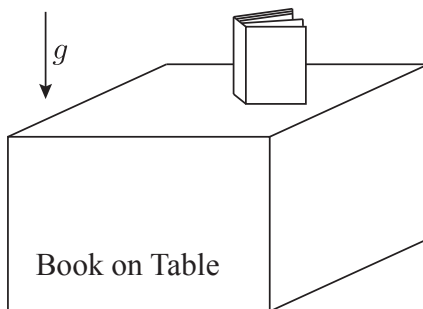


- Every body (mass) in a system is modeled as a particle (point mass). Each particle gets its own FBD.
- Each FBD should include *all* forces acting on the particle.

FBD Examples (1)



FBD Examples (2)



Degrees of Freedom (DOF) and Constraints

Degrees of Freedom

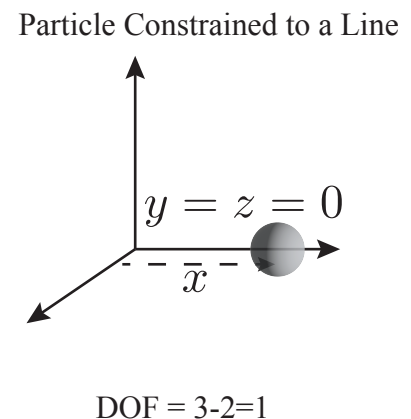
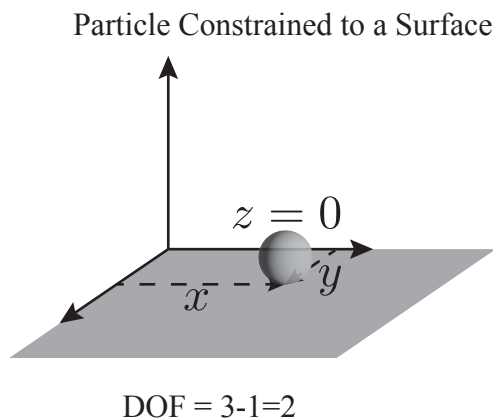
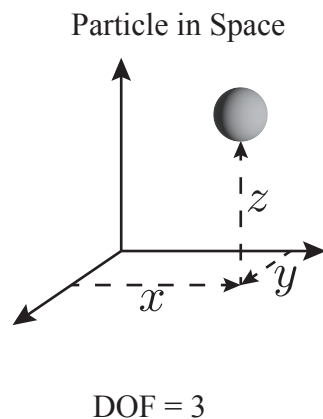
The number of independent coordinates needed to describe the position and orientation of every part of your system

- **NB:** Particles have no extent and therefore have **no** orientation
- A **constraint** removes a degree of freedom in your system
- We live in what appears to be a 3-dimensional universe and so, in general, need a minimum of 3 numbers (coordinates) to describe the position of one point relative to another

For a system of N particles with k constraints:

$$\text{DOF} = 3N - k$$

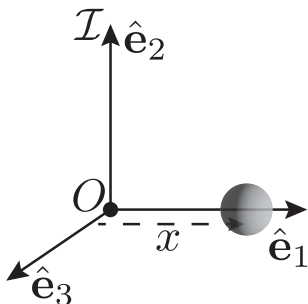
Particle Degrees of Freedom



Equations of Motion (EOM)

- Our goal (in most dynamics problems) is to find one equation for the second order derivative of the coordinate corresponding to each degree of freedom
- Collectively, these are known as the **Equations of Motion** of the system
 - Sometimes we can solve these analytically
 - More frequently we can only solve these numerically
- The solution to the equations of motion, along with their initial conditions, describe the motion of every part of our system *for all time*

1 DOF Systems



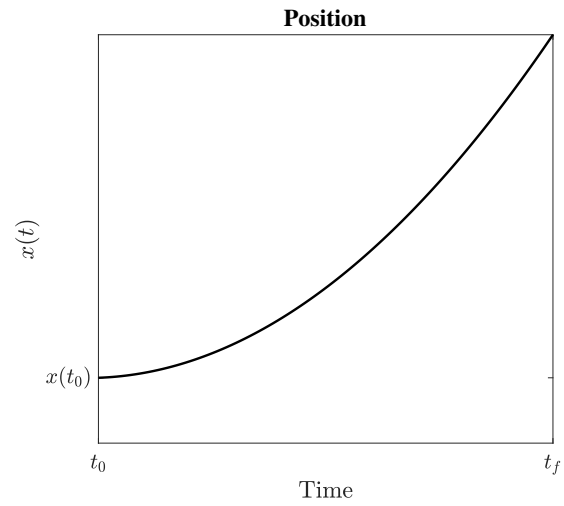
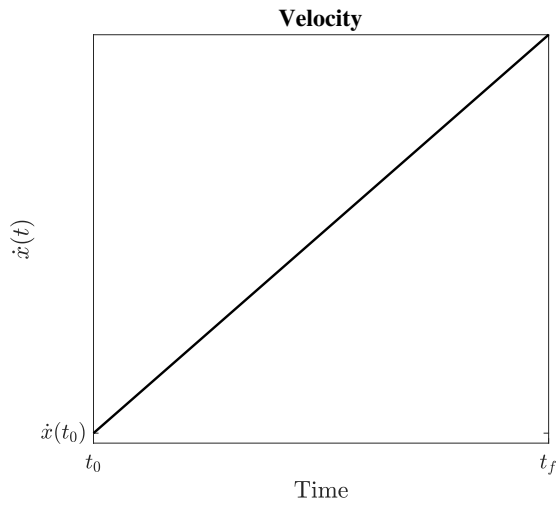
- Consider the 1-DOF system composed of a particle P constrained to move in a straight line
- The position of the particle can be described by the single coordinate x , and we can write: $\mathbf{r}_{P/O} = x\hat{\mathbf{e}}_1$
- The velocity and acceleration of the particle are given by:

$${}^{\mathcal{I}}\mathbf{v}_{P/O} = \frac{{}^{\mathcal{I}}d}{dt}\mathbf{r}_{P/O} = \dot{x}\hat{\mathbf{e}}_1 \quad \text{and} \quad {}^{\mathcal{I}}\mathbf{a}_{P/O} = \frac{{}^{\mathcal{I}}d^2}{dt^2}\mathbf{r}_{P/O} = \ddot{x}\hat{\mathbf{e}}_1$$

- The motion of this system takes place entirely in the $\hat{\mathbf{e}}_1$ direction *only*
- Because of this, for 1-DOF systems **only** it is okay to drop the vectors and move directly to scalar equations of motion
- If this particle experiences no forces in the $\hat{\mathbf{e}}_1$ direction, then its EOM is: $\ddot{x} = 0$
- NB: The particle *must* experience constraint forces in $\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in order constrain its motion to $\hat{\mathbf{e}}_1$, but we don't need to explicitly model these

Forced 1 DOF Systems

$$m_P \ddot{x} = F_x \quad \Rightarrow \quad \begin{cases} \dot{x}(t) &= \dot{x}(t_0) + \frac{F_x}{m_P} t \\ x(t) &= x(t_0) + \dot{x}(t_0) t + \frac{F_x}{m_P} \frac{t^2}{2} \end{cases}$$



Week 2

Dmitry Savransky

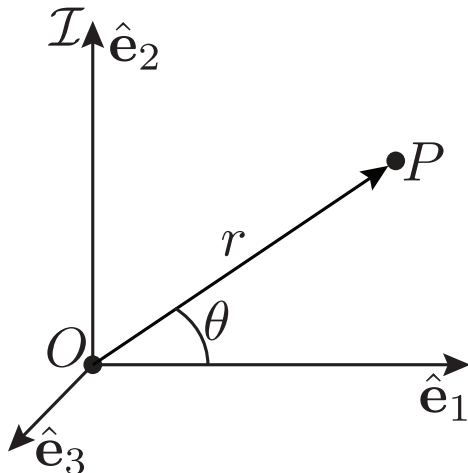
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Polar Coordinates

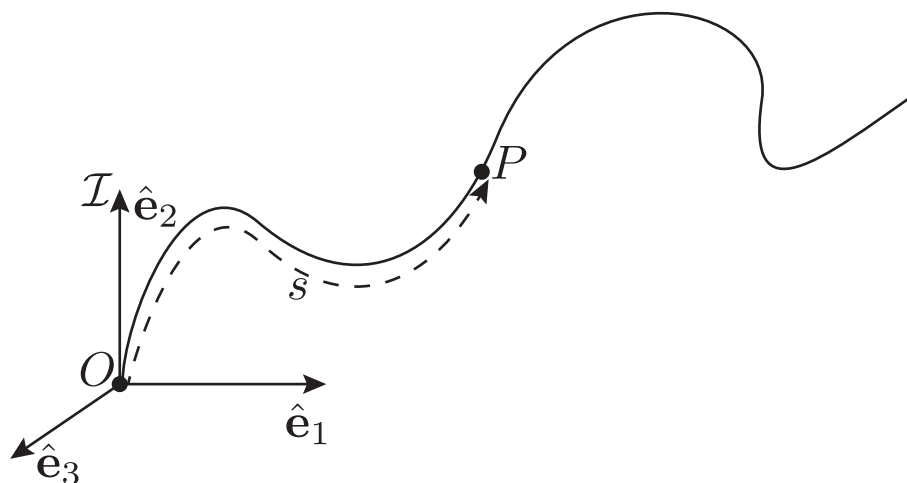
Recall: A single reference frame can have an infinite number of coordinate systems



NB: P is in the $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ plane

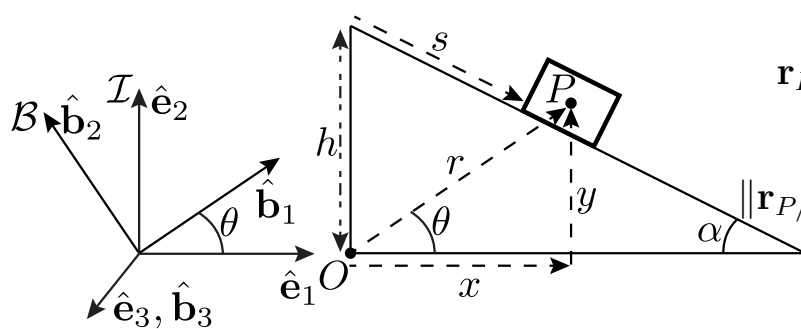
$$[\mathbf{r}_{P/O}]_{\mathcal{I}} = \begin{bmatrix} r \cos \theta \\ r \sin \theta \\ 0 \end{bmatrix}_{\mathcal{I}}$$

Path (Curvilinear) Coordinates



An exact knowledge of the path is required to use these coordinates

Recall the Box on a Ramp



$$\begin{aligned}\mathbf{r}_{P/O} &= x\hat{e}_1 + y\hat{e}_2 \\ &= r \cos \theta \hat{e}_1 + r \sin \theta \hat{e}_2 \\ \|\mathbf{r}_{P/O}\| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(r \cos \theta)^2 + (r \sin \theta)^2} = r \\ \theta &= \tan^{-1} \left(\frac{y}{x} \right)\end{aligned}$$

- x, y are Cartesian coordinates in $\mathcal{I}$
- r, θ are polar coordinates in $\mathcal{I}$
- r is also a Cartesian coordinate in $\mathcal{B}$
- s is a Path coordinate

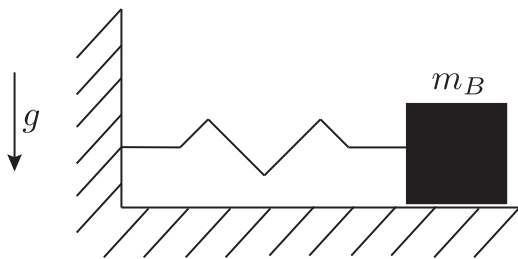
$$\begin{aligned}x &= s \cos(\alpha) \\ y &= h - s \sin(\alpha)\end{aligned}$$

$$r = \frac{h \cos(\alpha)}{\sin(\alpha + \theta)}$$

How to Solve a Dynamics Problem

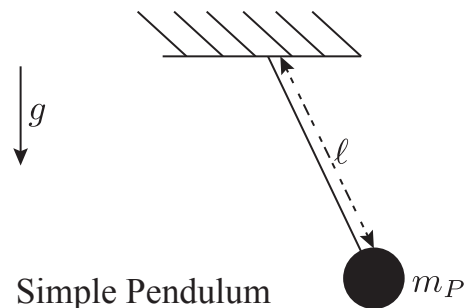
- 1 **Model the system** including all particles, rigid bodies, forces and assumptions
- 2 **Choose the coordinates**—at least one per DOF
- 3 **Define the reference frames**
- 4 **Draw the Free Body Diagrams**—one per particle/body
- 5 **Calculate the Kinematics**—position, velocity and acceleration (in an inertial frame) for each particle/body
- 6 **Write Newton's 2nd Law** for each particle/body
- 7 **Find the Equations of Motion**—unknowns are second derivatives of your coordinates and any constraint forces
- 8 **Integrate the Equations of Motion** to get the system trajectories

1. Model the system



Spring Mass on Frictionless Surface

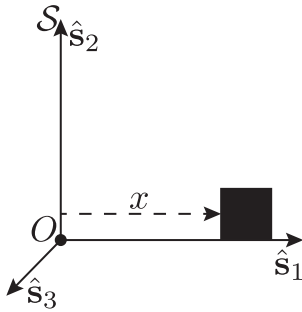
- Only external forces are due to gravity and the spring
- Massless, linear spring
- Constrained to move in 1D



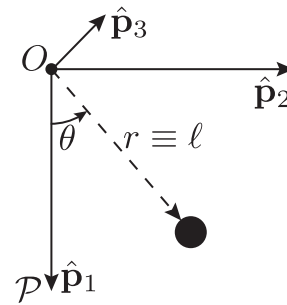
Simple Pendulum

- Only external forces are due to gravity
- Massless, rigid link
- Constrained to move in 2D

2-3. Define coordinates and reference frame(s)

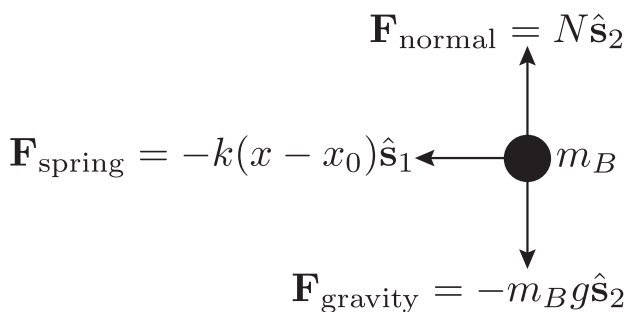


- 1-DOF so only 1 coordinate needed
- Cartesian coordinates (just x)
- $\mathcal{S} \triangleq (O, \hat{s}_1, \hat{s}_2, \hat{s}_3)$ is inertial

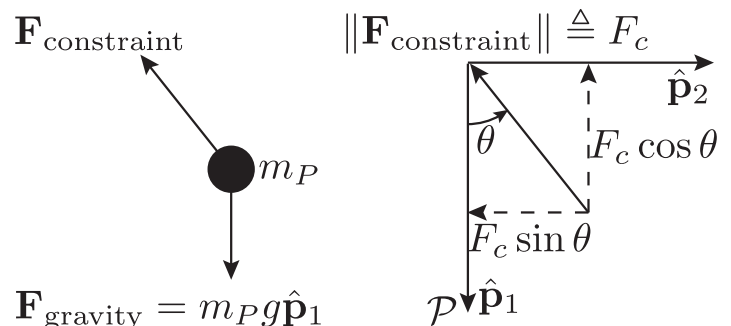


- 2D motion, but mass is always a distance ℓ from O , so 1-DOF
- Polar coordinates with $r \equiv \ell \forall t$
- $\mathcal{P} \triangleq (O, \hat{p}_1, \hat{p}_2, \hat{p}_3)$ is inertial

4. Draw the Free Body Diagrams



- Linear spring model = Hooke's Law
 - k = spring constant
 - x_0 = unstretched spring length (where the spring force equals zero)
- Normal force must be opposite the force of gravity (in the $\hat{s}_2$ direction)



- Constraint force required whenever modeling massless link

$$\mathbf{F}_{\text{constraint}} = -F_c \cos \theta \hat{p}_1 - F_c \sin \theta \hat{p}_2$$

5. Calculate the Kinematics

$\mathbf{r}_{B/O} = x\hat{\mathbf{s}}_1$ $\begin{aligned} {}^S\mathbf{v}_{B/O} &= \frac{{}^S d}{dt}\mathbf{r}_{B/O} = \frac{{}^S d}{dt}(x\hat{\mathbf{s}}_1) \\ &= \dot{x}\hat{\mathbf{s}}_1 \end{aligned}$ $\begin{aligned} {}^S\mathbf{a}_{B/O} &= \frac{{}^S d^2}{dt^2}\mathbf{r}_{B/O} = \frac{{}^S d^2}{dt^2}(x\hat{\mathbf{s}}_1) \\ &= \ddot{x}\hat{\mathbf{s}}_1 \end{aligned}$		$\mathbf{r}_{P/O} = \ell \cos \theta \hat{\mathbf{p}}_1 + \ell \sin \theta \hat{\mathbf{p}}_2$ ${}^{\mathcal{P}}\mathbf{v}_{P/O} = \frac{{}^{\mathcal{P}} d}{dt}\mathbf{r}_{P/O} = -\ell\dot{\theta} \sin(\theta)\hat{\mathbf{p}}_1 + \ell\dot{\theta} \cos(\theta)\hat{\mathbf{p}}_2$ $\begin{aligned} {}^{\mathcal{P}}\mathbf{a}_{P/O} &= \frac{{}^{\mathcal{P}} d^2}{dt^2}\mathbf{r}_{P/O} \\ &= \left(-\ell\ddot{\theta} \sin(\theta) - \ell\dot{\theta}^2 \cos(\theta)\right)\hat{\mathbf{p}}_1 \\ &\quad + \left(\ell\ddot{\theta} \cos(\theta) - \ell\dot{\theta}^2 \sin(\theta)\right)\hat{\mathbf{p}}_2 \end{aligned}$
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6-7. Write Newton's 2nd Law and find EOM

$\sum \mathbf{F}_B = m_B {}^S\mathbf{a}_{B/O} \implies$ $N\hat{\mathbf{s}}_2 - m_B g \hat{\mathbf{s}}_2 - k(x - x_0)\hat{\mathbf{s}}_1 = m_B \ddot{x}\hat{\mathbf{s}}_1$ $(1) \cdot \hat{\mathbf{s}}_1 \implies -k(x - x_0) = m_B \ddot{x}$ $(1) \cdot \hat{\mathbf{s}}_2 \implies -m_B g + N = 0$ $\ddot{x} = \frac{-k}{m_B}(x - x_0)$ $N = m_B g$	(1)	$\sum \mathbf{F}_P = m_P {}^{\mathcal{P}}\mathbf{a}_{P/O} \implies$ $-F_c \cos \theta \hat{\mathbf{p}}_1 + m_P g \hat{\mathbf{p}}_1 - F_c \sin \theta \hat{\mathbf{p}}_2 =$ $m_P \left(-\ell\ddot{\theta} \sin(\theta) - \ell\dot{\theta}^2 \cos(\theta)\right)\hat{\mathbf{p}}_1$ $+ m_P \left(\ell\ddot{\theta} \cos(\theta) - \ell\dot{\theta}^2 \sin(\theta)\right)\hat{\mathbf{p}}_2$ $\ddot{\theta} = -\frac{g}{\ell} \sin \theta$ $F_c = m_P \left(g \cos \theta + \ell\dot{\theta}^2\right)$
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Integrating the Spring-Mass Equations of Motion (1)

- We have found that the EOM for the spring-mass system is:

$$\ddot{x} = \frac{-k}{m}(x - x_0)$$

- $\int \ddot{x} \underbrace{dx}_{\equiv \dot{x} dt} = \int \frac{-k}{m}(x - x_0) dx \implies \frac{\dot{x}^2}{2} + c_1 = -\frac{k}{2m}(x - x_0)^2$

- @ $t = 0 \implies c_1 = -\frac{k}{2m}(x(0) - x_0)^2 - \frac{1}{2}(\dot{x}(0))^2$

- NB: $x_0 \neq x(t_0)$. x_0 = spring rest length. $x(0)$ = mass position at time $t = 0$

- $\dot{x} = \frac{dx}{dt} = \sqrt{-\frac{k}{m}(x - x_0)^2 - 2c_1} \implies \frac{dx}{\sqrt{-\frac{k}{m}(x - x_0)^2 - 2c_1}} = dt$

- Integrating both sides: $\int \frac{dx}{\sqrt{-\frac{k}{m}(x - x_0)^2 - 2c_1}} = t + c_2$

Simple Harmonic Motion

- To make the next step a bit easier, we introduce a change of variables:

$$z \triangleq x - x_0 \text{ and define a new constant } \omega_0 \triangleq \sqrt{\frac{k}{m}}$$

- The original EOM becomes: $\ddot{z} = -\omega_0^2 z$. This differential equation describes a **simple harmonic oscillator** (SHO) with a **natural frequency** of ω_0

- Assume a solution of the form $z(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)$

- $\dot{z} = -A\omega_0 \sin(\omega_0 t) + B\omega_0 \cos(\omega_0 t) \Rightarrow \ddot{z} = -A\omega_0^2 \cos(\omega_0 t) - B\omega_0^2 \sin(\omega_0 t) = -\omega_0^2 z$

- @ $t = 0 : z(0) = A \cdot 1 + B \cdot 0 \implies A = z(0)$

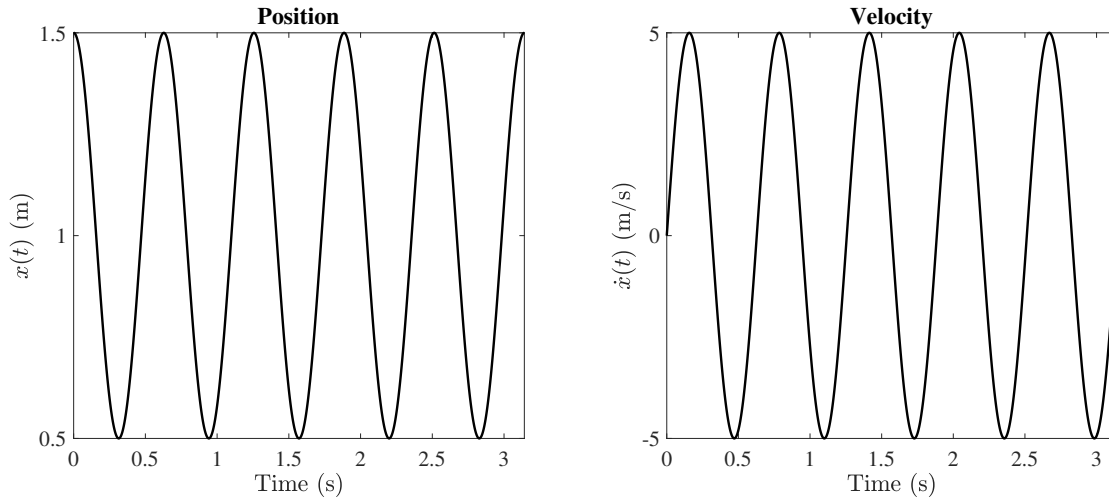
- @ $t = 0 : \dot{z}(0) = -A\omega_0 \cdot 0 + B\omega_0 \cdot 1 \implies B = \frac{\dot{z}(0)}{\omega_0}$

$$z(t) = z(0) \cos(\omega_0 t) + \frac{\dot{z}(0)}{\omega_0} \sin(\omega_0 t)$$

Integrating the Spring-Mass Equations of Motion (2)

- We can now write the position of the mass for all time as:

$$x(t) - x_0 = (x(0) - x_0) \cos \left(\sqrt{\frac{k}{m}} t \right) + \frac{\dot{x}(0)}{\sqrt{k/m}} \sin \left(\sqrt{\frac{k}{m}} t \right)$$



Simple Pendulum Small-Angle Approximation

Remember

The simple pendulum is **not** a simple harmonic oscillator (the EOMs don't match)

- But what if we assume θ is small? ($\sin(\theta) \approx \theta$)
- In the small-angle approximation, the pendulum's EOM becomes
$$\ddot{\theta} \approx -\frac{g}{\ell}\theta$$
- This is the equation of an SHO with natural frequency $\omega_0 = \sqrt{\frac{g}{\ell}}$
- But how can we tell how good this approximation is?

Numerical Integration for Initial Value Problems

- In general, numerical integrators are trying to solve an **Initial Value Problem** (IVP): For

$\dot{\mathbf{x}} = f(\mathbf{x}, t); \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n; f : \mathbb{R}^n \times [t_0, \infty) \rightarrow \mathbb{R}^n$, Find $\mathbf{x}(t); t \in [t_0, \pm\infty)$

- One of the simplest approaches is the Forward Euler Method:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t f(\mathbf{x}_k, t_k)$$

- The integrators we'll be using in this class are (mostly) based on the Runge-Kutta method:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \sum_{i=1}^s b_i g_i \begin{cases} g_1 &= f(\mathbf{x}_k, t_k) \\ g_2 &= f(\mathbf{x}_k + \Delta t(a_{21}g_1), t_k + c_2\Delta t) \\ g_3 &= f(\mathbf{x}_k + \Delta t(a_{31}g_1 + a_{32}g_2), t_k + c_3\Delta t) \\ &\dots \\ g_i &= f\left(\mathbf{x}_k + \Delta t \sum_{j=1}^s a_{ij}g_j, t_k + c_i\Delta t\right) \end{cases}$$

The values of the coefficients a_{ij} , b_i , c_i are determined by the order of the Runge-Kutta method.

Numerical Integrators in MATLAB

MATLAB Provides many different functions for solving systems of ordinary differential equations. A few of these are:

- ODE45 General-purpose, medium-order Runge-Kutta method. Good place to start for most problems.
- ODE23 General-purpose, lower-order Runge-Kutta method. Good for getting faster results with less precision.
- ODE78 and ODE89 High-order Runge-Kutta solvers. Mostly overkill for the kinds of problems we'll be solving, but useful when higher precision is needed.
- ODE113 Variable-order method (not Runge-Kutta), useful when high precision (low numerical error) is desired, and when the function f is expensive to compute.

For lots more detail, see:

www.mathworks.com/help/matlab/ordinary-differential-equations.html

Numerical Integration for 2nd order differential equations

- Our equations of motion typically have the form $\ddot{\mathbf{x}} = f(\mathbf{x}, \dot{\mathbf{x}}, t)$
- We can always turn these into first-order equations by defining a new state:

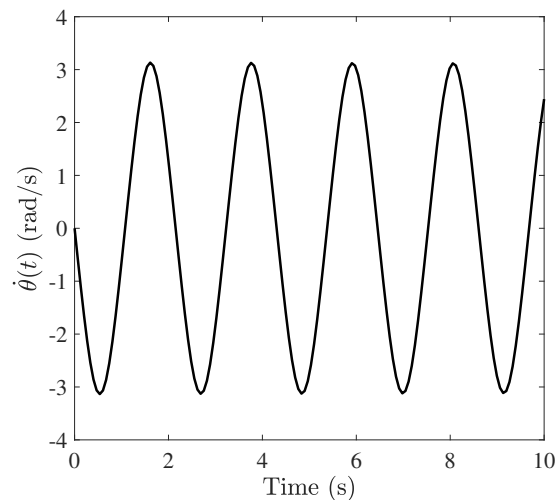
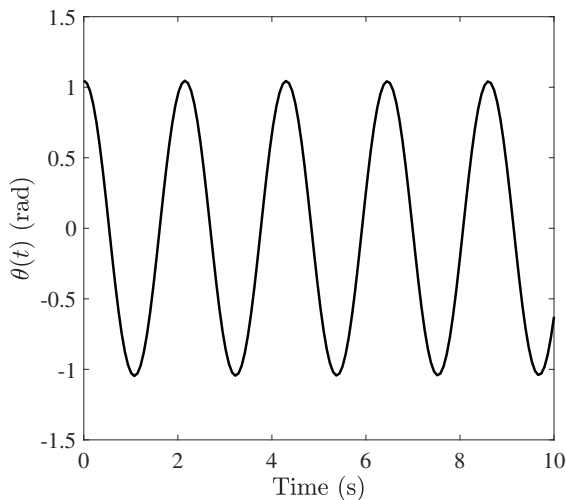
$$\mathbf{z} \triangleq \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} \implies \dot{\mathbf{z}} = \begin{bmatrix} \mathbf{z}_2 \\ f(\mathbf{z}_1, \mathbf{z}_2, t) \end{bmatrix}$$

Remember

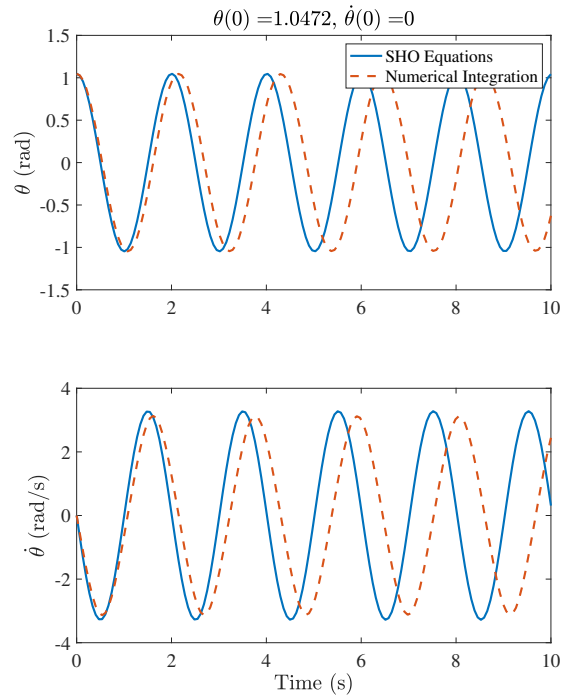
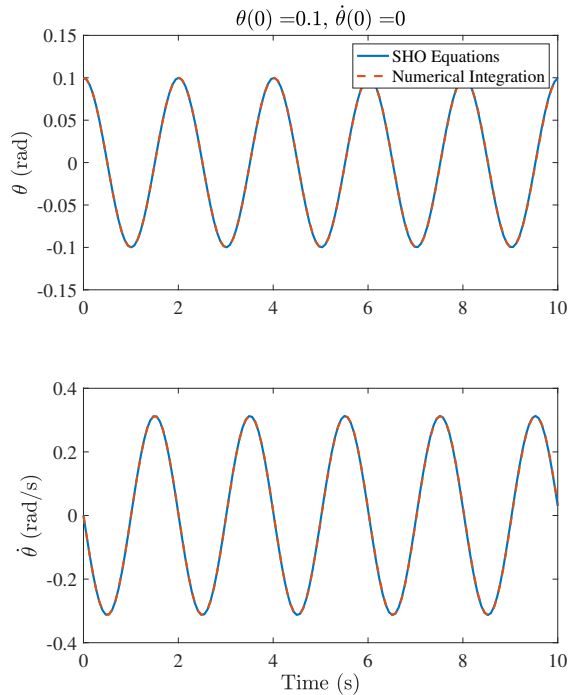
Numerical integration is *always* an approximation in some way (or in many different ways all at once). **Never** blindly trust the results.

Numerical Integration of the Simple Pendulum EOM

$$\ddot{\theta} = -\frac{g}{\ell} \sin \theta \implies \mathbf{z} \triangleq \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \equiv \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \implies \dot{\mathbf{z}} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell} \sin \theta \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{g}{\ell} \sin(z_1) \end{bmatrix}$$



Testing the Small-Angle Approximation



Springs and Dampers

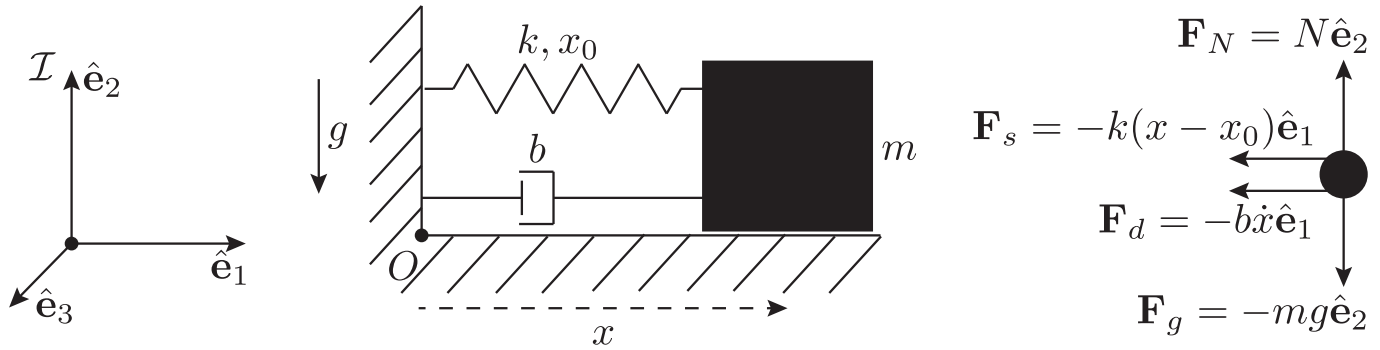


- A **linear spring** is described by spring constant k (mass $\times$ time $^{-2}$) and unstretched length x_0
- Force acts opposite direction stretching (spring wants to return to its unstretched length) with magnitude: $\|\mathbf{F}_{\text{spring}}\| = |k(x - x_0)|$ for stretched length x



- A **viscous damper** (dashpot) is described by damping constant b (force $\times$ time $\times$ length $^{-1}$)
- Force acts in direction opposite of velocity vector of damped body (damper wants to stop motion) with magnitude: $\|\mathbf{F}_{\text{damper}}\| = |b\dot{x}|$ for speed $\dot{x}$ of the damper arm with respect to damper body

Damped Harmonic Motion



- We now add a damper to our spring-mass system
- The coordinates, reference frame, and kinematics remain the same
- Our new equation of motion is:

$$m\ddot{x} = -k(x - x_0) - b\dot{x}$$

Damped Harmonic Motion Solution

- We again we introduce a change of variables: $z \triangleq x - x_0$, define the **natural frequency** $\omega_0 \triangleq \sqrt{\frac{k}{m}}$ and define the **damping factor (ratio)**

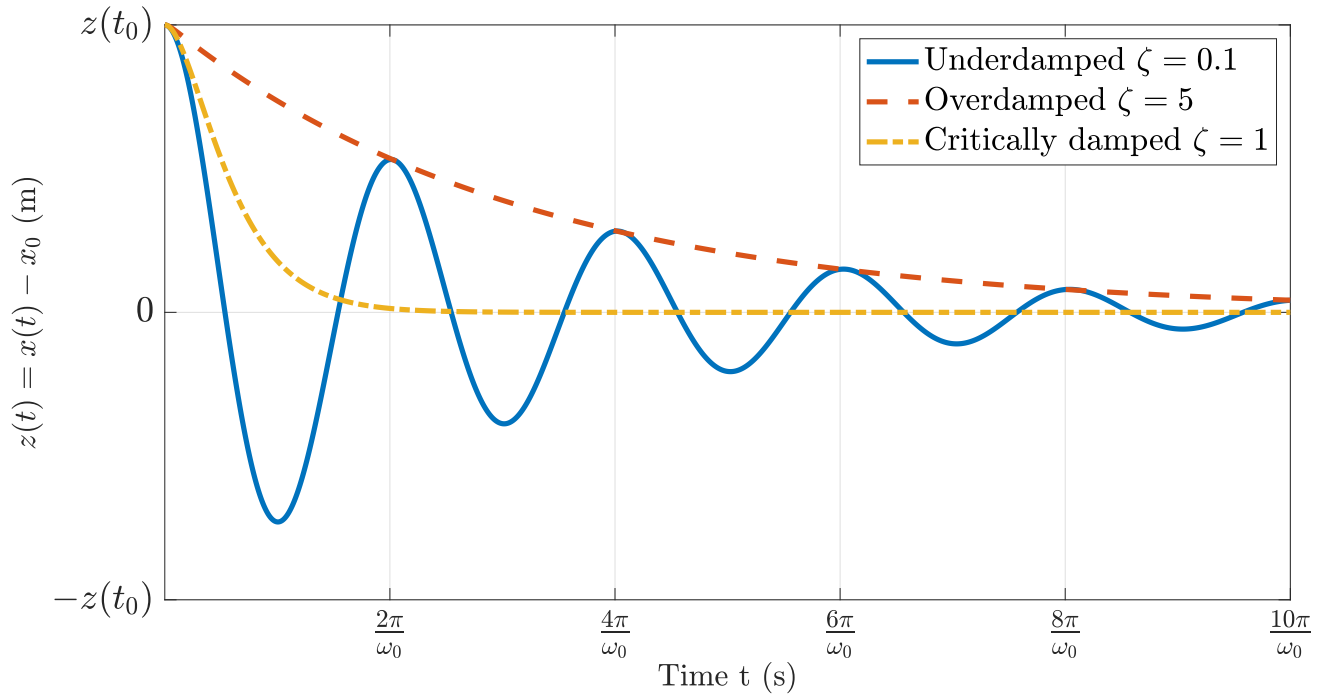
$$\zeta \triangleq \frac{b}{2m\omega_0}$$

- The original EOM becomes: $\ddot{z} + 2\zeta\omega_0\dot{z} + \omega_0^2 z = 0$
- The general solution for this equation is:

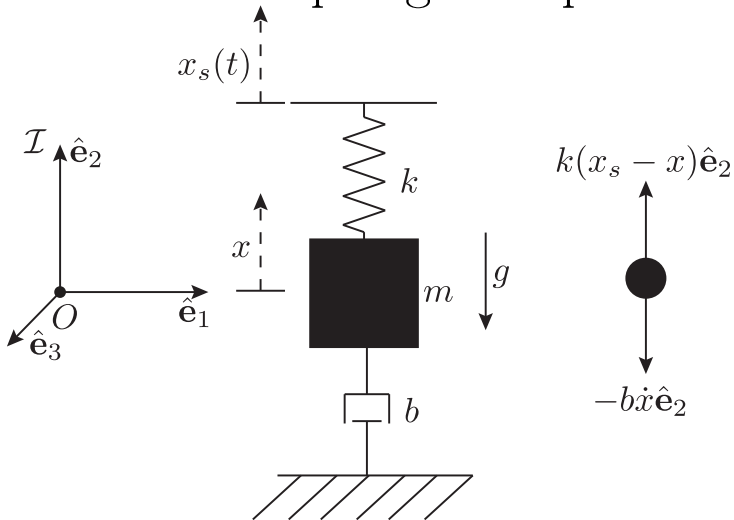
$$z(t) = e^{-\zeta\omega_0 t} \left(A e^{i\omega_0 t \sqrt{1-\zeta^2}} + B e^{-i\omega_0 t \sqrt{1-\zeta^2}} \right)$$

- This system can behave in one of three ways:
 - $\zeta < 1$ Underdamped: position oscillates around spring rest length
 - $\zeta > 1$ Overdamped: position gradually decays to rest length
 - $\zeta = 1$ Critically damped: position decays directly to spring rest length

Damped Harmonic Oscillation



Driven Mass-Spring-Damper



- The spring is attached to a platform that is driven with trajectory $x_s(t)$
- System has 1 DOF as $x_s(t)$ is known
- Mass position x is measured with respect to system equilibrium
- This allows FBD to lump together spring rest length and gravity, so gravity does not need to be treated separately

Newton's 2nd Law: $m\ddot{x}\hat{e}_2 = (-b\dot{x} + kx_s - kx)\hat{e}_2$

$\implies m\ddot{x} + b\dot{x} + kx = kx_s(t) \triangleq F_s(t)$ Forcing Function

Forced Damped SHO General Solution

- The general solution is the sum of the homogeneous (transient) and particular solutions: $x(t) = x_c(t) + x_p(t)$
- We have already found the homogeneous solution by studying the unforced case:

$$x_c(t) = e^{-\zeta\omega_0 t} [C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)] = A e^{-\zeta\omega_0 t} \cos(\omega_d t - \phi)$$

- $\omega_d = \omega_0 \sqrt{1 - \zeta^2}$ is the **damped frequency**, C_1, C_2, A are set by initial conditions with $A = \sqrt{C_1^2 + C_2^2}$, and $\phi = \tan^{-1} \left(\frac{C_2}{C_1} \right)$ is the **phase**
- As before, $\omega_0 \triangleq \sqrt{\frac{k}{m}}$ is the **natural frequency** and $\zeta \triangleq \frac{b}{2m\omega_0}$ is the **damping factor**

Forced Damped SHO Particular Solution

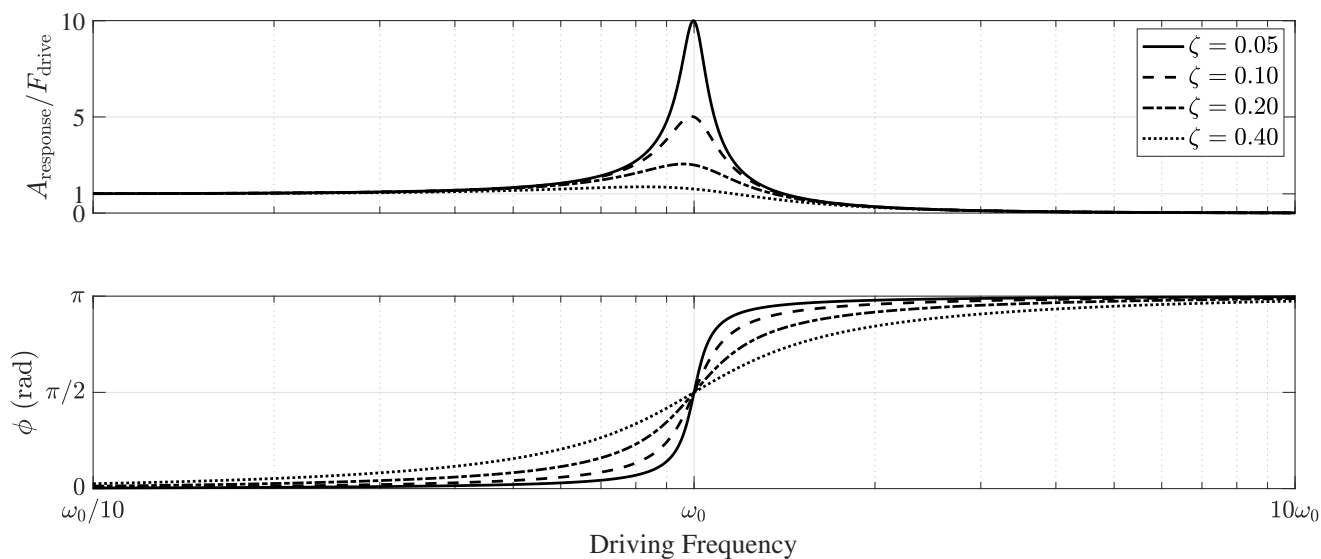
- When forcing function is sinusoidal with frequency ω , $x_p(t)$ will also be sinusoidal with the same frequency and phase lag ϕ :

$$F_s(t) \triangleq kx_s(t) = \underbrace{F_{\text{drive}}}_{\text{Forcing Amplitude}} \cos(\omega t) \implies x_p(t) = \underbrace{A_{\text{response}}}_{\text{Response Amplitude}} \cos(\omega t - \phi)$$

- We can find expressions for A_{response} and ϕ by plugging the particular solution back into the equation of motion and solving:

$$A = \frac{F_{\text{drive}}}{\sqrt{m^2 (\omega_0^2 - \omega^2)^2 + (b\omega)^2}} \qquad \phi = \tan^{-1} \left(\frac{b\omega}{k - m\omega^2} \right)$$

Forced Damped SHO Particular Solution



Resonance

- The less damping, the greater the change in phase, and the larger the response when the system is driven near its natural frequency
- **Resonance** is a large-amplitude response caused by a (relatively) small periodic forcing near the system's natural frequency
- We can find the resonant frequency by looking for the maxima of A_{response} :

$$\left. \frac{dA_{\text{response}}}{d\omega} \right|_{\omega=\omega_r} = 0 \quad \implies \quad \omega_r = \omega_0 \sqrt{1 - 2\zeta^2}$$

- For small damping factors: $\zeta \ll 1 \quad \implies \quad \omega_r \approx \omega_0$

Week 3

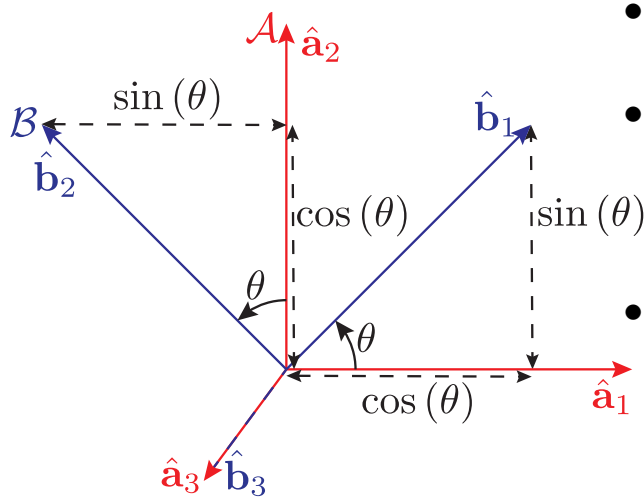
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MAE 2030, Spring 2026

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Coordinate Transformations



- $\mathcal{A}$ and $\mathcal{B}$ are related by a **simple rotation**
- One axis remains the same for both frames
 - This does **not** have to be one of the frame's defining unit vectors
- In *this* case:

$$\hat{\mathbf{b}}_1 = \cos(\theta)\hat{\mathbf{a}}_1 + \sin(\theta)\hat{\mathbf{a}}_2$$

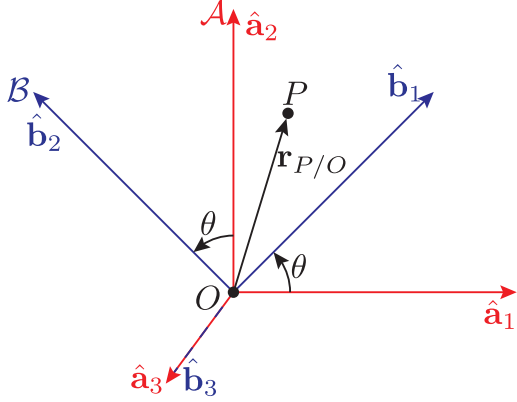
$$\hat{\mathbf{b}}_2 = -\sin(\theta)\hat{\mathbf{a}}_1 + \cos(\theta)\hat{\mathbf{a}}_2$$

- This is a **transformation table**:

$$\mathcal{A} \triangleq (\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3) \quad \mathcal{B} \triangleq (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$$
$$\hat{\mathbf{b}}_3 \equiv \hat{\mathbf{a}}_3$$

	$\hat{\mathbf{a}}_1$	$\hat{\mathbf{a}}_2$	$\hat{\mathbf{a}}_3$
$\hat{\mathbf{b}}_1$	$\cos \theta$	$\sin \theta$	0
$\hat{\mathbf{b}}_2$	$-\sin \theta$	$\cos \theta$	0
$\hat{\mathbf{b}}_3$	0	0	1

Coordinate Transformations Continued

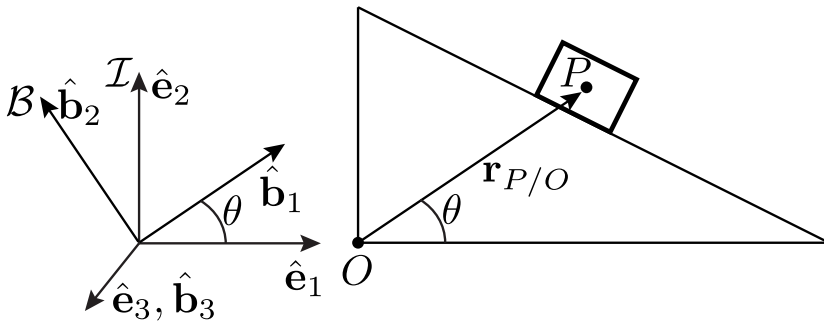


$$\begin{aligned}\mathbf{r}_{P/O} &\triangleq a_1 \hat{\mathbf{a}}_1 + a_2 \hat{\mathbf{a}}_2 + a_3 \hat{\mathbf{a}}_3 \\ &\triangleq b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3\end{aligned}$$

	$\hat{\mathbf{a}}_1$	$\hat{\mathbf{a}}_2$	$\hat{\mathbf{a}}_3$
$\hat{\mathbf{b}}_1$	$\cos \theta$	$\sin \theta$	0
$\hat{\mathbf{b}}_2$	$-\sin \theta$	$\cos \theta$	0
$\hat{\mathbf{b}}_3$	0	0	1

$$\begin{aligned}\mathbf{r}_{P/O} &= a_1 \underbrace{(\cos(\theta) \hat{\mathbf{b}}_1 - \sin(\theta) \hat{\mathbf{b}}_2)}_{\equiv \hat{\mathbf{a}}_1} + a_2 \underbrace{(\sin(\theta) \hat{\mathbf{b}}_1 + \cos(\theta) \hat{\mathbf{b}}_2)}_{\equiv \hat{\mathbf{a}}_2} + a_3 \underbrace{\hat{\mathbf{b}}_3}_{\equiv \hat{\mathbf{a}}_3} \\ &= \underbrace{(a_1 \cos(\theta) + a_2 \sin(\theta))}_{\equiv b_1} \hat{\mathbf{b}}_1 + \underbrace{(-a_1 \sin(\theta) + a_2 \cos(\theta))}_{\equiv b_2} \hat{\mathbf{b}}_2 + \underbrace{a_3}_{\equiv b_3} \hat{\mathbf{b}}_3\end{aligned}$$

Recall the Box on a Ramp



$$\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$$

$$\mathcal{B} \triangleq (O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$$

$$\hat{\mathbf{b}}_1 \equiv \hat{\mathbf{r}}_{P/O}$$

$$\frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O} \neq \frac{{}^{\mathcal{B}}d}{dt} \mathbf{r}_{P/O}$$

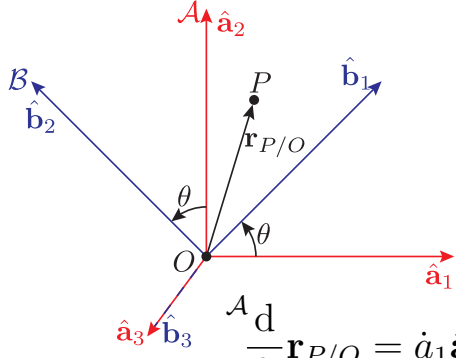
$$\mathbf{r}_{P/O} = x \hat{\mathbf{e}}_1 + y \hat{\mathbf{e}}_2 = r \hat{\mathbf{b}}_1$$

$$\frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O} = \frac{{}^{\mathcal{I}}d}{dt} (x \hat{\mathbf{e}}_1 + y \hat{\mathbf{e}}_2) = \dot{x} \hat{\mathbf{e}}_1 + \dot{y} \hat{\mathbf{e}}_2$$

$$\frac{{}^{\mathcal{B}}d}{dt} \mathbf{r}_{P/O} = \frac{{}^{\mathcal{B}}d}{dt} (r \hat{\mathbf{b}}_1) = \dot{r} \hat{\mathbf{b}}_1$$

$$\begin{aligned}\frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O} &= \frac{{}^{\mathcal{I}}d}{dt} (r \hat{\mathbf{b}}_1) \\ &= \dot{r} \hat{\mathbf{b}}_1 + r \frac{{}^{\mathcal{I}}d}{dt} (\hat{\mathbf{b}}_1)\end{aligned}$$

Vector Derivatives Across Frames (Take 1)



$$\mathbf{r}_{P/O} \triangleq a_1 \hat{\mathbf{a}}_1 + a_2 \hat{\mathbf{a}}_2 + a_3 \hat{\mathbf{a}}_3$$

$$\triangleq b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3$$

	$\hat{\mathbf{a}}_1$	$\hat{\mathbf{a}}_2$	$\hat{\mathbf{a}}_3$
$\hat{\mathbf{b}}_1$	$\cos \theta$	$\sin \theta$	0
$\hat{\mathbf{b}}_2$	$-\sin \theta$	$\cos \theta$	0
$\hat{\mathbf{b}}_3$	0	0	1

$${}^A \frac{d}{dt} \mathbf{r}_{P/O} = \dot{a}_1 \hat{\mathbf{a}}_1 + \dot{a}_2 \hat{\mathbf{a}}_2 + \dot{a}_3 \hat{\mathbf{a}}_3 = \frac{d}{dt} (b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3)$$

$$= \dot{b}_1 \hat{\mathbf{b}}_1 + b_1 \underbrace{{}^A \frac{d}{dt} \hat{\mathbf{b}}_1}_{\text{}} + \dot{b}_2 \hat{\mathbf{b}}_2 + b_2 \underbrace{{}^A \frac{d}{dt} \hat{\mathbf{b}}_2}_{\text{}} + \dot{b}_3 \hat{\mathbf{b}}_3 + b_3 \underbrace{{}^A \frac{d}{dt} \hat{\mathbf{b}}_3}_{\text{}}$$

$$\equiv \frac{d}{dt} (\cos(\theta) \hat{\mathbf{a}}_1 + \sin(\theta) \hat{\mathbf{a}}_2)$$

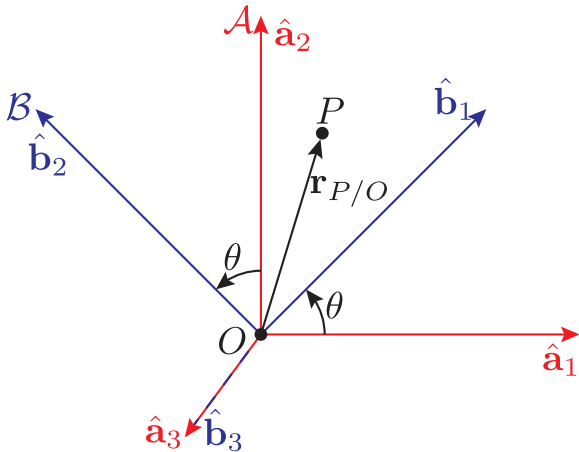
$${}^A \frac{d}{dt} \hat{\mathbf{b}}_1 \equiv \frac{d}{dt} (\cos(\theta) \hat{\mathbf{a}}_1 + \sin(\theta) \hat{\mathbf{a}}_2) = -\dot{\theta} \sin(\theta) \underbrace{\hat{\mathbf{a}}_1}_{\text{}} + \dot{\theta} \cos(\theta) \underbrace{\hat{\mathbf{a}}_2}_{\text{}}$$

$$\left(\cos(\theta) \hat{\mathbf{b}}_1 - \sin(\theta) \hat{\mathbf{b}}_2 \right) \quad \left(\sin(\theta) \hat{\mathbf{b}}_1 + \cos(\theta) \hat{\mathbf{b}}_2 \right)$$

$$= \dot{\theta} (-\sin(\theta) \cos \theta + \sin(\theta) \cos \theta) \hat{\mathbf{b}}_1 + \dot{\theta} (\sin^2(\theta) + \cos^2(\theta)) \hat{\mathbf{b}}_2$$

$$= \dot{\theta} \hat{\mathbf{b}}_2$$

Derivatives of Rotating Planar Unit Vectors



$${}^A \frac{d}{dt} \hat{\mathbf{b}}_1 = \dot{\theta} \hat{\mathbf{b}}_2$$

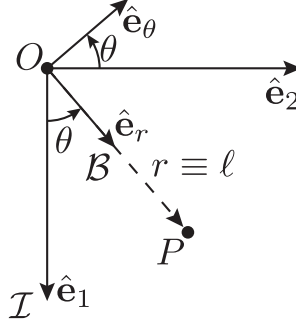
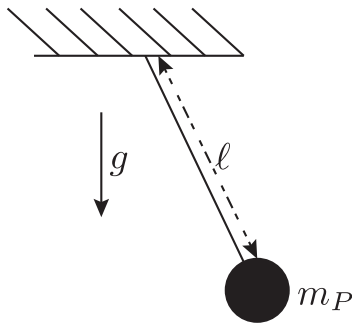
$${}^A \frac{d}{dt} \hat{\mathbf{b}}_2 = -\dot{\theta} \hat{\mathbf{b}}_1$$

$${}^A \frac{d}{dt} \hat{\mathbf{b}}_3 = 0$$

	$\hat{\mathbf{a}}_1$	$\hat{\mathbf{a}}_2$	$\hat{\mathbf{a}}_3$
$\hat{\mathbf{b}}_1$	$\cos \theta$	$\sin \theta$	0
$\hat{\mathbf{b}}_2$	$-\sin \theta$	$\cos \theta$	0
$\hat{\mathbf{b}}_3$	0	0	1

- This is a **general** result for *counter-clockwise* rotations about $\hat{\mathbf{a}}_3$
- For clockwise rotations, the signs on the expressions flip

Back to the Simple Pendulum



$$\mathbf{F}_{\text{constraint}} = -F_c \hat{\mathbf{e}}_r$$

$$\mathbf{F}_{\text{gravity}} = m_P g (\cos(\theta) \hat{\mathbf{e}}_r - \sin \theta \hat{\mathbf{e}}_\theta)$$

- $\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ is inertial ($\hat{\mathbf{e}}_3$ out of page)
- $\mathcal{B} \triangleq (O, \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$ is a non-inertial rotating frame
- $\hat{\mathbf{e}}_r \equiv \hat{\mathbf{r}}_{P/O}$
- $r \equiv \ell$ is a polar coordinate in $\mathcal{I}$ and Cartesian in $\mathcal{B}$

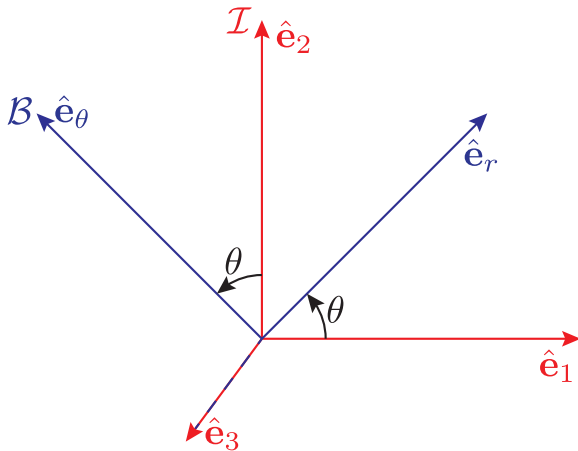
$$\mathbf{r}_{P/O} = \ell \hat{\mathbf{e}}_r$$

$${}^{\mathcal{I}}\mathbf{v}_{P/O} = \ell \dot{\theta} \hat{\mathbf{e}}_\theta$$

$${}^{\mathcal{I}}\mathbf{a}_{P/O} = \ell \ddot{\theta} \hat{\mathbf{e}}_\theta - \ell \dot{\theta}^2 \hat{\mathbf{e}}_r$$

$$\ddot{\theta} = -\frac{g}{\ell} \sin \theta$$

Derivatives of Rotating Planar Unit Vectors



	$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
$\hat{\mathbf{e}}_r$	$\cos \theta$	$\sin \theta$	0
$\hat{\mathbf{e}}_\theta$	$-\sin \theta$	$\cos \theta$	0
$\hat{\mathbf{e}}_3$	0	0	1

$${}^{\mathcal{I}}\frac{d}{dt}\hat{\mathbf{e}}_r = \dot{\theta}\hat{\mathbf{e}}_\theta$$

$${}^{\mathcal{I}}\frac{d}{dt}\hat{\mathbf{e}}_\theta = -\dot{\theta}\hat{\mathbf{e}}_r$$

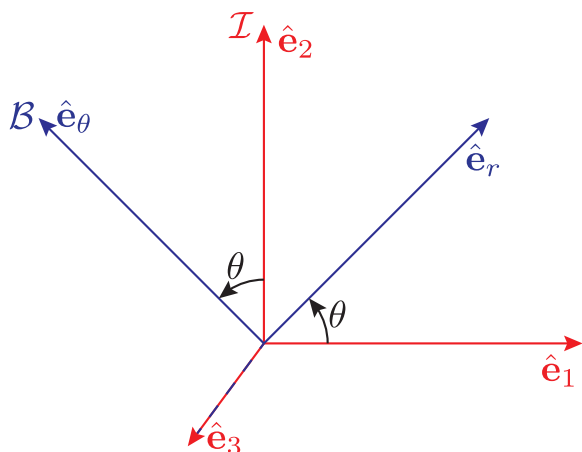
$${}^{\mathcal{I}}\frac{d}{dt}\hat{\mathbf{e}}_3 = 0$$

- $\dot{\theta}$ is the **rotation rate** of frame $\mathcal{B}$ in frame $\mathcal{I}$, about **rotation axis** $\hat{\mathbf{e}}_3$
- $\dot{\theta}$ is taken to be positive for counter-clockwise rotations and negative for clockwise rotations

Simple Angular Velocity

The **angular velocity vector**

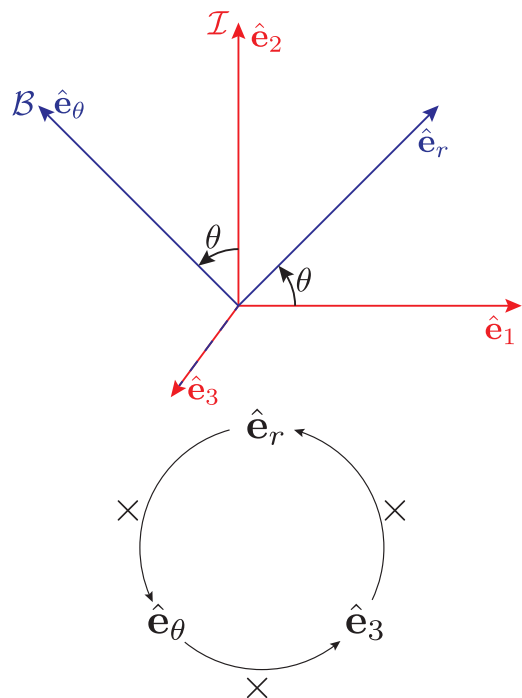
has a magnitude equal to the rotation rate and direction equal to the rotation axis



$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \triangleq \dot{\theta} \hat{\mathbf{e}}_3$$

- Rotation rate is positive for counter-clockwise rotations and negative for clockwise rotations
- Clockwise vs. counter-clockwise is determined by looking *down* the axis of rotation
- ${}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{I}} = -{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$

Utility of the Angular Velocity Vector



$$\begin{aligned} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} &\triangleq \dot{\theta} \hat{\mathbf{e}}_3 \\ {}^{\mathcal{I}} \frac{d}{dt} \hat{\mathbf{e}}_r &= {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{e}}_r = \dot{\theta} \hat{\mathbf{e}}_{\theta} \\ {}^{\mathcal{I}} \frac{d}{dt} \hat{\mathbf{e}}_{\theta} &= {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{e}}_{\theta} = -\dot{\theta} \hat{\mathbf{e}}_r \end{aligned}$$

This is a Completely General Result!

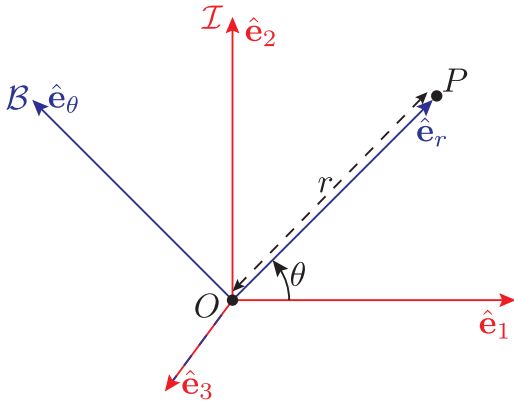
For *any* pair of frames $\mathcal{A}$ and $\mathcal{B}$, with angular velocity vector ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}$: the time derivative in frame $\mathcal{A}$ of any unit vector $\hat{\mathbf{b}}_i$ that is part of the $\mathcal{B}$ frame definition will be:

$${}^{\mathcal{A}}\frac{d}{dt}\hat{\mathbf{b}}_i = {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_i$$

- Corollary: if two frames are not rotating with respect to one another (have zero angular velocity), then all of their unit vectors will have zero time derivatives in both frames
- NB: This is *not* physics—this is a mathematical statement only. It does not depend on any physical model, and is always 100% true

Acceleration in a Rotating Frame

- Point P is constrained to the $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ plane of inertial frame $\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$



- Define rotating frame $\mathcal{B}$ to track point P , such that $\hat{\mathbf{e}}_r \equiv \hat{\mathbf{r}}_{P/O}$ and ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\hat{\mathbf{e}}_3$

$$\begin{aligned} \mathbf{r}_{P/O} &= r\hat{\mathbf{e}}_r \\ {}^{\mathcal{I}}\mathbf{v}_{P/O} &= \frac{{}^{\mathcal{I}}d}{dt}(r\hat{\mathbf{e}}_r) = \dot{r}\hat{\mathbf{e}}_r + r\underbrace{\frac{{}^{\mathcal{I}}d}{dt}\hat{\mathbf{e}}_r}_{{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{e}}_r} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta \end{aligned}$$

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{a}_{P/O} &= \ddot{r}\hat{\mathbf{e}}_r + \underbrace{\dot{r}\frac{{}^{\mathcal{I}}d}{dt}\hat{\mathbf{e}}_r}_{{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{e}}_r} + \dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta + r\ddot{\theta}\hat{\mathbf{e}}_\theta + r\dot{\theta}\underbrace{\frac{{}^{\mathcal{I}}d}{dt}\hat{\mathbf{e}}_\theta}_{{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{e}}_\theta} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta \\ {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{e}}_r &= \dot{\theta}\hat{\mathbf{e}}_\theta \quad {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{e}}_\theta = -\dot{\theta}\hat{\mathbf{e}}_r \end{aligned}$$

Fictitious Forces

- Consider what happens if we apply Newton's second law to the rotating frame acceleration assuming zero net forces on P :

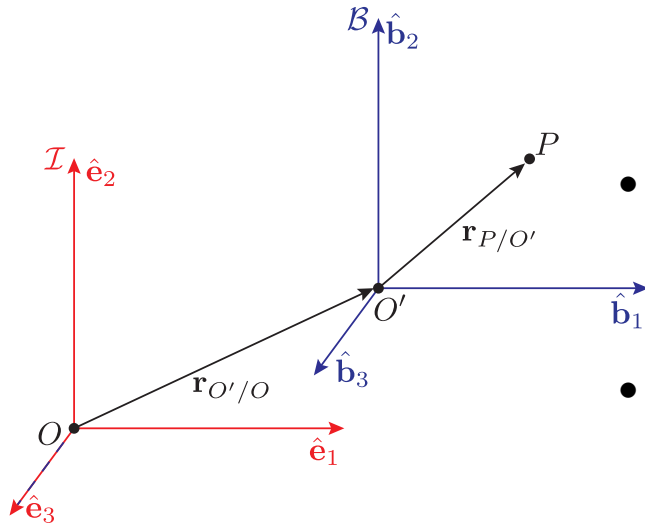
$$\mathbf{F}_p = \mathbf{0} = m_P \left[\left(\ddot{r} - r\dot{\theta}^2 \right) \hat{\mathbf{e}}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) \hat{\mathbf{e}}_\theta \right] \Rightarrow \begin{cases} \ddot{r} = r\dot{\theta}^2 & \text{Centripetal} \\ \ddot{\theta} = -2\frac{\dot{r}\dot{\theta}}{r} & \text{Coriolis} \end{cases}$$

- These terms *look* like forces, but are absolutely **not**
- These terms arise because we are doing dynamics using a non-inertial reference frame—they are purely kinematics

Nomenclature

- A mass moving on a circular path experiences a centripetal acceleration in the direction of the center of the circle ($-\hat{\mathbf{e}}_r$ direction) with a magnitude of $r\dot{\theta}^2$
- Centripetal means “center seeking” (towards the rotation axis)
 - Force that keeps an object on a circular trajectory, corresponds to acceleration arising naturally from taking inertial derivatives of rotating frame unit vectors
- Centrifugal means “center fleeing” (away from the rotation axis)
 - Force that appears to affect objects traveling in straight lines in an inertial frame, when measured in a rotating frame (e.g. assuming that the rotating Earth represents an inertial reference frame)
 - Reaction force arising from Newton's 3rd law (due to a centripetal force)
- A mass moving on a circular path with a decreasing radius experiences an additional Coriolis acceleration in the direction tangential to the path (in the $\hat{\mathbf{e}}_\theta$ direction) with a magnitude of $2\dot{r}\dot{\theta}$

Translating Frames



$$\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \quad \mathcal{B} \triangleq (O', \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$$

$$\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} = \mathbf{0}$$

$$\mathbf{r}_{P/O} = \mathbf{r}_{P/O'} + \mathbf{r}_{O'/O}$$

$$\mathcal{I}\mathbf{v}_{P/O} = \mathcal{I}\mathbf{v}_{P/O'} + \mathcal{I}\mathbf{v}_{O'/O}$$

$$\mathcal{I}\mathbf{a}_{P/O} = \mathcal{I}\mathbf{a}_{P/O'} + \mathcal{I}\mathbf{a}_{O'/O}$$

- If $\mathcal{I}\mathbf{v}_{O'/O}$ is constant (or zero):
 - Frame $\mathcal{B}$ moves away from frame $\mathcal{I}$ at constant speed (or is stationary)
 - $\mathcal{I}\mathbf{a}_{O'/O} = 0 \implies \mathcal{I}\mathbf{a}_{P/O} \equiv \mathcal{I}\mathbf{a}_{P/O'}$
- Define $\mathbf{r}_{P/O'} \triangleq b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3$
 - ${}^{\mathcal{B}}\mathbf{v}_{P/O'} = \dot{b}_1 \hat{\mathbf{b}}_1 + \dot{b}_2 \hat{\mathbf{b}}_2 + \dot{b}_3 \hat{\mathbf{b}}_3$
 - $\mathcal{I}\mathbf{v}_{P/O'} = \dot{b}_1 \hat{\mathbf{b}}_1 + \dot{b}_2 \hat{\mathbf{b}}_2 + \dot{b}_3 \hat{\mathbf{b}}_3 + \underbrace{\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}}_{\equiv 0}$
 - $\mathcal{I}\mathbf{v}_{P/O'} \equiv {}^{\mathcal{B}}\mathbf{v}_{P/O'}$ and $\mathcal{I}\mathbf{a}_{P/O'} \equiv {}^{\mathcal{B}}\mathbf{a}_{P/O'}$
 - $\mathcal{I}\mathbf{a}_{P/O} = {}^{\mathcal{B}}\mathbf{a}_{P/O'} + \mathcal{I}\mathbf{a}_{O'/O}$

Translating Frames

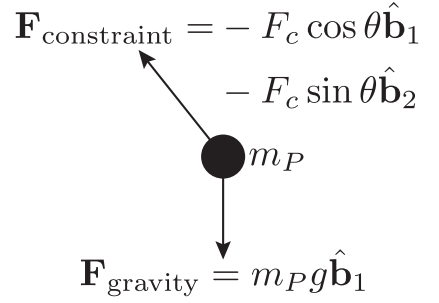
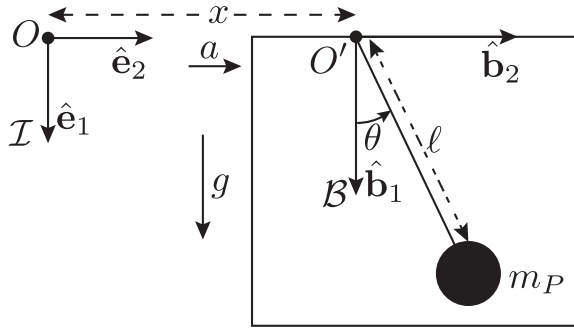
- Any non-rotating, non-accelerating frame (moving at zero or constant velocity with respect to an inertial frame) is itself inertial
- If a frame $\mathcal{B}$ is non-rotating but linearly accelerating with respect to an inertial frame $\mathcal{I}$, Newton's second law becomes:

$$\sum \mathbf{F}_P = m_P ({}^{\mathcal{B}}\mathbf{a}_{P/O'} + \mathcal{I}\mathbf{a}_{O'/O})$$

- If we wish to express our accelerations with respect to frame $\mathcal{B}$, in this case we need to correct for the inertial acceleration of the frame:

$$m_P {}^{\mathcal{B}}\mathbf{a}_{P/O'} = \sum \mathbf{F}_P - m_P \mathcal{I}\mathbf{a}_{O'/O}$$

Pendulum in an Accelerating Box

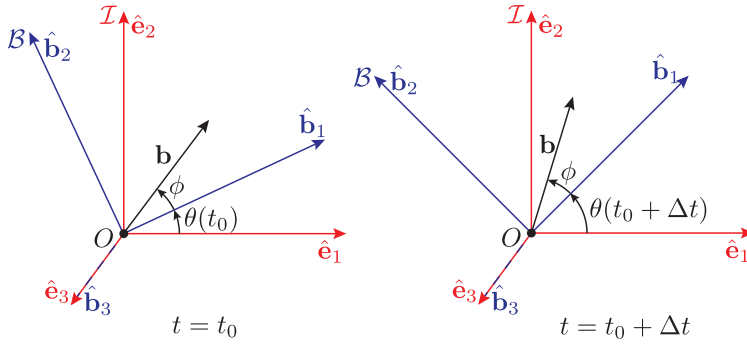


- $\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ is inertial ($\hat{\mathbf{e}}_3$ out of page)
- $\mathcal{B} \triangleq (O', \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$ is a non-inertial accelerating frame (${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \mathbf{0}$)
- $r \equiv \ell, \theta$ are polar coordinates in $\mathcal{B}$
 x is a Cartesian coordinate in $\mathcal{I}$

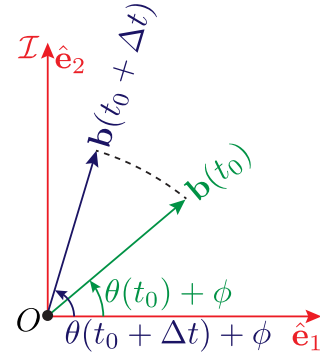
$$\begin{aligned}\mathbf{r}_{P/O'} &= \ell \cos(\theta) \hat{\mathbf{b}}_1 + \ell \sin(\theta) \hat{\mathbf{b}}_2 \\ {}^{\mathcal{B}}\mathbf{v}_{P/O'} &= -\ell \dot{\theta} \sin(\theta) \hat{\mathbf{b}}_1 + \ell \dot{\theta} \cos(\theta) \hat{\mathbf{b}}_2 \\ {}^{\mathcal{B}}\mathbf{a}_{P/O'} &= \left(-\ell \ddot{\theta} \sin(\theta) - \ell \dot{\theta}^2 \cos(\theta) \right) \hat{\mathbf{b}}_1 \\ &\quad + \left(\ell \ddot{\theta} \cos(\theta) - \ell \dot{\theta}^2 \sin(\theta) \right) \hat{\mathbf{b}}_2 \\ {}^{\mathcal{I}}\mathbf{a}_{O'/O} &= a \hat{\mathbf{e}}_2 \equiv a \hat{\mathbf{b}}_2\end{aligned}$$

$$\begin{aligned}m_P {}^{\mathcal{B}}\mathbf{a}_{P/O'} &= \sum \mathbf{F}_P - m_P {}^{\mathcal{I}}\mathbf{a}_{O'/O} \\ m_P \left[\left(-\ell \ddot{\theta} \sin(\theta) - \ell \dot{\theta}^2 \cos(\theta) \right) \hat{\mathbf{b}}_1 + \left(\ell \ddot{\theta} \cos(\theta) - \ell \dot{\theta}^2 \sin(\theta) \right) \hat{\mathbf{b}}_2 \right] &= \\ &\quad - F_c \cos \theta \hat{\mathbf{b}}_1 - F_c \sin \theta \hat{\mathbf{b}}_2 + m_P g \hat{\mathbf{b}}_1 - m_P a \hat{\mathbf{b}}_2 \implies \\ \ddot{\theta} &= -\frac{a \cos(\theta) + g \sin(\theta)}{\ell} \qquad F_c = m_P \left(-a \sin(\theta) + \ell \dot{\theta}^2 + g \cos(\theta) \right)\end{aligned}$$

Rotating Frames Revisited



- Vector $\mathbf{b}$ has constant magnitude r and is **fixed** in frame $\mathcal{B}$, which has angular velocity with respect to frame $\mathcal{I}$ of ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\hat{\mathbf{e}}_3$
- θ changes in time, but ϕ (the angle between $\mathbf{b}$ and $\hat{\mathbf{b}}_1$) is constant



$$\mathbf{b} = r \cos(\theta + \phi) \hat{\mathbf{e}}_1 + r \sin(\theta + \phi) \hat{\mathbf{e}}_2$$

$${}^{\mathcal{I}}\frac{d}{dt}\mathbf{b} = -r\dot{\theta} \sin(\theta + \phi) \hat{\mathbf{e}}_1 + r\dot{\theta} \cos(\theta + \phi) \hat{\mathbf{e}}_2$$

A Bit of Derivation

- Let's see what happens when we cross $\mathbf{b}$ with its $\mathcal{I}$ frame derivative:

$$\begin{aligned} \mathbf{b} \times {}^{\mathcal{I}}\frac{d}{dt}\mathbf{b} &= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ r \cos(\theta + \phi) & r \sin(\theta + \phi) & 0 \\ -r\dot{\theta} \sin(\theta + \phi) & r\dot{\theta} \cos(\theta + \phi) & 0 \end{vmatrix} \\ &= \left(r^2 \dot{\theta} \sin^2(\phi + \theta) + r^2 \dot{\theta} \cos^2(\phi + \theta) \right) \hat{\mathbf{e}}_3 = r^2 \dot{\theta} \hat{\mathbf{e}}_3 \equiv r^2 {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \end{aligned}$$

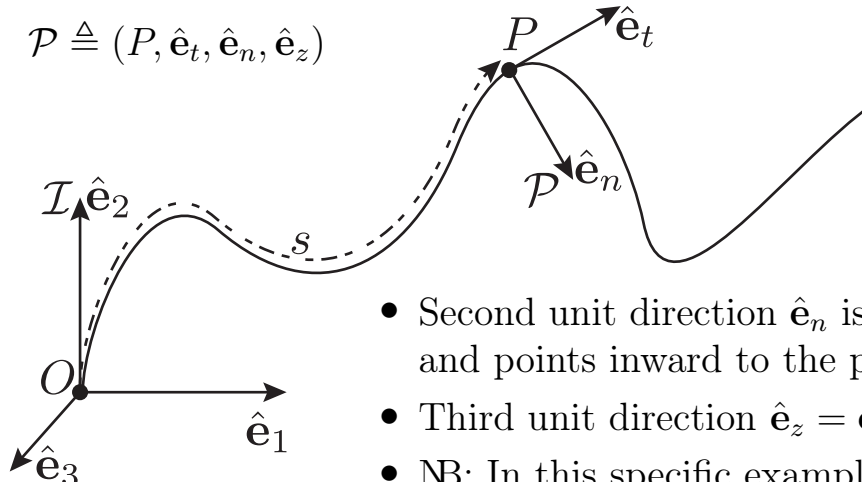
- We can now cross both sides of this expression with $\mathbf{b}$ to find:

$$r^2 {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{b} = \left(\mathbf{b} \times {}^{\mathcal{I}}\frac{d}{dt}\mathbf{b} \right) \times \mathbf{b} = \underbrace{(\mathbf{b} \cdot \mathbf{b})}_{\equiv r^2} \frac{d}{dt}\mathbf{b} - \underbrace{\left(\mathbf{b} \cdot \frac{d}{dt}\mathbf{b} \right)}_{\equiv 0} \mathbf{b}$$

- This works for **any** vector $\mathbf{b}$ fixed in $\mathcal{B}$: ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{b} = \frac{d}{dt}\mathbf{b}$

The Path Frame

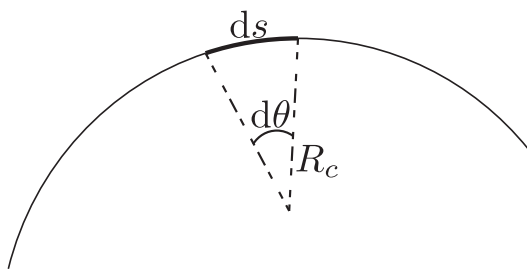
$\mathcal{P} \triangleq (P, \hat{\mathbf{e}}_t, \hat{\mathbf{e}}_n, \hat{\mathbf{e}}_z)$



- Define **path frame** $\mathcal{P}$ whose coordinate origin (P) follows some path
- First unit direction $\hat{\mathbf{e}}_t$ is always tangent to the path such that $\hat{\mathbf{e}}_t \equiv \hat{\mathbf{v}}_{P/O}$
- Second unit direction $\hat{\mathbf{e}}_n$ is always normal to the path and points inward to the path curvature
- Third unit direction $\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_n$
- NB: In this specific example, $\hat{\mathbf{e}}_z = -\hat{\mathbf{e}}_3$

- By the frame definition we can write: ${}^I\mathbf{v}_{P/O} = \dot{s}\hat{\mathbf{e}}_t$ for path coordinate s
- The acceleration is then: ${}^I\mathbf{a}_{P/O} = \ddot{s}\hat{\mathbf{e}}_t + \dot{s} \frac{d}{dt}\hat{\mathbf{e}}_t = \ddot{s}\hat{\mathbf{e}}_t + \dot{s} ({}^I\boldsymbol{\omega}^{\mathcal{P}} \times \hat{\mathbf{e}}_t)$

Radius of Curvature



- We define ${}^I\boldsymbol{\omega}^{\mathcal{P}} \triangleq \dot{\theta}\hat{\mathbf{e}}_z$ so ${}^I\boldsymbol{\omega}^{\mathcal{P}} \times \hat{\mathbf{e}}_t = \dot{\theta}\hat{\mathbf{e}}_n$
 - Consider an infinitesimal section of the path ds : this can be modeled as an infinitesimal section of a circle of radius R_c subtending angle $d\theta$ such that $ds = d\theta R_c$
 - R_c is the **radius of curvature**: the radius of the circular arc that best approximates our curve at our current location
 - NB: R_c is **not** constant throughout an arbitrary path
- $$ds = d\theta R_c \implies d\theta = \frac{ds}{R_c} \implies \frac{d\theta}{dt} = \frac{1}{R_c} \frac{ds}{dt} \implies \dot{\theta} = \frac{\dot{s}}{R_c}$$

$${}^I\mathbf{a}_{P/O} = \ddot{s}\hat{\mathbf{e}}_t + \frac{\dot{s}^2}{R_c}\hat{\mathbf{e}}_n$$

Finding the Radius of Curvature

- Assume the path is described in Cartesian coordinates by function $y(x)$ such that $ds = \sqrt{dx^2 + dy^2}$. Dividing through by dt :

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx} \frac{dx}{dt}\right)^2} \Rightarrow \dot{s} = \sqrt{\dot{x}^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

- Assuming the path lies in the $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ plane:

$$\mathcal{I}_{\mathbf{v}_{P/O}} = \dot{x}\hat{\mathbf{e}}_1 + \dot{y}\hat{\mathbf{e}}_2 = \dot{x} \left(\hat{\mathbf{e}}_1 + \frac{dy}{dx}\hat{\mathbf{e}}_2 \right) \equiv \dot{s}\hat{\mathbf{e}}_t$$

$$\hat{\mathbf{e}}_t = \frac{\mathcal{I}_{\mathbf{v}_{P/O}}}{\dot{s}} = \dot{x} \frac{\hat{\mathbf{e}}_1 + \frac{dy}{dx}\hat{\mathbf{e}}_2}{\sqrt{\dot{x}^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}}$$

Finding the Radius of Curvature (continued)

- Recall that $\hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_z = -\hat{\mathbf{e}}_n$ and $\hat{\mathbf{e}}_z = \pm\hat{\mathbf{e}}_3$

$$\hat{\mathbf{e}}_n = -\hat{\mathbf{e}}_t \times \pm\hat{\mathbf{e}}_3 = -\dot{x} \frac{\hat{\mathbf{e}}_1 + \frac{dy}{dx}\hat{\mathbf{e}}_2}{\sqrt{\dot{x}^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}} \times \pm\hat{\mathbf{e}}_3 = \pm \frac{\hat{\mathbf{e}}_2 - \frac{dy}{dx}\hat{\mathbf{e}}_1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

- We can similarly directly differentiate the $\hat{\mathbf{e}}_t$ expression:

$$\frac{\mathcal{I}d}{dt}\hat{\mathbf{e}}_t = \frac{\mathcal{I}d}{dt} \left(\frac{\hat{\mathbf{e}}_1 + \frac{dy}{dx}\hat{\mathbf{e}}_2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = \frac{\dot{x} \frac{d^2y}{dx^2} \left(-\frac{dy}{dx}\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \right)}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}} = \pm \frac{\dot{x} \frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} \hat{\mathbf{e}}_n$$

Finding the Radius of Curvature (continued some more)

- We have already figured out that ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{P}} = \dot{s}/R_c \hat{\mathbf{e}}_z$ so:

$${}^{\mathcal{I}}\frac{d}{dt}\hat{\mathbf{e}}_t = \frac{\dot{s}}{R_c}\hat{\mathbf{e}}_n \implies \frac{\dot{s}}{R_c} = \pm \frac{\dot{x} \frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{\dot{x}^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}}{R_c}$$

- R_c is defined to be strictly positive, so we can finally write:

$$R_c = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}$$

Week 4

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Vector Products as Matrix Multiplications

The true utility of our matrix (column-array) notation for vector components is that it allows us to express all vector operations as linear algebraic matrix operations. This allows us to efficiently offload a lot of the tedious work involved in solving dynamics problems onto computers. For vector addition, we simply add (or subtract) each pair of elements of the vector array, one by one. Similarly, for scalar multiplication we scale each component individually by the scalar multiple. However, in the case of the vector products, the operations must be carried out as matrix multiplications. One key thing to remember is that the operator definitions given here *only* work for vector components written as a column matrix—they would be fundamentally different if we were to write our vectors as row matrices.

Vector Products as Matrix Multiplications

$$\begin{aligned} [\mathbf{a} \cdot \mathbf{b}]_{\mathcal{I}} &= [\mathbf{a}]_{\mathcal{I}}^T [\mathbf{b}]_{\mathcal{I}} \\ [\mathbf{a} \times \mathbf{b}]_{\mathcal{I}} &= [\mathbf{a} \times]_{\mathcal{I}} [\mathbf{b}]_{\mathcal{I}} \\ [\mathbf{b} \times \mathbf{a}]_{\mathcal{I}} &= [\mathbf{b} \times]_{\mathcal{I}} [\mathbf{a}]_{\mathcal{I}} = -[\mathbf{a} \times]_{\mathcal{I}} [\mathbf{b}]_{\mathcal{I}} \end{aligned} \quad \text{where} \quad [\mathbf{a} \times]_{\mathcal{I}} \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_{\mathcal{I}}$$

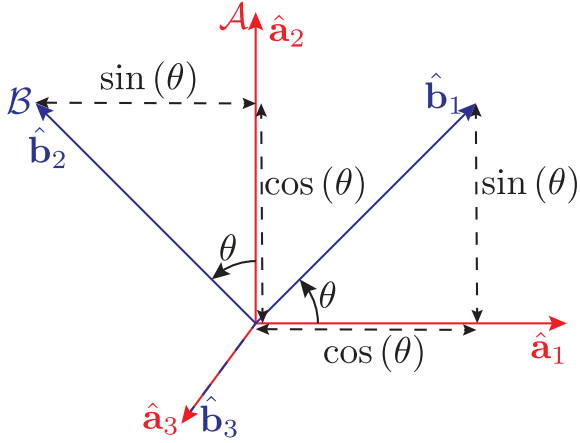
$$\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \quad [\mathbf{a}]_{\mathcal{I}} \triangleq \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{I}} \quad [\mathbf{b}]_{\mathcal{I}} \triangleq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}} \quad \begin{aligned} \mathbf{a} &= \sum_i a_i \hat{\mathbf{e}}_i \Rightarrow a_i = \mathbf{a} \cdot \hat{\mathbf{e}}_i \\ \mathbf{b} &= \sum_i b_i \hat{\mathbf{e}}_i \Rightarrow b_i = \mathbf{b} \cdot \hat{\mathbf{e}}_i \end{aligned}$$

$$[\mathbf{a} \cdot \mathbf{b}]_{\mathcal{I}} = \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{I}} \right)^T \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}_{\mathcal{I}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$[\mathbf{a} \times \mathbf{b}]_{\mathcal{I}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_{\mathcal{I}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}_{\mathcal{I}}$$

$$[\mathbf{b} \times \mathbf{a}]_{\mathcal{I}} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}_{\mathcal{I}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} -a_2 b_3 + a_3 b_2 \\ a_1 b_3 - a_3 b_1 \\ -a_1 b_2 + a_2 b_1 \end{bmatrix}_{\mathcal{I}}$$

The Direction Cosine Matrix (DCM)



$$\mathcal{A} \triangleq (\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3) \quad \mathcal{B} \triangleq (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$$

$$\hat{\mathbf{b}}_3 \equiv \hat{\mathbf{a}}_3$$

- Recall the **transformation table**:

	$\hat{\mathbf{a}}_1$	$\hat{\mathbf{a}}_2$	$\hat{\mathbf{a}}_3$
$\hat{\mathbf{b}}_1$	$\cos \theta$	$\sin \theta$	0
$\hat{\mathbf{b}}_2$	$-\sin \theta$	$\cos \theta$	0
$\hat{\mathbf{b}}_3$	0	0	1

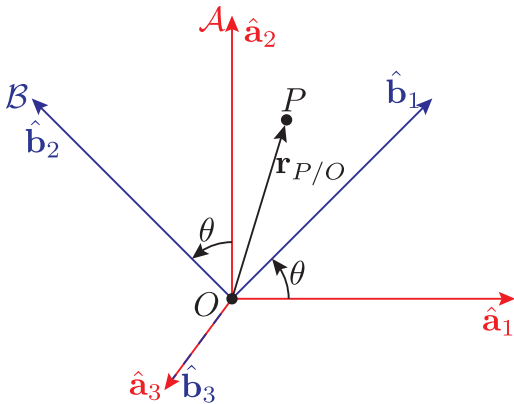
- This looks a lot like a matrix (because it is)

- Define: ${}^{\mathcal{B}}C^{\mathcal{A}} \triangleq \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- ${}^{\mathcal{B}}C^{\mathcal{A}}$ is a **direction cosine matrix**

$$[\mathbf{r}]_{\mathcal{B}} = {}^{\mathcal{B}}C^{\mathcal{A}} [\mathbf{r}]_{\mathcal{A}}$$

DCM in Action

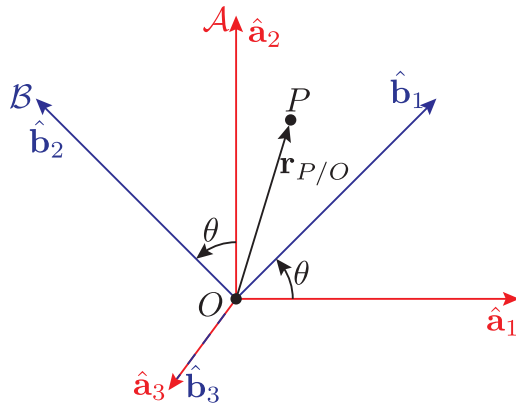


$$\mathbf{r}_{P/O} \triangleq a_1 \hat{\mathbf{a}}_1 + a_2 \hat{\mathbf{a}}_2 + a_3 \hat{\mathbf{a}}_3$$

$$\triangleq b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3$$

$$\begin{aligned} [\mathbf{r}_{P/O}]_{\mathcal{B}} &= {}^{\mathcal{B}}C^{\mathcal{A}} [\mathbf{r}_{P/O}]_{\mathcal{A}} \\ &= \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{A}} \\ &= \begin{bmatrix} a_1 \cos(\theta) + a_2 \sin(\theta) \\ -a_1 \sin(\theta) + a_2 \cos(\theta) \\ a_3 \end{bmatrix}_{\mathcal{B}} \equiv \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{B}} \end{aligned}$$

DCM Inverses



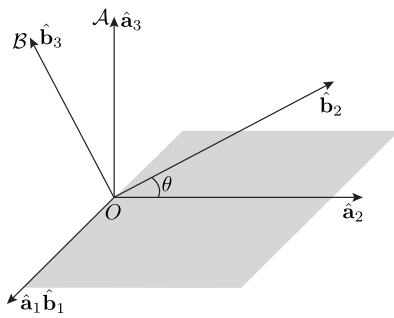
$$\begin{aligned}\mathbf{r}_{P/O} &\triangleq a_1 \hat{\mathbf{a}}_1 + a_2 \hat{\mathbf{a}}_2 + a_3 \hat{\mathbf{a}}_3 \\ &\triangleq b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3\end{aligned}$$

DCMs are Orthogonal Matrices:

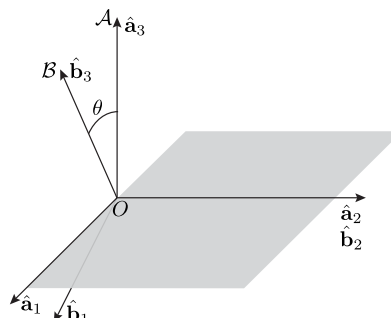
$${}^{\mathcal{A}}C^{\mathcal{B}} = ({}^{\mathcal{B}}C^{\mathcal{A}})^{-1} = ({}^{\mathcal{B}}C^{\mathcal{A}})^T$$

$$\begin{aligned}[\mathbf{r}_{P/O}]_{\mathcal{A}} &= \underbrace{({}^{\mathcal{B}}C^{\mathcal{A}})^T}_{\equiv {}^{\mathcal{A}}C^{\mathcal{B}}} [\mathbf{r}_{P/O}]_{\mathcal{B}} \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{B}} \\ &= \begin{bmatrix} b_1 \cos(\theta) - b_2 \sin(\theta) \\ b_1 \sin(\theta) + b_2 \cos(\theta) \\ b_3 \end{bmatrix}_{\mathcal{A}} \equiv \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{A}}\end{aligned}$$

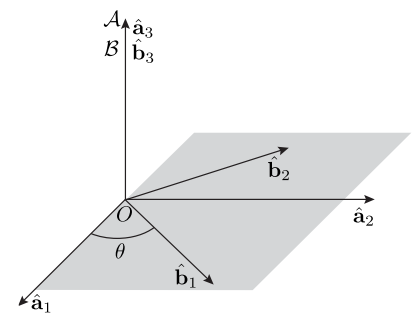
Simple Direction Cosine Matrices



$$\begin{aligned}{}^{\mathcal{B}}C^{\mathcal{A}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \\ &\triangleq C_1(\theta)\end{aligned}$$



$$\begin{aligned}{}^{\mathcal{B}}C^{\mathcal{A}} &= \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \\ &\triangleq C_2(\theta)\end{aligned}$$

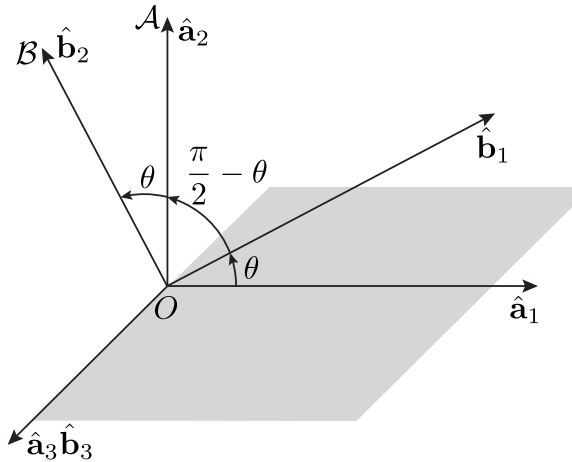


$$\begin{aligned}{}^{\mathcal{B}}C^{\mathcal{A}} &= \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\triangleq C_3(\theta)\end{aligned}$$

Why are They Called DCM?

Each entry of a DCM is the cosine of the angle between each pair of unit vectors of the two frames the DCM maps between:

$$\left. \begin{array}{l} \mathcal{A} = (O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3) \\ \mathcal{B} = (O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3) \end{array} \right\} [\mathcal{A}^{\mathcal{B}}]_{ij} = \hat{\mathbf{a}}_i \cdot \hat{\mathbf{b}}_j \implies {}^{\mathcal{B}}\mathcal{C}^{\mathcal{A}} = ({}^{\mathcal{A}}\mathcal{C}^{\mathcal{B}})^T \implies [{}^{\mathcal{B}}\mathcal{C}^{\mathcal{A}}]_{ij} = \hat{\mathbf{b}}_i \cdot \hat{\mathbf{a}}_j$$



$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_1 = \cos \theta$$

$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_2 = \cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta$$

$$\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_1 = \cos \left(\theta + \frac{\pi}{2} - \theta + \theta \right) = -\sin \theta$$

$$\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_2 = \cos \theta$$

$$\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_3 = 1$$

$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{a}}_3 = \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_3 = \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_1 = \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_2 = \cos \left(\frac{\pi}{2} \right) = 0$$

Forces We've Seen So Far

- Gravity: Acts in direction of attracting body, with magnitude mg
- Spring: Acts in opposite direction of stretched spring, with magnitude $k(x - x_0)$ for spring constant k , spring rest length x_0 , and spring extension length x
- Damper: Acts in opposite direction of velocity vector of damped body, with magnitude $b\dot{x}$ for speed damping constant b and body speed $\dot{x}$
- Normal and constraint forces: Act to hold systems together, or impose constraints, usually due to 3rd law
- Fictitious forces: not forces at all, but kinematic terms due to rotating and accelerating frames

Forces can be classified as **field** forces (acting simultaneously on the whole body, like gravity) or **contact** forces (acting on a specific part of a body, like friction).

Friction

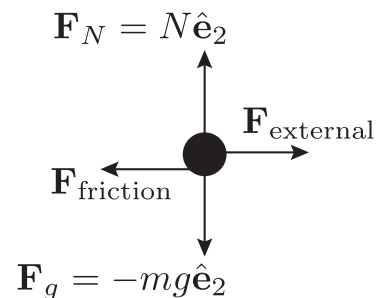
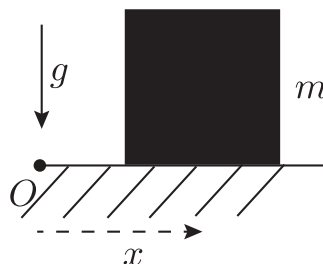
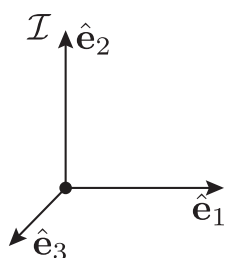
- **Friction** is the resistance to relative motion of two surfaces in contact
 - If an object is in motion it experiences **kinetic friction**: force is always opposite the direction of motion
 - If an object is motionless it experiences **static friction**: a force greater than the static friction force must be applied for the object to start moving

Exact modeling of friction is tremendously difficult!

All of our friction models are vast simplifications

Coulomb Friction

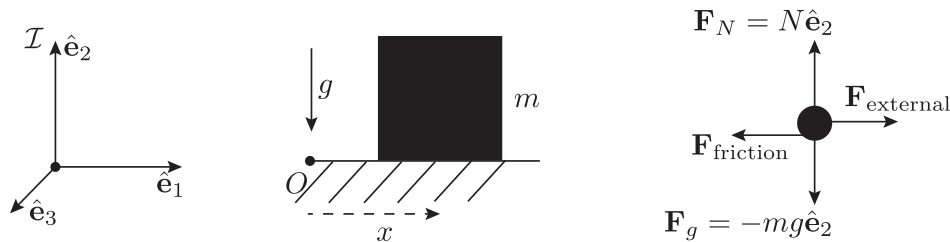
- $\|\mathbf{F}_{\text{friction}}\| \leq \mu \|\mathbf{F}_{\text{Normal}}\|$ where μ is the **coefficient of friction**
- μ is determined empirically and is typically split into static (μ_s) and kinetic (μ_k or μ_c) parts



- If $\dot{x} = 0$, need an external force with magnitude greater than the static friction force to start the object moving:

$$\mathbf{F}_{\text{friction}} = -\mathbf{F}_{\text{external}} = -\mu_s mg \hat{\mathbf{F}}_{\text{external}}$$
- μ_s increases proportionally to $\|\mathbf{F}_{\text{external}}\|$ while the object is stationary

Coulomb Friction Continued



- If $\dot{x} \neq 0$: $\|\mathbf{F}_{\text{friction}}\| = \mu_c \|\mathbf{F}_{\text{Normal}}\|$
- If we assume that an object is set in motion ($\dot{x} \neq 0$) and no further external forces (other than friction) act upon it ($\|\mathbf{F}_{\text{external}}\| = 0$) then:

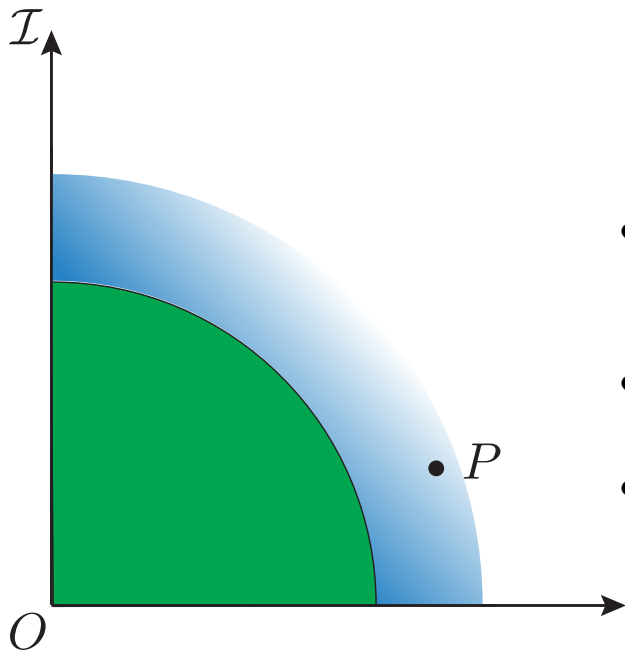
$$\sum \mathbf{F}_P = m {}^I \mathbf{a}_{P/O} \implies m \ddot{x} \hat{\mathbf{e}}_1 = -mg \hat{\mathbf{e}}_2 + N \hat{\mathbf{e}}_2 - \|\mathbf{F}_{\text{friction}}\| \hat{\mathbf{e}}_1 \quad \begin{cases} m \ddot{x} = -\|\mathbf{F}_{\text{friction}}\| \\ -mg + N = 0 \end{cases}$$

$$\ddot{x} = -\mu_c g$$

Viscous Friction

- We've already introduced a different model of friction via the damper
- The **viscous friction** mode predicts a friction force of the form:
 $\mathbf{F}_{\text{friction}} = -b \mathbf{v}$ where b is the friction coefficient and $\mathbf{v}$ is the *relative* velocity of the body with respect to the surface causing friction
- NB: Both the viscous and Coulomb friction models predict forces independent of positions. Coulomb friction is also independent of velocity, and will be our standard friction model

Drag (Fluid Resistance)



Cross-sectional Area $\rightarrow$ Relative Velocity $\rightarrow$

$$\mathbf{F}_{\text{drag}} = -\frac{1}{2} C_D A \rho v_{\text{rel}}^2 \hat{\mathbf{v}}_{\text{rel}}$$

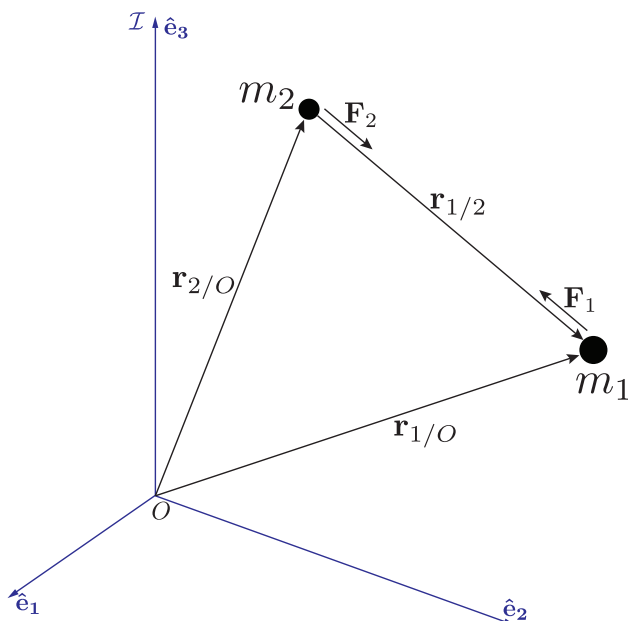
Drag Coefficient $\rightarrow$

Fluid Density $\rightarrow$

- The drag force acts opposite the direction of relative velocity of the body to the fluid it is traveling within: $\mathbf{v}_{\text{rel}} = {}^I\mathbf{v}_{P/O} - {}^I\mathbf{v}_{\text{fluid}/O}$
- If the fluid is stationary, then $\mathbf{v}_{\text{rel}}$ is the body's inertial velocity
- A common measure of a body's ability to overcome air resistance is its

Ballistic Coefficient $\triangleq \frac{m}{C_D A}$

Newton's Law of Gravity



- Our constant gravity force model ($\|\mathbf{F}_{\text{gravity}}\| = mg$) is an approximation that works reasonably well for objects near the surface of the Earth
- In addition to his 3 laws, Newton also posited a law of gravity:

$$\mathbf{F}_1 = -\mathbf{F}_2 = -\frac{G m_1 m_2}{\|\mathbf{r}_{1/2}\|^3} \mathbf{r}_{1/2}$$

- G is the universal gravitational constant $\approx 6.67430 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$

Recall Newton's Second Law

$$\begin{array}{c} \text{Inertial Frame Derivative} \quad \text{Inertially Fixed Point} \quad \text{Mass (Assumed Constant)} \\ \mathbf{F}_P = \frac{\mathcal{I} d}{dt} (\mathcal{I} \mathbf{p}_{P/O}) = \frac{\mathcal{I} d}{dt} (m_P \mathcal{I} \mathbf{v}_{P/O}) = m_P \mathcal{I} \mathbf{a}_{P/O} \\ \text{Resultant Force on } P \quad \text{Linear Momentum} \quad \text{Inertial Velocity and Acceleration} \end{array}$$

- Integrating with respect to time:

$$\begin{aligned} \int_{t_1}^{t_2} \mathbf{F}_P dt &= \int_{t_1}^{t_2} \frac{\mathcal{I} d}{dt} (\mathcal{I} \mathbf{p}_{P/O}) dt = \mathcal{I} \mathbf{p}_{P/O}(t_2) - \mathcal{I} \mathbf{p}_{P/O}(t_1) \\ &\Rightarrow m_P \mathcal{I} \mathbf{v}_{P/O}(t_2) = m_P \mathcal{I} \mathbf{v}_{P/O}(t_1) + \int_{t_1}^{t_2} \mathbf{F}_P dt \end{aligned}$$

- No force means no change in linear momentum. This is a **conservation** law

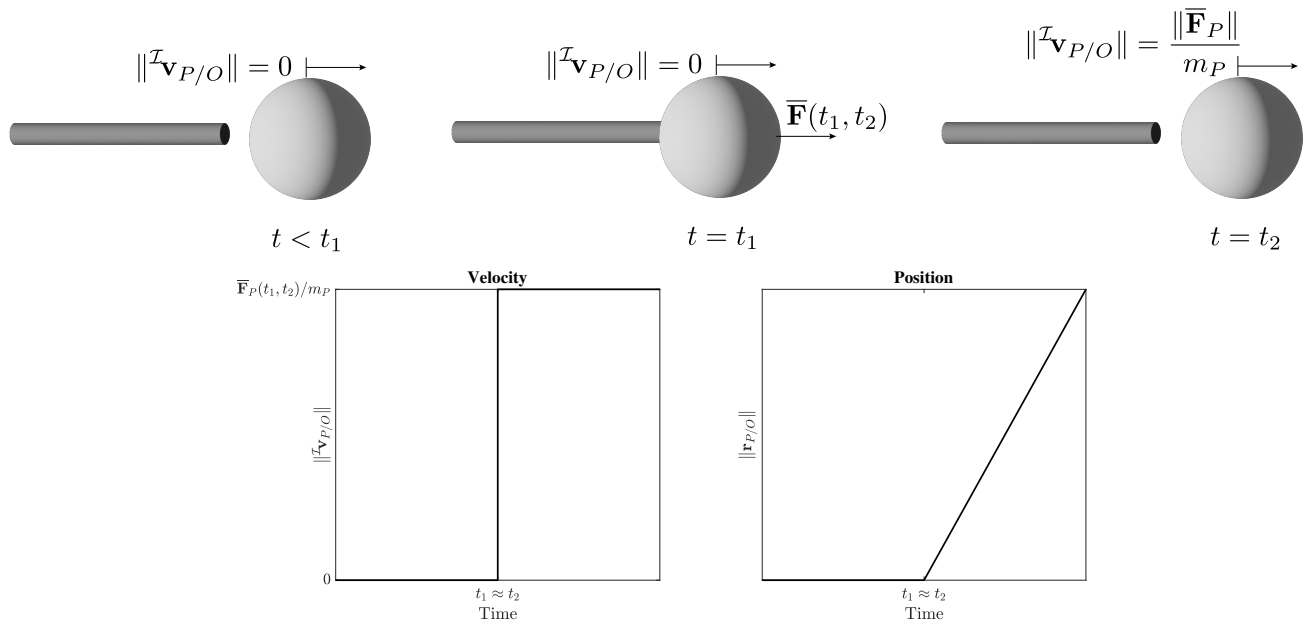
Linear Impulse

- Define **linear impulse**:

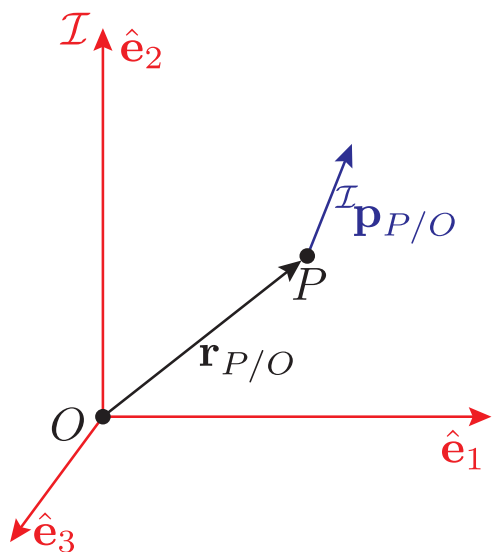
$$\bar{\mathbf{F}}_P(t_1, t_2) \triangleq \int_{t_1}^{t_2} \mathbf{F}_P dt$$

- Therefore: $\mathcal{I} \mathbf{p}_{P/O}(t_2) = \mathcal{I} \mathbf{p}_{P/O}(t_1) + \bar{\mathbf{F}}_P(t_1, t_2)$
- What if a force is applied over a *very short* period of time ($\Delta t \triangleq t_2 - t_1 \ll 1$)?
 - We can approximate $\mathbf{F}_P$ as independent of position and velocity of P and thereby avoid solving the complicated integral for the linear impulse
 - This helps us model how things *start* moving: we frequently approximate changes in momentum due to short applications of forces to be instantaneous

A (very) simple model of Pool



Angular Momentum



Angular Momentum of P relative to O in frame $\mathcal{I}$:

$$\mathcal{I}\mathbf{h}_{P/O} \triangleq \mathbf{r}_{P/O} \times \mathcal{I}\mathbf{p}_{P/O} = \mathbf{r}_{P/O} \times m_P \mathcal{I}\mathbf{v}_{P/O}$$

- Angular momentum is *always* relative to some point
- Angular momentum is orthogonal to both the position and velocity of the body

Moment (Torque)

- What happens when we differentiate ${}^{\mathcal{I}}\mathbf{h}_{P/O}$?

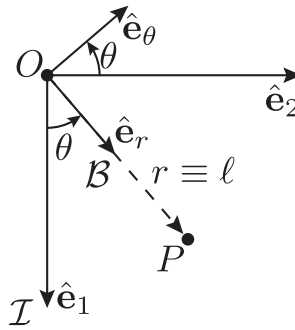
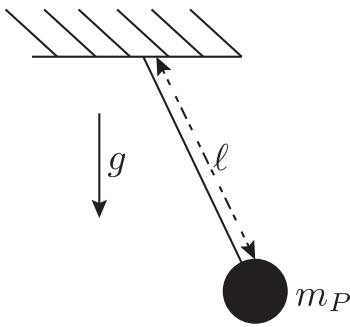
$$\begin{aligned} \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{h}_{P/O} &= \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} (\mathbf{r}_{P/O} \times {}^{\mathcal{I}}\mathbf{p}_{P/O}) = \underbrace{\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} (\mathbf{r}_{P/O}) \times m_P {}^{\mathcal{I}}\mathbf{v}_{P/O}}_{\equiv {}^{\mathcal{I}}\mathbf{v}_{P/O} \times m_P {}^{\mathcal{I}}\mathbf{v}_{P/O}} + \underbrace{\mathbf{r}_{P/O} \times \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} (m_P {}^{\mathcal{I}}\mathbf{v}_{P/O})}_{\mathbf{r}_{P/O} \times m_P \underbrace{{}^{\mathcal{I}}\mathbf{a}_{P/O}}_{\equiv \mathbf{F}_P}} \\ &\equiv \mathbf{r}_{P/O} \times m_P \mathbf{a}_{P/O} \equiv \mathbf{r}_{P/O} \times \mathbf{F}_P \end{aligned}$$

- Define the **Moment** (equivalently **Torque**): $\mathbf{M}_{P/O} \equiv \boldsymbol{\tau} \triangleq \mathbf{r}_{P/O} \times \mathbf{F}_P$

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{h}_{P/O} = \mathbf{M}_{P/O}$$

- NB: This is literally just a restatement of Newton's second law
- The moment is the tendency of a force to rotate an object about an axis

Return of the Simple Pendulum



$$\begin{aligned} \mathbf{F}_{\text{constraint}} &= -F_c \hat{\mathbf{e}}_r \\ \mathbf{F}_{\text{gravity}} &= m_P g (\cos(\theta) \hat{\mathbf{e}}_r - \sin \theta \hat{\mathbf{e}}_\theta) \end{aligned}$$

$$\begin{aligned} \mathbf{r}_{P/O} &= \ell \hat{\mathbf{e}}_r \Rightarrow {}^{\mathcal{I}}\mathbf{v}_{P/O} = \ell \dot{\theta} \hat{\mathbf{e}}_\theta \\ {}^{\mathcal{I}}\mathbf{h}_{P/O} &= \mathbf{r}_{P/O} \times m_P {}^{\mathcal{I}}\mathbf{v}_{P/O} \\ &= \ell \hat{\mathbf{e}}_r \times m_P \ell \dot{\theta} \hat{\mathbf{e}}_\theta = m_P \ell^2 \dot{\theta} \hat{\mathbf{e}}_3 \end{aligned}$$

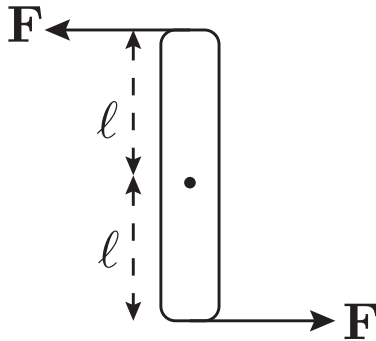
$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{h}_{P/O} = m_P \ell^2 \ddot{\theta} \hat{\mathbf{e}}_3$$

$$\begin{aligned} \mathbf{M}_{P/O} &= \mathbf{r}_{P/O} \times \mathbf{F}_P \\ &= \ell \hat{\mathbf{e}}_r \times (-F_c \hat{\mathbf{e}}_r + m_P g (\cos \theta \hat{\mathbf{e}}_r - \sin \theta \hat{\mathbf{e}}_\theta)) \\ &= -\ell m_P g \sin \theta \hat{\mathbf{e}}_3 \end{aligned}$$

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{h}_{P/O} = \mathbf{M}_{P/O} \Rightarrow \ddot{\theta} = -\frac{g}{\ell} \sin \theta$$

Conservation Laws

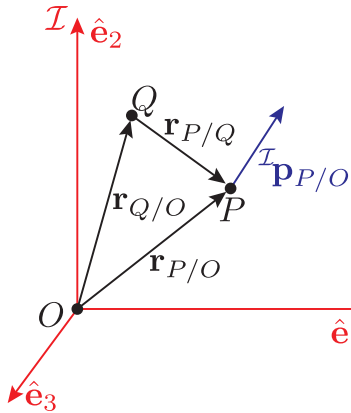
- Conservation of Linear Momentum: $\sum \mathbf{F}_P = 0 \Rightarrow \mathcal{I}\mathbf{p}_{P/O} = \text{constant}$
- Conservation of Angular Momentum: $\sum \mathbf{M}_{P/O} = 0 \Rightarrow \mathcal{I}\mathbf{h}_{P/O} = \text{constant}$
- NB: You can have conserved angular momentum even while the linear momentum of a system is changing



- This is an example of a **Force Couple**:

$$\sum \mathbf{F} = 0 \quad \text{but} \quad \sum \mathbf{M}_{P/O} \neq 0$$

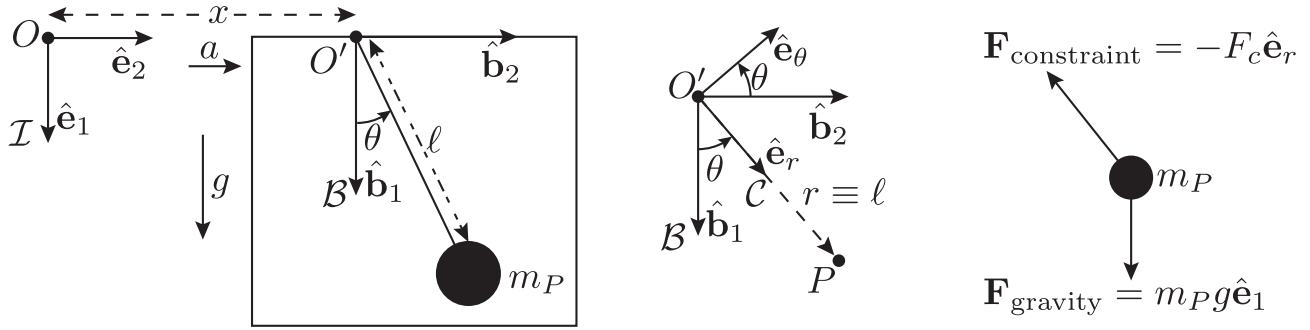
Angular Momentum About an Arbitrary Point



- Remember: Angular momentum is always relative to some point
- This point does *not* have to be a coordinate origin
- By definition: $\mathcal{I}\mathbf{h}_{P/Q} \triangleq \mathbf{r}_{P/Q} \times m_P \mathcal{I}\mathbf{v}_{P/Q}$
- $\mathbf{r}_{P/Q} = \mathbf{r}_{P/O} - \mathbf{r}_{Q/O} \Rightarrow \mathcal{I}\mathbf{v}_{P/Q} = \mathcal{I}\mathbf{v}_{P/O} - \mathcal{I}\mathbf{v}_{Q/O}$
- $\mathcal{I}\mathbf{h}_{P/Q} = (\mathbf{r}_{P/O} - \mathbf{r}_{Q/O}) \times m_P (\mathcal{I}\mathbf{v}_{P/O} - \mathcal{I}\mathbf{v}_{Q/O})$
 $\equiv 0$

$$\begin{aligned} \mathcal{I} \frac{d}{dt} \mathcal{I}\mathbf{h}_{P/Q} &= \overbrace{\frac{d}{dt} (\mathbf{r}_{P/O} - \mathbf{r}_{Q/O}) \times m_P (\mathcal{I}\mathbf{v}_{P/O} - \mathcal{I}\mathbf{v}_{Q/O})}^{\equiv \mathbf{F}_P} \\ &+ (\mathbf{r}_{P/O} - \mathbf{r}_{Q/O}) \times m_P \frac{d}{dt} (\mathcal{I}\mathbf{v}_{P/O} - \mathcal{I}\mathbf{v}_{Q/O}) = \mathbf{r}_{P/Q} \times \left(\overbrace{m_P \mathcal{I}\mathbf{a}_{P/O} - m_P \mathcal{I}\mathbf{a}_{Q/O}} \right) \\ \mathcal{I} \frac{d}{dt} \mathcal{I}\mathbf{h}_{P/Q} &= \mathbf{M}_{P/Q} - \mathbf{r}_{P/Q} \times m_P \mathcal{I}\mathbf{a}_{Q/O} \end{aligned}$$

Pendulum in an Accelerating Box Redux



- New frame $\mathcal{C} \triangleq (O', \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$ moves with the box *and* the pendulum
- $\mathcal{I}\boldsymbol{\omega}^{\mathcal{C}} \equiv \mathcal{B}\boldsymbol{\omega}^{\mathcal{C}} = \dot{\theta}\hat{\mathbf{e}}_3$
- $r \equiv \ell, \theta$ are polar coordinates in $\mathcal{B}$
 x is a Cartesian coordinate in $\mathcal{I}$
 r is Cartesian in $\mathcal{C}$

$$\begin{aligned}\mathbf{r}_{P/O'} &= \ell \hat{\mathbf{e}}_r \\ \mathcal{I}\mathbf{v}_{P/O'} &= \ell (\mathcal{I}\boldsymbol{\omega}^{\mathcal{C}} \times \hat{\mathbf{e}}_r) = \ell \dot{\theta} \hat{\mathbf{e}}_\theta \\ \mathcal{I}\mathbf{h}_{P/O'} &= \mathbf{r}_{P/O'} \times m_P \mathcal{I}\mathbf{v}_{P/O'} \\ &= m_P \ell^2 \dot{\theta} \hat{\mathbf{e}}_3 \\ \mathbf{M}_{P/O'} &= \mathbf{r}_{P/O'} \times (-F_c \hat{\mathbf{e}}_r + m_P g \hat{\mathbf{e}}_1) \\ &= -\ell m_P g \sin(\theta) \hat{\mathbf{e}}_3\end{aligned}$$

$$\mathcal{I} \frac{d}{dt} \mathcal{I}\mathbf{h}_{P/O'} = \mathbf{M}_{P/O'} - \mathbf{r}_{P/O'} \times m_P \overbrace{\mathcal{I}\mathbf{a}_{O'/O}}^{= a \hat{\mathbf{e}}_2}$$

$$\mathcal{I} \frac{d}{dt} (m_P \ell^2 \dot{\theta} \hat{\mathbf{e}}_3) = -\ell m_P g \sin(\theta) \hat{\mathbf{e}}_3 - \ell \hat{\mathbf{e}}_r \times m_P a \hat{\mathbf{e}}_2$$

$$m_P \ell^2 \ddot{\theta} \hat{\mathbf{e}}_3 = -\ell m_P g \sin(\theta) \hat{\mathbf{e}}_3 - \ell m_P a \cos(\theta) \hat{\mathbf{e}}_3 \Rightarrow \ddot{\theta} = -\frac{a \cos(\theta) + g \sin(\theta)}{\ell}$$

Angular Impulse

- Just as we did with linear impulse, we can define **angular impulse**:

$$\overline{\mathbf{M}}_{P/O}(t_1, t_2) \triangleq \int_{t_1}^{t_2} \mathbf{M}_{P/O} dt$$

- Therefore: $\mathcal{I}_{\mathbf{h}_{P/O}}(t_2) = \mathcal{I}_{\mathbf{h}_{P/O}}(t_1) + \overline{\mathbf{M}}_{P/O}(t_1, t_2)$
- Everything we said about linear impulse applies equally to angular impulse

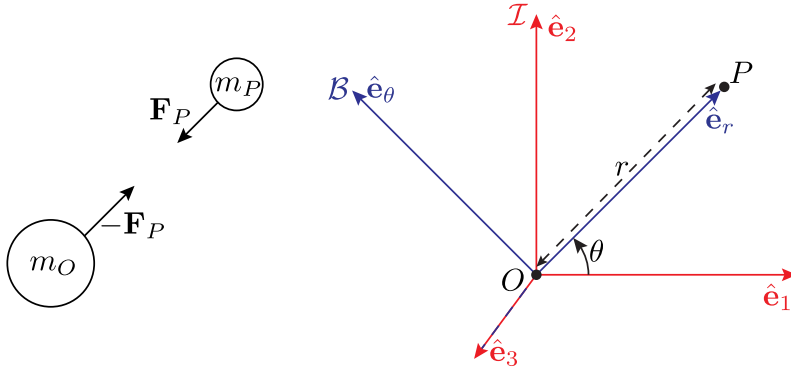
Central Force Problems

- A **central force** $\mathbf{F}_P$ is one where:

$$\mathbf{M}_{P/O} = \mathbf{r}_{P/O} \times \mathbf{F}_P = 0, \text{ for } \mathbf{r}_{P/O} \neq 0$$

- This means that $\mathcal{I}_{\mathbf{h}_{P/O}}$ is constant in time (both magnitude and direction)
- By the properties of the cross product, this also tells us that, in this case, $\mathbf{F}_P$ is parallel (or anti-parallel) to $\mathbf{r}_{P/O}$: the force acts in the direction of the position vector

Two-Body Problem



$$\mathbf{r}_{P/O} = r\hat{\mathbf{e}}_r \quad \mathcal{I}_{\omega}^{\mathcal{B}} = \dot{\theta}\hat{\mathbf{e}}_3$$

$$\mathcal{I}\mathbf{v}_{P/O} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta$$

$$\mathcal{I}\mathbf{a}_{P/O} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{e}}_\theta$$

- Consider two masses interacting via gravity *only*
- We place the larger mass m_O at the coordinate origin and the smaller mass m_P a distance r away
- By Newton's law of gravity:

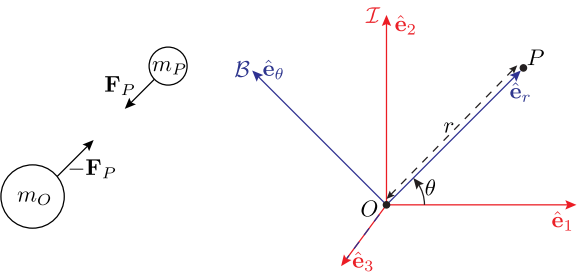
$$\mathbf{F}_p = -\frac{Gm_Pm_O}{\|\mathbf{r}_{P/O}\|^3}\mathbf{r}_{P/O}$$

$$m_P\mathcal{I}\mathbf{a}_{P/O} = -\underbrace{\frac{Gm_Pm_O}{r^2}}_{\mathbf{F}_P}\hat{\mathbf{e}}_r$$

$$\ddot{r} = -\frac{Gm_O}{r^2} + r\dot{\theta}^2$$

$$\ddot{\theta} = -2\frac{\dot{r}\dot{\theta}}{r}$$

Two-Body Equations



$$\mathcal{I}\mathbf{h}_{P/O} = r\hat{\mathbf{e}}_r \times m_P(\dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta) = m_P r^2 \dot{\theta} \hat{\mathbf{e}}_3$$

- Define **specific angular momentum**:

$$h \triangleq \frac{\|\mathcal{I}\mathbf{h}_{P/O}\|}{m_P} = r^2\dot{\theta}$$

- Substituting h into the equations of motion:

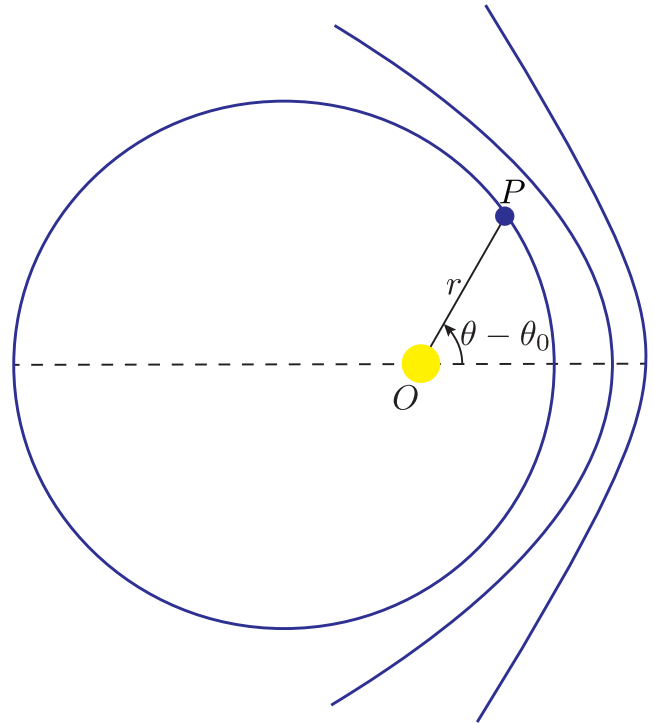
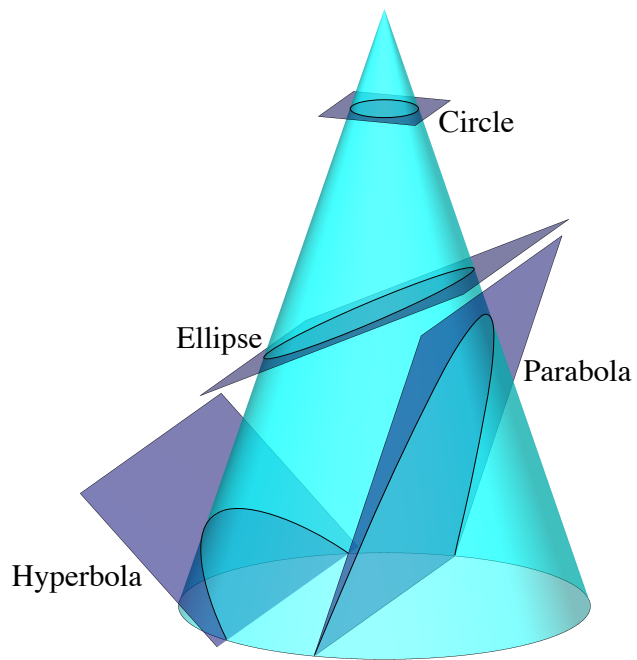
$$\ddot{\theta} = -2\frac{\dot{r}\dot{\theta}}{r^3} \quad \text{and} \quad \ddot{r} = -\frac{Gm_O}{r^2} + \frac{h^2}{r^3}$$

- These equations are uncoupled and fully solvable, with solution:

$$r(\theta) = \frac{h^2/Gm_O}{1 + e \cos(\theta - \theta_0)}$$

- e and θ_0 are constants. e is the **eccentricity** and determines the shape of the orbit. θ_0 encodes the relationship of θ and time.

Two-Body Orbits



Week 5

Dmitry Savransky

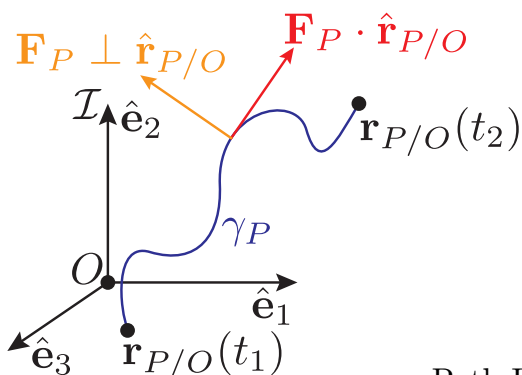
Cornell University

MAE 2030, Spring 2026

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Work

A force does **work** on a body when it displaces the body along a trajectory



The work performed on particle P by force $\mathbf{F}_P$ from time t_1 to t_2 is the integral of the component of the force tangential ($\mathbf{F}_P \cdot \hat{\mathbf{r}}_{P/O}$) to P 's trajectory (γ_P) from position $\mathbf{r}_{P/O}(t_1)$ to $\mathbf{r}_{P/O}(t_2)$:

$$W_P^{\mathbf{F}_P}(\mathbf{r}_{P/O}; \gamma_P) \triangleq \int_{\gamma_P} \mathbf{F}_P \cdot \underbrace{\mathcal{I} d\mathbf{r}_{P/O}}_{\text{Differential Displacement}}$$

Path Integral over trajectory

- The component of force perpendicular to the trajectory ($\mathbf{F}_P \perp \hat{\mathbf{r}}_{P/O}$) does **no** work

More Work

- Work is **not** (in general) simply force times distance: it depends on the specific trajectory
- Work is a scalar quantity, but depends on integration of vector quantities in an inertial frame
- Work is only done when the body is moving. As long as the body does not stop moving between t_1 and t_2 (e.g. $\mathcal{I}\mathbf{v}_{P/O}(t) \neq \mathbf{0}$ for $t_1 \leq t \leq t_2$), we can change the work integral to be with respect to time:

$$W_P^{\mathbf{F}_P}(\mathbf{r}_{P/O}; \gamma_P) = \int_{t_1}^{t_2} \mathbf{F}_P \cdot \mathcal{I}\mathbf{v}_{P/O} dt$$

Power

- The **power** done by a force acting on a body is equal to the instantaneous rate of change of the work done by the force:

$$P_P^{\mathbf{F}_P}(t_1) \triangleq \left. \frac{d}{dt} \right|_{t=t_1} W_P^{\mathbf{F}_P}(\mathbf{r}_{P/O}; \gamma_P)$$

- If the velocity is always non-zero:

$$P_P^{\mathbf{F}_P}(t_1) = \mathbf{F}_P(t_1) \cdot \mathcal{I}\mathbf{v}_{P/O}(t_1)$$

Units

- Force is mass times acceleration
 - SI units are Newtons: $N = \text{kg m s}^{-2}$
- Work is force integrated over displacement
 - SI units are Joules: $J = N \text{ m}$
- Power is work per unit time
 - SI units are Watts: $W = J \text{ s}^{-1}$

Total Work

- We can use the definition of work along with Newton's second law to define **total work**—the work due to the sum of all forces acting on a body:

$$\begin{aligned} W_P^{\text{tot}}(\mathbf{r}_{P/O}; \gamma_P) &\triangleq W_P^{\sum \mathbf{F}_P}(\mathbf{r}_{P/O}; \gamma_P) = \int_{\gamma_P} \overbrace{m_P \mathcal{I} \mathbf{a}_{P/O}}^{\equiv \sum \mathbf{F}_P} \cdot \mathcal{I} d\mathbf{r}_{P/O} \\ &= m_P \int_{\gamma_P} \mathcal{I} \frac{d}{dt} \mathcal{I} \mathbf{v}_{P/O} \cdot \mathcal{I} d\mathbf{r}_{P/O} = m_P \int_{\gamma_P} \mathcal{I} d\mathbf{v}_{P/O} \cdot \underbrace{\frac{1}{dt} \mathcal{I} d\mathbf{r}_{P/O}}_{\equiv \mathcal{I} \mathbf{v}_{P/O}} \end{aligned}$$

A Brief Aside

- Note that differentials essentially follow the same rules as derivatives (including chain rule and product rule). Therefore:

$$\begin{aligned} & \equiv \|\mathcal{I}\mathbf{v}_{P/O}\|^2 \\ \mathcal{I}d(\overbrace{\mathcal{I}\mathbf{v}_{P/O} \cdot \mathcal{I}\mathbf{v}_{P/O}}) &= \mathcal{I}d(\mathcal{I}\mathbf{v}_{P/O}) \cdot \mathcal{I}\mathbf{v}_{P/O} + \mathcal{I}\mathbf{v}_{P/O} \cdot \mathcal{I}d(\mathcal{I}\mathbf{v}_{P/O}) \\ &= 2(\mathcal{I}d(\mathcal{I}\mathbf{v}_{P/O}) \cdot \mathcal{I}\mathbf{v}_{P/O}) \end{aligned}$$

- Why is this important? Because we can now write our total work expression as:

$$W_P^{\text{tot}}(\mathbf{r}_{P/O}; \gamma_P) = m_P \int_{\gamma_P} \mathcal{I}d\mathbf{v}_{P/O} \cdot \mathcal{I}\mathbf{v}_{P/O} = \frac{m_P}{2} \int_{\gamma_P} \mathcal{I}d\|\mathcal{I}\mathbf{v}_{P/O}\|^2$$

- We are now integrating an exact differential and therefore our integral no longer has path dependence!

Kinetic Energy (KE)

- Our total work expression is now:

$$W_P^{\text{tot}} = \frac{m_P}{2} \int_{\mathcal{I}\mathbf{v}_{P/O}(t_1)}^{\mathcal{I}\mathbf{v}_{P/O}(t_2)} \mathcal{I}d\|\mathcal{I}\mathbf{v}_{P/O}\|^2 = \underbrace{\frac{m_P}{2} \|\mathcal{I}\mathbf{v}_{P/O}(t_2)\|^2}_{\triangleq T_{P/O}(t_2)} - \underbrace{\frac{m_P}{2} \|\mathcal{I}\mathbf{v}_{P/O}(t_1)\|^2}_{\triangleq T_{P/O}(t_1)}$$

- $T_{P/O} \triangleq \frac{1}{2}m_p(\mathcal{I}\mathbf{v}_{P/O} \cdot \mathcal{I}\mathbf{v}_{P/O})$ is the **kinetic energy** of our particle/body
- The change in kinetic energy from time t_1 to t_2 is equal to the total work done on the particle/body in that interval:

$$T_{P/O}(t_2) = T_{P/O}(t_1) + W_P^{\text{tot}}(t_1, t_2)$$

- Even though work is path dependent, the change in kinetic energy (and total work) depends only on path endpoints
- The velocity used to calculate kinetic energy must be with respect to an inertial frame

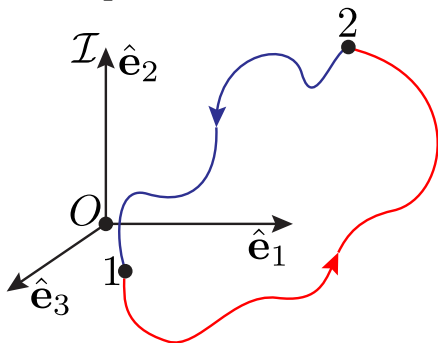
Work and Impulse

- Recall that our model of impulse is that the body does not move while the impulse is being applied
- This implies that no work is being done
- To fix this, we define work due to an impulse by the change in velocity/KE:

$$\begin{aligned} m_P \mathcal{I} \mathbf{v}_{P/O}(t_2) &= m_P \mathcal{I} \mathbf{v}_{P/O}(t_1) + \bar{\mathbf{F}}_P(t_1, t_2) \\ \implies W_{\bar{\mathbf{F}}_P}(t_1, t_2) &= T_{P/O}(t_2) - T_{P/O}(t_1) \end{aligned}$$

Conservative Forces

- We now consider a special class of forces called **conservative forces**
- Work done by a conservative force depends only on the *endpoints* of the path



- If the force on the particle is non-conservative, different amounts of work will be done on $1 \rightarrow 2$ vs $2 \rightarrow 1$
- If the force is conservative, the work done on both paths is the same and total work over the round trip ($1 \rightarrow 2 \rightarrow 1$) is zero

$$\oint \mathbf{F}_P \cdot \mathcal{I} d\mathbf{r}_{P/O} = 0 \iff \mathbf{F}_P \text{ is conservative}$$

Closed Path Integral

Potential Energy

- It is useful to split total work as:

$$W_P^{(\text{tot})}(\mathbf{r}_{P/O}; \gamma_P) = \underbrace{W_P^{(c)}(t_1, t_2)}_{\text{Work due to conservative forces}} + \underbrace{W_P^{(nc)}(\mathbf{r}_{P/O}; \gamma_P)}_{\text{Work due to non-conservative forces}}$$

- The work due to conservative forces is how we define **potential energy**:

$$U_{P/O}^{\mathbf{F}_P^{(\text{cons})}}(\mathbf{r}_{P/O}) \triangleq - \int \mathbf{F}_P^{(\text{cons})} \cdot \mathcal{I} d\mathbf{r}_{P/O} \implies \mathbf{F}_P^{(\text{cons})} = -\nabla U_{P/O}^{\mathbf{F}_P^{(\text{cons})}}$$

- The change in potential energy is equal to the magnitude of the work done by conservative forces:

$$U_{P/O}(t_2) = U_{P/O}(t_1) - W_P^{(c)}(t_1, t_2)$$

Total Energy

- The **total energy** of a system is the sum of the kinetic and potential energy:

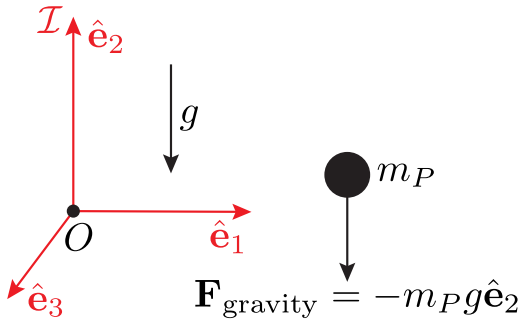
$$E_{P/O}(t) \triangleq T_{P/O}(t) + U_{P/O}(t)$$

- Recalling our previous definitions:

$$\begin{aligned} E_{P/O}(t_2) &= \overbrace{T_{P/O}(t_2)}^{T_{P/O}(t_1) + W_P^{\text{tot}}(t_1, t_2)} + \overbrace{U_{P/O}(t_2)}^{U_{P/O}(t_1) - W_P^{(c)}(t_1, t_2)} \\ &= \underbrace{T_{P/O}(t_1) + U_{P/O}(t_1)}_{E_{P/O}(t_1)} + \underbrace{W_P^{\text{tot}}(t_1, t_2) - W_P^{(c)}(t_1, t_2)}_{W_P^{(nc)}(\mathbf{r}_{P/O}; \gamma_P)} \end{aligned}$$

- Change in total energy is due to non-conservative forces *only*

Gravity



- Constant field forces (modeled as independent of mass location) are conservative
- An example of this is our gravity model for objects near the surface of the Earth
- g is approximated as constant so the force is constant at all positions

- The potential energy associated with this force is:

$$\begin{aligned} U_{P/O}^{\mathbf{F}_g}(\mathbf{r}_{P/O}) &= - \int \mathbf{F}_g \cdot \mathcal{I} d\mathbf{r}_{P/O} = - \int (-m_P g \hat{\mathbf{e}}_2) \cdot \mathcal{I} d(x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2 + z\hat{\mathbf{e}}_3) \\ &= m_P g \int \hat{\mathbf{e}}_2 \cdot \underbrace{\mathcal{I} d(y\hat{\mathbf{e}}_2)}_{\equiv dy\hat{\mathbf{e}}_2} = m_P g y + c \end{aligned}$$

Gravity Continued

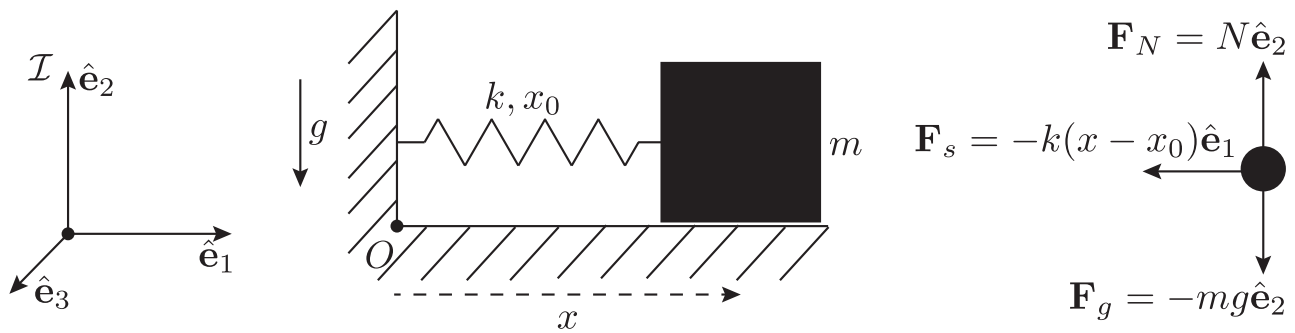
- NB: We get to choose the value of the constant of integration to make the potential energy behave the way we want it to (e.g. so that it equals zero at some convenient location)
- For gravity, we typically choose c such that $U_{P/O}^{\mathbf{F}_g}(\mathbf{r}_{P/O}) = 0$ for $y = 0$. Thus:

$$U_{P/O}^{\mathbf{F}_g}(\mathbf{r}_{P/O}) = m_P g y$$

- Potential due to constant gravity is proportional to height
- We can also check that we can recover the force from the potential:

$$\mathbf{F}_g = -\nabla U_{P/O}^{\mathbf{F}_g} = - \left[\frac{\partial (m_P g y)}{\partial x} \hat{\mathbf{e}}_1 + \frac{\partial (m_P g y)}{\partial y} \hat{\mathbf{e}}_2 + \frac{\partial (m_P g y)}{\partial z} \hat{\mathbf{e}}_3 \right] = -m_P g \hat{\mathbf{e}}_2$$

Linear Springs



- Define $x' \triangleq (x - x_0)$ such that $\mathbf{F}_s = -kx'\hat{\mathbf{e}}_1$
- $U_{P/O}^{\mathbf{F}_s}(\mathbf{r}_{P/O}) = - \int \mathbf{F}_s \cdot \mathcal{I} d\mathbf{r}_{P/O} = - \int (-kx'\hat{\mathbf{e}}_1) \cdot \mathcal{I} d((x' + x_0)\hat{\mathbf{e}}_1) = \frac{k(x')^2}{2} + c$

Linear Springs Continued

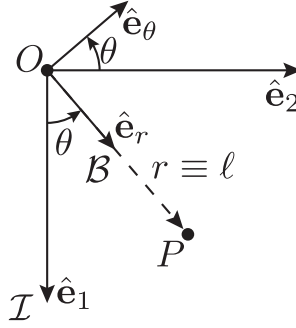
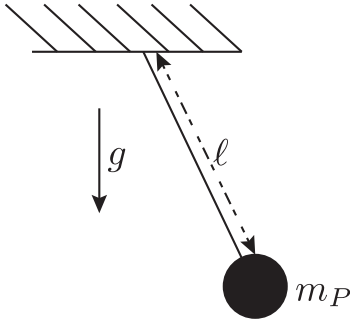
- Select c such that $U_{P/O}^{\mathbf{F}_s} = 0$ when $x' = 0$:

$$U_{P/O}^{\mathbf{F}_s}(\mathbf{r}_{P/O}) = \frac{1}{2}k(x - x_0)^2$$

- Checking the reverse operation:

$$\begin{aligned} \mathbf{F}_s &= -\nabla U_{P/O}^{\mathbf{F}_s} = - \left[\frac{\partial \left(\frac{k(x')^2}{2} \right)}{\partial x} \hat{\mathbf{e}}_1 + \frac{\partial \left(\frac{k(x')^2}{2} \right)}{\partial y} \hat{\mathbf{e}}_2 + \frac{\partial \left(\frac{k(x')^2}{2} \right)}{\partial z} \hat{\mathbf{e}}_3 \right] = -kx'\hat{\mathbf{e}}_1 \\ &= -k(x - x_0)\hat{\mathbf{e}}_1 \end{aligned}$$

Our Old Friend, the Simple Pendulum



$$\mathbf{F}_{\text{constraint}} = -F_c \hat{\mathbf{e}}_r$$

$$\mathbf{F}_{\text{gravity}} = m_P g (\cos(\theta) \hat{\mathbf{e}}_r - \sin \theta \hat{\mathbf{e}}_\theta)$$

$$\mathbf{r}_{P/O} = \ell \hat{\mathbf{e}}_r \Rightarrow \mathcal{I} \mathbf{v}_{P/O} = \ell \dot{\theta} \hat{\mathbf{e}}_\theta$$

$$T_{P/O} = \frac{m_P}{2} (\mathcal{I} \mathbf{v}_{P/O} \cdot \mathcal{I} \mathbf{v}_{P/O})$$

$$= \frac{m_P}{2} \ell^2 \dot{\theta}^2$$

- Remember: we get to choose *where* the gravitational potential energy is zero ($U_{P/O}^{\mathbf{F}_g} = m_P g y + c$)
- Let's pick a zero point at the height of the attachment point ($\theta = \pm \frac{\pi}{2}$)
 $U_{P/O}^{\mathbf{F}_g} = m_P g (\mathbf{r}_{P/O} \cdot -\hat{\mathbf{e}}_1) = -m_P g \ell \cos \theta$

The Simple Pendulum with Energy, Continued

- The constraint force is a normal force (perpendicular to the velocity) and therefore does no work (we know that it is normal because it is always in the $\hat{\mathbf{e}}_r$ direction while the velocity is always in the $\hat{\mathbf{e}}_\theta$ direction)
- Let's check anyway:

$$\int_{t_1}^{t_2} (-F_c \hat{\mathbf{e}}_r) \cdot \mathcal{I} d\mathbf{r}_{P/O} = \int_{t_1}^{t_2} (-F_c \hat{\mathbf{e}}_r) \cdot \underbrace{\mathcal{I} \mathbf{v}_{P/O} dt}_{\equiv \frac{\mathcal{I}}{dt} \mathbf{r}_{P/O} dt} = - \int_{t_1}^{t_2} F_c \ell \dot{\theta} (\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta) dt = 0$$

- There are no other forces so:
 $E_{P/O} = T_{P/O} + U_{P/O} = \frac{m_P}{2} \ell^2 \dot{\theta}^2 - m_P g \ell \cos \theta$
- All forces are conservative, and so $E_{P/O}$ is conserved for all time

Some More of the Simple Pendulum with Energy

- Let's take a look at the initial energy:

$$E_{P/O}(t_0) = \frac{1}{2}m_p\ell^2\dot{\theta}(t_0)^2 - m_pg\ell \cos \theta(t_0) = \text{constant}$$

- Solving for $\dot{\theta}^2$:

$$\dot{\theta}^2 = 2 \left(\frac{E_{P/O}(t_0)}{m_p\ell^2} + \frac{g}{\ell} \cos \theta \right)$$

- We now have a first order differential equation rather than a second order one, as we got from Newton's second law. Are they the same?

$$\frac{d}{dt}\dot{\theta}^2 = 2\ddot{\theta}\dot{\theta} = -2\frac{g}{\ell} \sin \theta \dot{\theta} \implies \ddot{\theta} = -\frac{g}{\ell} \sin \theta$$

Integrating the Simple Pendulum Energy Equation

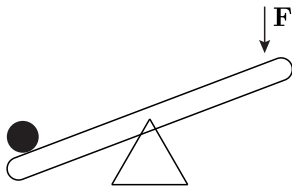
- Even more interestingly, we can directly integrate our $\dot{\theta}$ expression:

$$\frac{\dot{\theta}}{\sqrt{2 \left(\frac{E_{P/O}(0)}{m_p\ell^2} + \frac{g}{\ell} \cos \theta \right)}} = 1$$

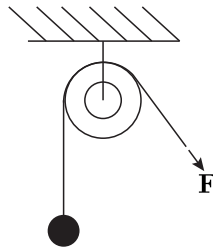
$$\int_0^t \frac{\dot{\theta} dt}{\sqrt{2 \left(\frac{E_{P/O}(0)}{m_p\ell^2} + \frac{g}{\ell} \cos \theta \right)}} = \int_0^t 1 dt \Rightarrow t = \int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\sqrt{2 \left(\frac{E_{P/O}(0)}{m_p\ell^2} + \frac{g}{\ell} \cos \theta \right)}}$$

- This is an example of an **elliptic integral**, and has known solutions (see also `ellipke` in MATLAB)

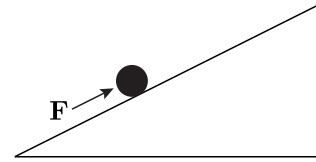
Simple Machines and Mechanical Advantage



Lever



Pulley

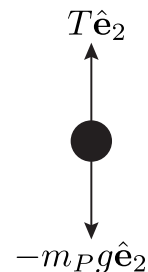
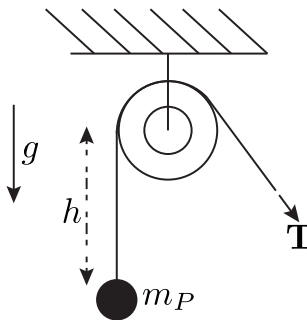
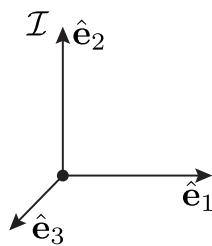


Inclined Plane (Ramp)

- A **simple machine** is a device that changes the magnitude and/or direction of a force
- Simple machines can produce **mechanical advantage** (force magnification)

$$MA = \frac{\|\mathbf{F}_{\text{out}}\|}{\|\mathbf{F}_{\text{in}}\|}$$

Pulleys



- A **stationary pulley** (one fixed in the inertial frame) is a simple machine that can change the direction of an applied force
- We model the pulley system by assuming the rope is massless and inelastic
- The rope directly transfers the tension force to the mass
- The rope acts as a position constraint: the mass moves up the exact distance the rope is pulled. **NB**: regardless of how the rope is pulled, the mass moves *only* in $\hat{e}_2$ (γ_P is a straight line)

Pulleys

- While the mass effectively has three degrees of freedom, our model means that it only moves in one dimension, and so we will only use a single coordinate to describe the kinematics:

$$\begin{aligned} \mathbf{r}_{P/O} = y\hat{\mathbf{e}}_2 &\implies \mathcal{I}_{\mathbf{a}_{P/O}} = \ddot{y}\hat{\mathbf{e}}_2 \\ (T - m_P g)\hat{\mathbf{e}}_y = m_P \ddot{y}\hat{\mathbf{e}}_y &\implies \ddot{y} = \frac{T}{m_P} - g \end{aligned}$$

Pulley Work

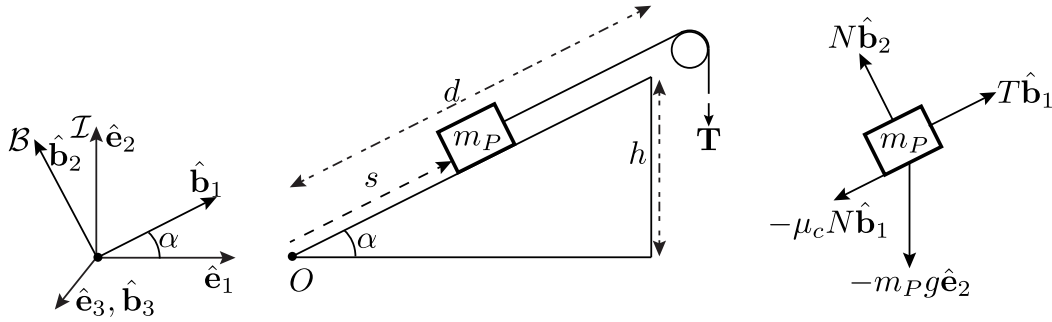
- If $T = m_P g$ then $\ddot{y} = 0 \implies \dot{y} = \text{constant} = \dot{y}(0) \triangleq v_0$
- In this case, the work associated with hoisting the mass a height h is

$$W_P^{(\mathbf{T})}(\mathbf{r}_{P/O}; \gamma_P) = \int_0^h m_P g \, dy = m_P g h$$

- The total work is $W_P^{(\text{tot})}(\mathbf{r}_{P/O}; \gamma_P) = \int_0^h (m_P g - m_P g) \, dy = 0$
- This is internally consistent, as $W_P^{(\text{tot})} = \Delta \text{KE}$ and the mass's velocity is constant, so KE does not change.
- Don't forget, the mass has to start moving somehow. We model this with an initial impulse $\bar{\mathbf{F}}_P$:

$$W_P^{\bar{\mathbf{F}}_P}(t_1, t_2) = T_{P/O}(t_2) - T_{P/O}(t_1) = \frac{m_P}{2} v_0^2 - 0 = \frac{m_P}{2} v_0^2$$

Combining Simple Machines



- Assume friction between the ramp and block, and that the block is pulled at a constant rate (${}^{\mathcal{I}}\mathbf{a}_{P/O} = \mathbf{0}$)

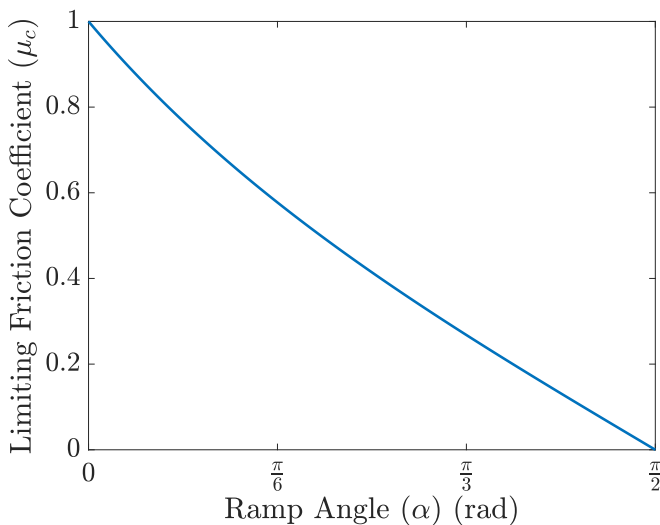
$$T\hat{\mathbf{b}}_1 - \mu_c N\hat{\mathbf{b}}_1 + N\hat{\mathbf{b}}_2 - m_P g (\sin(\alpha)\hat{\mathbf{b}}_1 + \cos(\alpha)\hat{\mathbf{b}}_2) = \mathbf{0} \begin{cases} N = m_P g \cos(\alpha) \\ T = m_P g (\cos(\alpha)\mu_c + \sin(\alpha)) \end{cases}$$

- We can achieve advantage over the pulley by itself if $T < m_P g$

$$m_P g (\cos(\alpha)\mu_c + \sin(\alpha)) < m_P g \Rightarrow \mu_c < \frac{1 - \sin \alpha}{\cos \alpha}$$

$$\text{MA} = \frac{\|\mathbf{F}_{\text{out}}\|}{\|\mathbf{F}_{\text{in}}\|} = \frac{m_P g}{m_P g (\cos(\alpha)\mu_c + \sin(\alpha))} = \frac{1}{\cos(\alpha)\mu_c + \sin(\alpha)}$$

Pulley + Inclined Plane Mechanical Advantage



- For a given value of the ramp angle (α), there is a limiting value of μ_c such that the mechanical advantage of the system is ≥ 1
- The decreased required force magnitude for this system comes at the cost of time: if the velocity of the block is the same as in the simple pulley case, then it will take longer to get the block to the same height

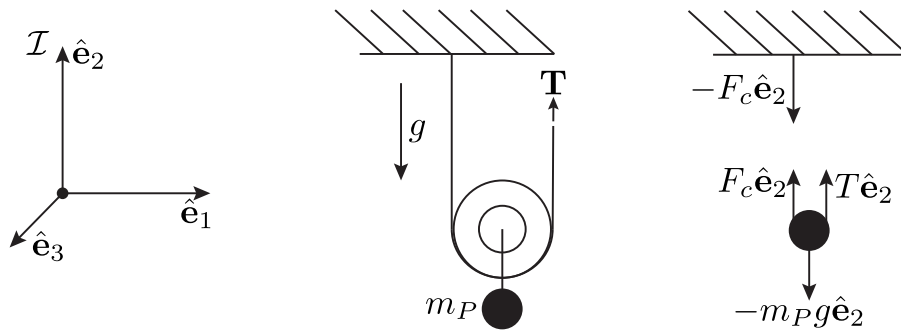
Pulley + Inclined Plane Work

- The work done by the tension force is:

$$\begin{aligned}
 W_P^{(\mathbf{T})}(\mathbf{r}_{P/O}; \gamma_P) &= \int_{\gamma_P} T \hat{\mathbf{b}}_1 \cdot \underbrace{\mathcal{I} d \mathbf{r}_{P/O}}_{= s \hat{\mathbf{b}}_1} = \int_0^d m_P g (\cos(\alpha) \mu_c + \sin(\alpha)) ds \\
 &= m_P g d (\cos(\alpha) \mu_c + \sin(\alpha)) = m_P g h \left(1 + \frac{\mu_c}{\tan(\alpha)} \right)
 \end{aligned}$$

- Smaller values of α require more work. As $\alpha \rightarrow 0$, $W_P^{(\mathbf{T})} \rightarrow \infty$

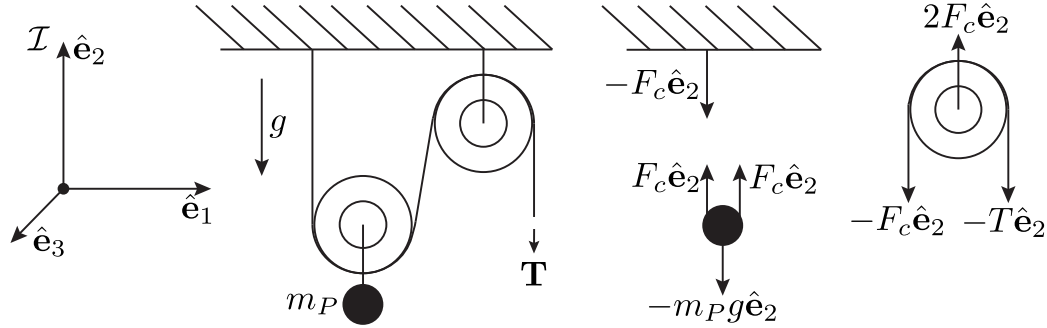
Movable Pulleys



- Another way of getting advantage out of pulleys is to allow them to move
- In this case, all forces are in the $\hat{\mathbf{e}}_2$ direction, so we can directly write:

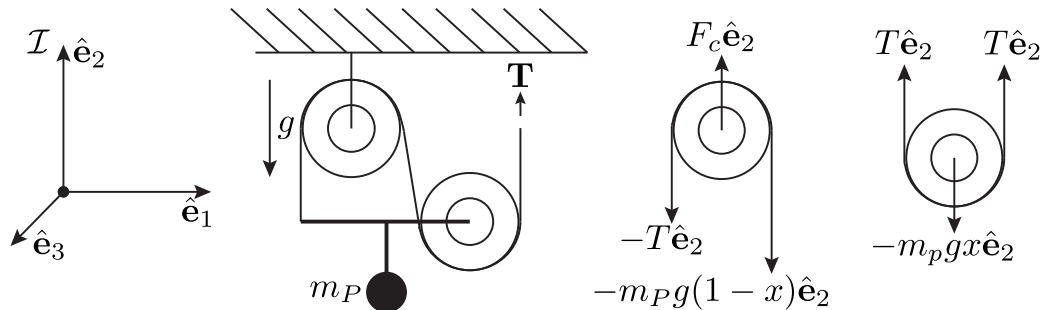
$$F_c + T = m_P g$$
- The tension force must be constant throughout the rope, so $F_c = T$
- Therefore: $T = \frac{m_P g}{2} \implies \text{MA} = \frac{m_P g}{m_P g/2} = 2$

Block and Tackle



- A **block and tackle** is a system of two or more pulleys connected by one rope
- We can add a stationary pulley to our movable one (this configuration is sometimes called a ‘gun tackle’)
- The stationary pulley only redirects the tension force, but provides no additional advantage
- The tension magnitude is still $T = \frac{m_P g}{2}$ so the mechanical advantage is still 2

Traveling Gun Tackle



- We can change the gun tackle setup to distribute the weight of the mass between the stationary and movable pulley
- Cannot assume that the weight is equally distributed, so introduce a variable x representing the fraction of weight on the movable pulley
- The tension in the rope is still constant so:

$$\left. \begin{array}{l} T + T = m_P g x \\ T = m_P g (1 - x) \end{array} \right\} 3T = m_P g \implies \text{MA} = 3$$

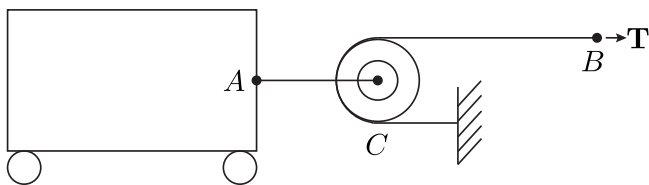
More on Tackle

- In general, the mechanical advantage of a tackle system is equal to the number of rope segments that support the load
- In theory: the more lengths you add, the higher the mechanical advantage
- In practice: real pulleys and ropes have mass and friction
- We can achieve a very general understanding of many simple machines by considering the transmission of power. Assuming no losses:

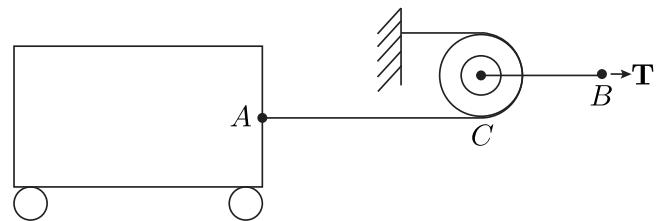
$$P_{\text{in}} = P_{\text{out}} \implies \mathbf{F}_{\text{in}} \cdot \mathcal{I} \mathbf{v}_{\text{in}} = \mathbf{F}_{\text{out}} \cdot \mathcal{I} \mathbf{v}_{\text{out}}$$

- If $\mathbf{F}$ is in the direction of the velocity (generally true for simple machines) then this is a scalar equation and tells us that if force is amplified then the motion is attenuated and vice versa

Carts + Pulleys



Force multiplied and motion attenuated: cart moves with $\frac{1}{2}$ speed



Force attenuated and motion multiplied: cart moves twice as fast as point B

Week 6

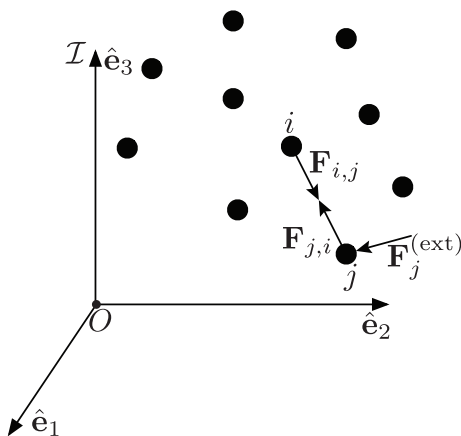
Dmitry Savransky

Cornell University

MAE 2030, Spring 2026

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Multiparticle Systems



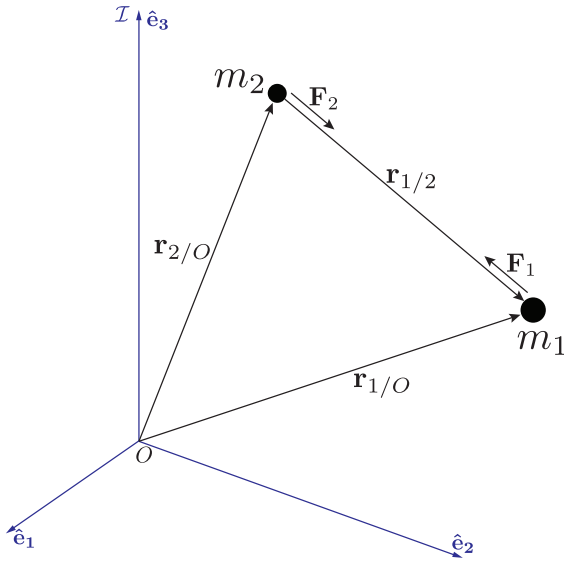
$$m_j {}^{\mathcal{I}}\mathbf{a}_{j/O} = \sum \mathbf{F}_j^{(\text{ext})} + \sum_{i=1}^N \mathbf{F}_{j,i}$$

- Remember: Newton's 2nd law applies only to point masses (particles)
- In a collection of multiple interacting masses, we can apply the 2nd law to each one
- Consider a collection of N particles
 - Each particle may have an **external force** acting on it (e.g. $\mathbf{F}_j^{(\text{ext})}$)
 - Pairs of particles may also force one another—these are **internal forces** (e.g. $\mathbf{F}_{j,i} = -\mathbf{F}_{i,j}$)
 - A particle cannot force itself, so $\mathbf{F}_{i,i} = \mathbf{0} \quad \forall i$

Recall Newton's Law of Gravity

- In general:

$$\mathbf{F}_{j,i} = -\frac{Gm_i m_j}{\|\mathbf{r}_{j/i}\|^3} \mathbf{r}_{j/i}$$

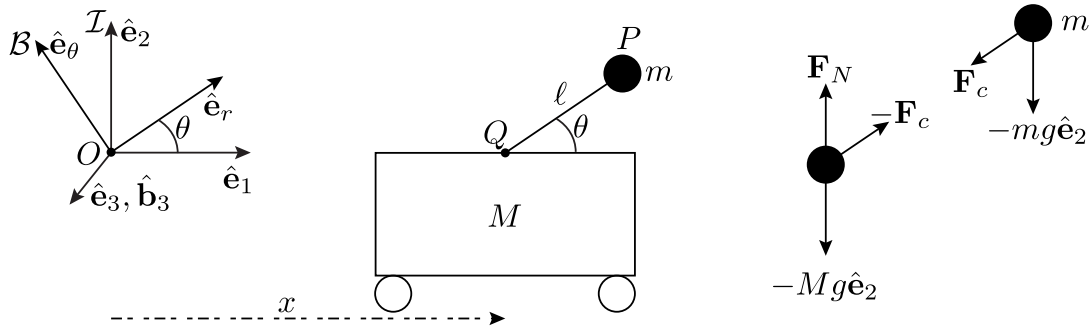


- A system of N bodies interacting only due to gravity can be modeled as a collection of particles with only internal forces
- For each one, we get a differential equation of the form:

$$m_j \frac{d^2}{dt^2} \mathbf{r}_{j/O} = - \sum_{i=1}^N \frac{Gm_i m_j}{\|\mathbf{r}_{j/O} - \mathbf{r}_{i/O}\|^3} \mathbf{r}_{j/i}$$

- To solve this system, you need to numerically integrate N coupled ODEs

Pendulum on a Cart



- Constraints between particles can be treated as internal forces
- Consider an inverted simple pendulum attached to a cart. We model the pendulum link as massless, and the cart as sliding without friction
- $\mathcal{I} \triangleq (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ is inertial; $\mathcal{B} \triangleq (\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$ is non-inertial, with $\hat{\mathbf{e}}_r \equiv \hat{\mathbf{r}}_{P/Q}$
- $\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\hat{\mathbf{e}}_3$

$$\begin{aligned} \mathbf{r}_{Q/O} &= x\hat{\mathbf{e}}_1 & \mathbf{r}_{P/O} &= x\hat{\mathbf{e}}_1 + \ell\hat{\mathbf{e}}_r \\ \mathcal{I}\mathbf{v}_{Q/O} &= \dot{x}\hat{\mathbf{e}}_1 & \mathcal{I}\mathbf{v}_{P/O} &= \dot{x}\hat{\mathbf{e}}_1 + \ell\dot{\theta}\hat{\mathbf{e}}_\theta \\ \mathcal{I}\mathbf{a}_{Q/O} &= \ddot{x}\hat{\mathbf{e}}_1 & \mathcal{I}\mathbf{a}_{P/O} &= \ddot{x}\hat{\mathbf{e}}_1 + \ell\ddot{\theta}\hat{\mathbf{e}}_\theta - \ell\dot{\theta}^2\hat{\mathbf{e}}_r \end{aligned}$$

$$\begin{aligned} -Mg\hat{\mathbf{e}}_2 + F_N\hat{\mathbf{e}}_2 + F_c\hat{\mathbf{e}}_r &= M\ddot{x}\hat{\mathbf{e}}_1 \\ -mg\hat{\mathbf{e}}_2 - F_c\hat{\mathbf{e}}_r &= \\ m \left(\ddot{x}\hat{\mathbf{e}}_1 + \ell\ddot{\theta}\hat{\mathbf{e}}_\theta - \ell\dot{\theta}^2\hat{\mathbf{e}}_r \right) \end{aligned}$$

Pendulum on a Cart

$$-Mg\hat{\mathbf{e}}_2 + F_N\hat{\mathbf{e}}_2 + F_c(\cos(\theta)\hat{\mathbf{e}}_1 + \sin(\theta)\hat{\mathbf{e}}_2) = M\ddot{x}\hat{\mathbf{e}}_1$$

$$-mg(\sin(\theta)\hat{\mathbf{e}}_r + \cos(\theta)\hat{\mathbf{e}}_\theta) - F_c\hat{\mathbf{e}}_r = m\left(\ddot{x}(\cos(\theta)\hat{\mathbf{e}}_r - \sin(\theta)\hat{\mathbf{e}}_\theta) + \ell\ddot{\theta}\hat{\mathbf{e}}_\theta - \ell\dot{\theta}^2\hat{\mathbf{e}}_r\right)$$

$$\left. \begin{aligned} -Mg + F_N + F_c \sin(\theta) &= 0 \\ F_c \cos(\theta) &= M\ddot{x} \\ -mg \sin(\theta) - F_c &= m\left(\ddot{x} \cos(\theta) - \ell\dot{\theta}^2\right) \\ -mg \cos \theta &= m\left(-\ddot{x} \sin(\theta) + \ell\ddot{\theta}\right) \end{aligned} \right\} \begin{aligned} F_c &= \frac{M\ddot{x}}{\cos(\theta)} \\ \ddot{x} &= \frac{m\left(\ell\dot{\theta}^2 - g \sin(\theta)\right) \cos(\theta)}{M + m \cos^2(\theta)} \\ \ddot{\theta} &= -\frac{g \cos(\theta)}{\ell} + \frac{m\left(\ell\dot{\theta}^2 - g \sin(\theta)\right) \sin(\theta) \cos(\theta)}{\ell(M + m \cos^2(\theta))} \end{aligned}$$

Linear Momentum of Collections of Particles

- Recall that we can write the change in linear momentum due to a force as:

$$m_P \mathcal{I} \mathbf{v}_{P/O}(t_2) = m_P \mathcal{I} \mathbf{v}_{P/O}(t_1) + \underbrace{\int_{t_1}^{t_2} \mathbf{F}_P dt}_{\bar{\mathbf{F}}_P(t_1, t_2)}$$

- For particle j of a collection of N particles:

$$m_j \mathcal{I} \mathbf{v}_{j/O}(t_2) = m_j \mathcal{I} \mathbf{v}_{j/O}(t_1) + \bar{\mathbf{F}}_j^{(\text{ext})}(t_1, t_2) + \sum_{i=1}^N \bar{\mathbf{F}}_{j,i}(t_1, t_2)$$

- This still gives us N equations, and so is not a great improvement. What if we sum over all of the particles to get the **total linear momentum**:

$$\sum_{j=1}^N m_j \mathcal{I} \mathbf{v}_{j/O} \triangleq \mathcal{I} \mathbf{p}_O$$

Total Linear Momentum in Time

- How does the total linear momentum change in time?

$$\mathcal{I}\mathbf{p}_O(t_2) = \mathcal{I}\mathbf{p}_O(t_1) + \sum_{j=1}^N \bar{\mathbf{F}}_j^{(\text{ext})}(t_1, t_2) + \underbrace{\sum_{j=1}^N \sum_{i=1}^N \bar{\mathbf{F}}_{j,i}(t_1, t_2)}_{=0}$$

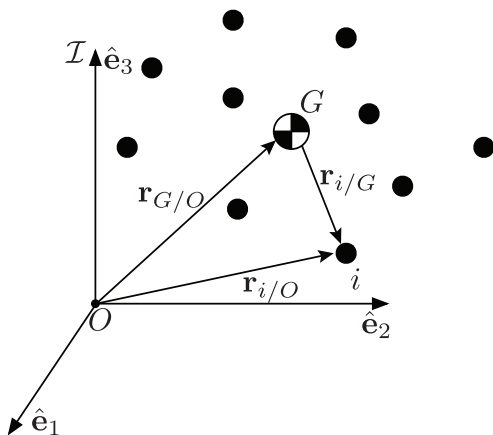
- Remember: $\mathbf{F}_{j,i} = -\mathbf{F}_{i,j}$, so summing over all internal forces cancels them out
- We define $\bar{\mathbf{F}}^{(\text{ext})}(t_1, t_2) \triangleq \sum \bar{\mathbf{F}}_j^{(\text{ext})}(t_1, t_2)$ so we can write:

$$\mathcal{I}\mathbf{p}_O(t_2) = \mathcal{I}\mathbf{p}_O(t_1) + \bar{\mathbf{F}}^{(\text{ext})}(t_1, t_2)$$

- Collections of particles obey Newton's 2nd law! Similarly, collections obey momentum conservation:

$$\bar{\mathbf{F}}^{(\text{ext})}(t_1, t_2) = 0 \Leftrightarrow \mathcal{I}\mathbf{p}_O(t_2) = \mathcal{I}\mathbf{p}_O(t_1)$$

Center of Mass (COM)



$$\mathbf{r}_{G/O} \triangleq \frac{1}{m_G} \sum_{i=1}^N m_i \mathbf{r}_{i/O} \text{ where } m_G \triangleq \sum_{i=1}^N m_i$$

- Consider particle i : $\mathbf{r}_{i/O} = \mathbf{r}_{G/O} + \mathbf{r}_{i/G}$
- Rewrite the COM definition as:

$$\begin{aligned} m_G \mathbf{r}_{G/O} &= \sum_{i=1}^N m_i \mathbf{r}_{i/O} = \sum_{i=1}^N m_i (\mathbf{r}_{i/G} + \mathbf{r}_{G/O}) \\ &= \sum_{i=1}^N m_i \mathbf{r}_{i/G} + \underbrace{\sum_{i=1}^N m_i \mathbf{r}_{G/O}}_{\equiv m_G \mathbf{r}_{G/O}} \end{aligned}$$

- The **center of mass** of a collection of particles is the mass-weighted average of all particle positions

This is the **center of mass corollary**:

$$\sum_{i=1}^N m_i \mathbf{r}_{i/G} = \mathbf{0}$$

Newton's 2nd Law for Collections of Particles

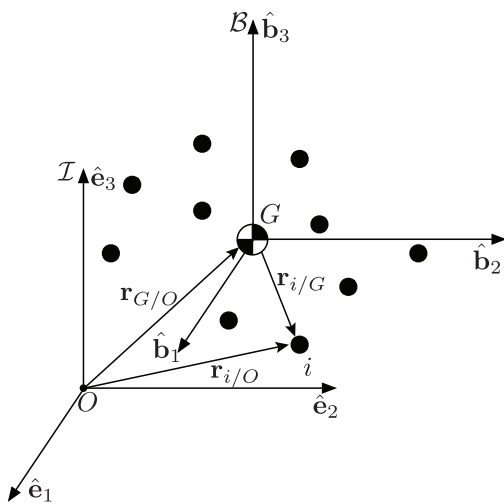
$$\begin{aligned} \sum_{i=1}^N m_i \frac{\mathcal{I} d^2}{dt^2} \overbrace{(\mathbf{r}_{i/G} + \mathbf{r}_{G/O})}^{\mathbf{r}_{i/O}} &= \sum_{i=1}^N \mathbf{F}_i^{(\text{ext})} \\ \Rightarrow \underbrace{\sum_{i=1}^N m_i \frac{\mathcal{I} d^2}{dt^2} \mathbf{r}_{G/O}}_{m_G \frac{\mathcal{I} d^2}{dt^2} \mathbf{r}_{G/O}} + \underbrace{\sum_{i=1}^N m_i \frac{\mathcal{I} d^2}{dt^2} \mathbf{r}_{i/G}}_{\frac{\mathcal{I} d^2}{dt^2} \left(\underbrace{\sum_{i=1}^N m_i \mathbf{r}_{i/G}}_{\equiv \mathbf{0}} \right)} &= \underbrace{\sum_{i=1}^N \mathbf{F}_i^{(\text{ext})}}_{\triangleq \mathbf{F}_G^{(\text{ext})}} \end{aligned}$$

Newton's 2nd Law for Collections of Particles

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{p}_G = \mathbf{F}_G^{(\text{ext})} \text{ where } {}^{\mathcal{I}}\mathbf{p}_G \triangleq m_G \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \mathbf{r}_{G/O}$$

- Newton's 2nd Law for a collection of particles is describing a particle with the mass of the whole collection, located at the collection's center of mass
- This is what makes the center of mass special
- We can now treat many particles as a single point mass (and in the future will show that this is also true for extended bodies)

Motion Relative to the Center of Mass



We attach a moving (but *not* rotating) frame $\mathcal{B}$ to the COM

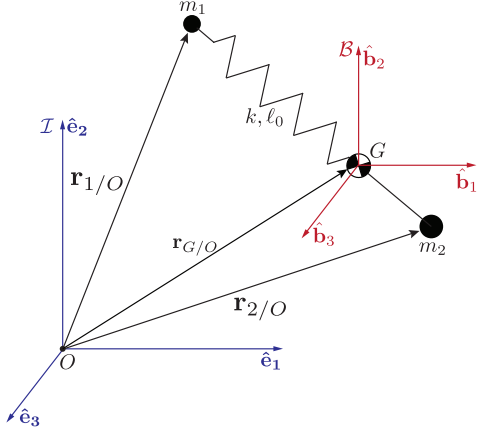
- Recall: $m_P {}^{\mathcal{B}}\mathbf{a}_{P/O'} = \sum \mathbf{F}_P - m_P {}^{\mathcal{I}}\mathbf{a}_{O'/O}$
- Thus, for every particle, we can write:

$$m_i {}^{\mathcal{B}}\mathbf{a}_{i/G} = \mathbf{F}_i^{(\text{ext})} + \sum_{j=1}^N \mathbf{F}_{i,j} - m_i {}^{\mathcal{I}}\mathbf{a}_{G/O}$$

- If $\mathbf{F}_i^{(\text{ext})} = \mathbf{0} \forall i$ then ${}^{\mathcal{I}}\mathbf{a}_{G/O} = \mathbf{0}$
- In the case of zero external forces:

$${}^{\mathcal{B}}\mathbf{a}_{i/G} = \frac{1}{m_i} \sum_{j=1}^N \mathbf{F}_{i,j}$$

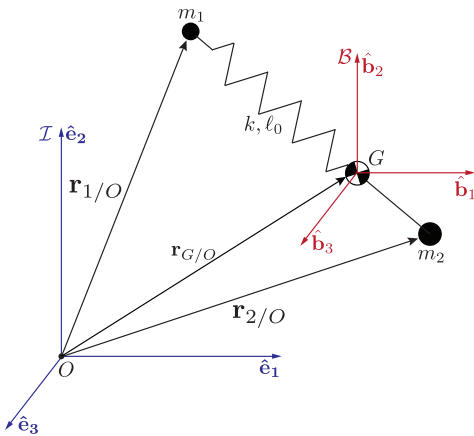
Two-Mass-Spring System



- Consider two masses connected by a spring that cannot bend or flex, sliding without friction on a flat surface
- The center of mass G always lies on the line between m_1 and m_2

- By our new version of Newton's 2nd law: $\frac{\mathcal{I}}{dt^2} \mathbf{r}_{G/O} = \frac{\mathbf{F}_G^{(\text{ext})}}{m_G}$
- Apply impulse $\bar{\mathbf{F}}_1(t_1, t_2)$ to m_1 :
 $m_G \mathcal{I} \mathbf{v}_{G/O}(t \geq t_2) = \bar{\mathbf{F}}_1(t_1, t_2) + m_G \mathcal{I} \mathbf{v}_{G/O}(t_1)$
- The COM travels at constant velocity

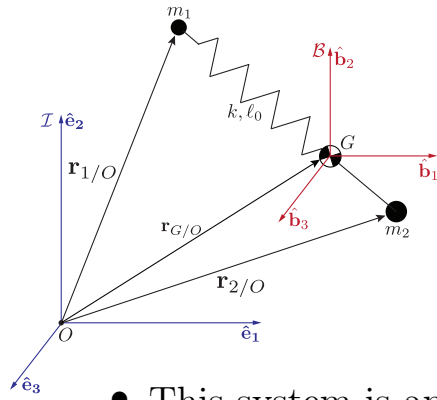
Two-Mass-Spring System: Motion of the COM



$$\begin{aligned} {}^B \mathbf{a}_{i/G} &= \frac{1}{m_i} \left(\mathbf{F}_i^{(\text{ext})} + \sum_{j=1}^N \mathbf{F}_{i,j} \right) - \mathcal{I} \mathbf{a}_{G/O} \\ {}^B \mathbf{a}_{1/G} &= \frac{\mathbf{F}_{1,2}}{m_1} \\ &= -\frac{k}{m_1} (\|\mathbf{r}_{1/G} - \mathbf{r}_{2/G}\| - \ell_0) \hat{\mathbf{r}}_{1/2} \\ {}^B \mathbf{a}_{2/G} &= \frac{\mathbf{F}_{2,1}}{m_2} \\ &= -\frac{k}{m_2} (\|\mathbf{r}_{1/G} - \mathbf{r}_{2/G}\| - \ell_0) \hat{\mathbf{r}}_{2/1} \end{aligned}$$

$$\hat{\mathbf{r}}_{1/2} = -\hat{\mathbf{r}}_{2/1} = \frac{\mathbf{r}_{1/O} - \mathbf{r}_{2/O}}{\|\mathbf{r}_{1/O} - \mathbf{r}_{2/O}\|} = \frac{\mathbf{r}_{1/G} - \mathbf{r}_{2/G}}{\|\mathbf{r}_{1/G} - \mathbf{r}_{2/G}\|}$$

Two-Mass-Spring System: Motion of the Particles



$$\begin{aligned} \mathbf{r}_{2/1} &= \mathbf{r}_{2/G} - \mathbf{r}_{1/G} \Rightarrow \frac{{}^B d^2}{dt^2} \mathbf{r}_{2/1} = {}^B \mathbf{a}_{2/G} - {}^B \mathbf{a}_{1/G} \\ &= [(\|\mathbf{r}_{2/1}\| - \ell_0) \hat{\mathbf{r}}_{2/1}] \left(\underbrace{-\frac{k}{m_2} - \frac{k}{m_1}}_{-\frac{k(m_1 + m_2)}{m_1 m_2}} \right) \end{aligned}$$

- This system is an SHO with $\omega_0 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$. **Reduced mass:**

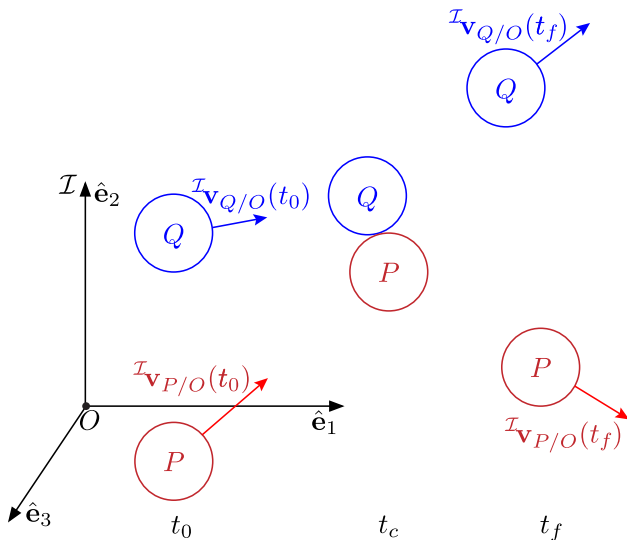
$$\frac{m_1 m_2}{m_1 + m_2}$$

- We initially set m_1 in motion via a linear impulse: $\mathcal{I}_{\mathbf{v}_{1/O}}(t_2) = \frac{\bar{\mathbf{F}}_1(t_1, t_2)}{m_1}$
- $\mathcal{I}_{\mathbf{v}_{2/O}}(t_2) = \mathbf{0}$ so $\mathcal{I}_{\mathbf{v}_{2/1}}(t_2) = \mathcal{I}_{\mathbf{v}_{2/O}}(t_2) - \mathcal{I}_{\mathbf{v}_{1/O}}(t_2) = -\frac{\bar{\mathbf{F}}_1(t_1, t_2)}{m_1}$

Modeling Impacts and Collisions

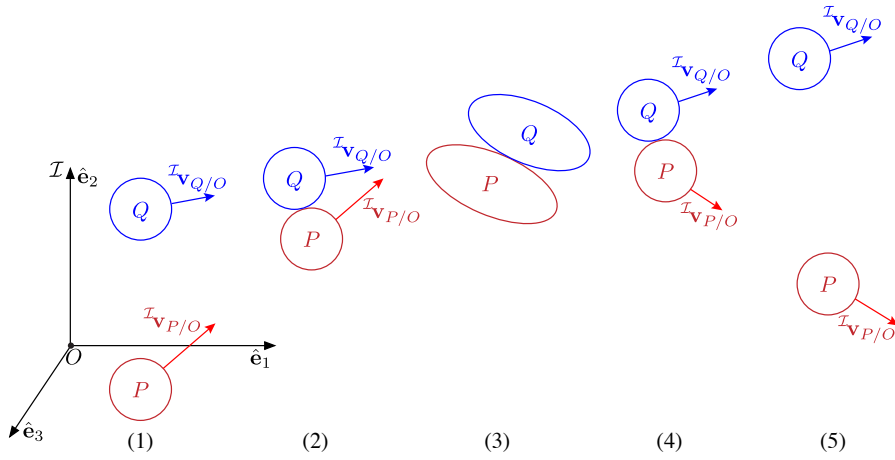
Fundamental question: Given the velocities of two masses before a collision, what will be their velocities afterwards?

- Collisions are incredibly complex. We are going to need a lot of assumptions to model them:



- The collision occurs over a very short time interval (instantaneous velocity changes)
- Total linear momentum is conserved (no external forces during collision)
- Infinitesimal particles can 'collide' (collision is overlap in position)
- No frictional forces between colliding bodies (unless the bodies stick together)

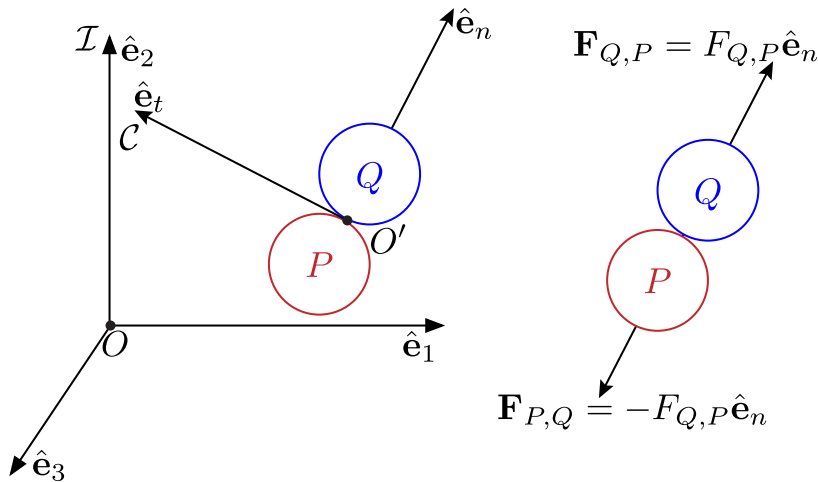
Collision Steps



- 1 Pre-impact
($t < t_1$)
- 2 Impact, deformation
(compression)
($t_1 < t < t_c$)
- 3 Compressed
($t = t_c$)
- 4 Restitution
(decompression)
($t_c < t < t_2$)
- 5 Post-impact
($t > t_2$)

- NB: A collision can change *both* the magnitude and direction of the velocities of *all* colliding bodies

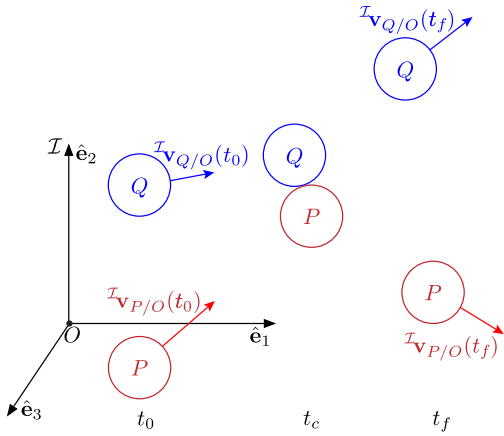
Collision Frame



- Define a **collision frame**:
 $C \triangleq (O', \hat{e}_n, \hat{e}_t, \hat{e}_3)$
- O' is the impact point
- $\hat{e}_n \triangleq \frac{\mathcal{I}_{\mathbf{v}_{P/Q}}(t_1)}{\|\mathcal{I}_{\mathbf{v}_{P/Q}}(t_1)\|} = \mathcal{I}_{\hat{\mathbf{v}}_{P/Q}}(t_1)$
- NB: The collision frame is *not* the path frame

- The internal force of the collision acts along $\hat{e}_n$ to push the particles apart
- The particles impart an equal and opposite linear impulse $\bar{\mathbf{F}}_{P,Q}(t_1, t_2)$ on one another

Linear Momentum is Conserved in Collisions



$$m_P \mathcal{I} \mathbf{v}_{P/O}(t_2) = m_P \mathcal{I} \mathbf{v}_{P/O}(t_1) - \bar{\mathbf{F}}_{Q,P}(t_1, t_2)$$

$$m_Q \mathcal{I} \mathbf{v}_{Q/O}(t_2) = m_Q \mathcal{I} \mathbf{v}_{Q/O}(t_1) + \bar{\mathbf{F}}_{Q,P}(t_1, t_2)$$

- We split the velocities into components in the collision frame:

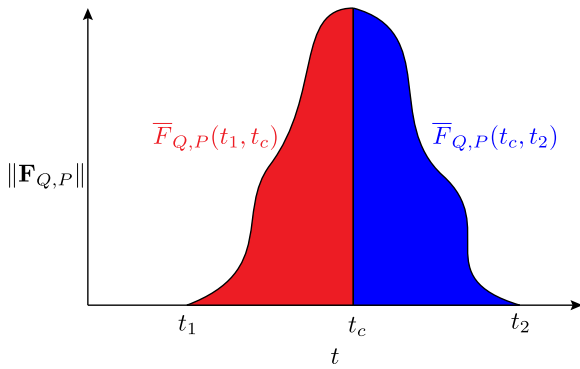
$$\mathcal{I} \mathbf{v}_{P/O} \triangleq v_P \hat{\mathbf{e}}_n + u_P \hat{\mathbf{e}}_t$$

$$\mathcal{I} \mathbf{v}_{Q/O} \triangleq v_Q \hat{\mathbf{e}}_n + u_Q \hat{\mathbf{e}}_t$$

- The impulse only acts in the $\hat{\mathbf{e}}_n$ direction, so there are no changes to the velocity in $\hat{\mathbf{e}}_t$: $u_P(t_2) \equiv u_P(t_1)$ and $u_Q(t_2) \equiv u_Q(t_1)$

$$\left. \begin{aligned} m_P v_P(t_2) &= m_P v_P(t_1) - \bar{F}_{Q,P} \\ m_Q v_Q(t_2) &= m_Q v_Q(t_1) + \bar{F}_{Q,P} \end{aligned} \right\} m_P v_P(t_2) + m_Q v_Q(t_2) = m_P v_P(t_1) + m_Q v_Q(t_1)$$

Coefficient of Restitution



- The distribution of the final velocity between P and Q is determined by the magnitude of the interaction forces during the deformation and restoration phases
- Define the **coefficient of restitution**:

$$e \triangleq \frac{\bar{F}_{Q,P}(t_c, t_2)}{\bar{F}_{Q,P}(t_1, t_c)}$$

- e is a property of materials and a measure of elasticity:
 - $e = 1$ **Elastic Collision**: all deformed momentum is returned
 - $e = 0$ **Inelastic (Plastic) Collision**: all normal momentum is lost and the particles stick together
 - $0 < e < 1$ Normal velocities are reduced (energy is lost)

$$e = \frac{\overline{F}_{Q,P}(t_c, t_2)}{\overline{F}_{Q,P}(t_1, t_c)} = \frac{v_P(t_2) - v_P(t_c)}{v_P(t_c) - v_P(t_1)} = \frac{v_Q(t_2) - v_Q(t_c)}{v_Q(t_c) - v_Q(t_1)} \implies$$

$$e = \frac{v_P(t_2) - v_Q(t_2)}{v_Q(t_1) - v_P(t_1)}$$

Solving Collision Problems

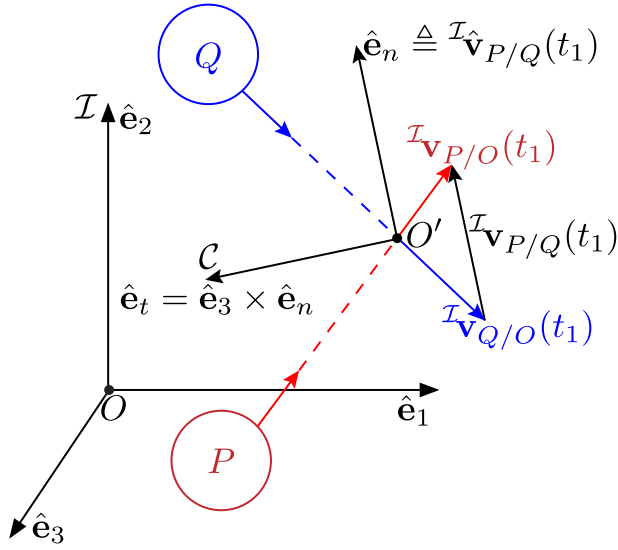
- Given initial velocities and the value of e , we can find the final velocities:

$$\begin{aligned} \mathcal{I}_{\mathbf{v}_{P/O}}(t_2) &= u_P(t_1)\hat{\mathbf{e}}_t + \left(\frac{m_P - em_Q}{m_P + m_Q}v_P(t_1) + \frac{m_Q(1+e)}{m_P + m_Q}v_Q(t_1) \right) \hat{\mathbf{e}}_n \\ \mathcal{I}_{\mathbf{v}_{Q/O}}(t_2) &= u_Q(t_1)\hat{\mathbf{e}}_t + \left(\frac{m_Q - em_P}{m_P + m_Q}v_Q(t_1) + \frac{m_P(1+e)}{m_P + m_Q}v_P(t_1) \right) \hat{\mathbf{e}}_n \end{aligned}$$

- If $e = 0$, our original fourth assumption is broken (no friction), so these equations do not hold. Instead, we use conservation of momentum and the assumption that the particles have the same velocity post-impact:

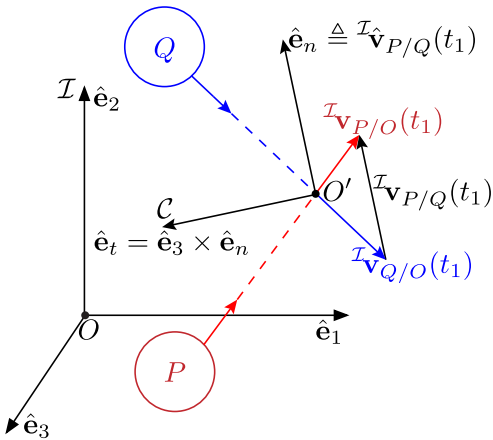
$$\mathcal{I}_{\mathbf{v}_{PQ/O}}(t_2) = \mathcal{I}_{\mathbf{v}_{P/O}}(t_2) = \mathcal{I}_{\mathbf{v}_{Q/O}}(t_2) = \frac{m_P \mathcal{I}_{\mathbf{v}_{P/O}}(t_1) + m_Q \mathcal{I}_{\mathbf{v}_{Q/O}}(t_1)}{m_P + m_Q}$$

Oblique Collisions



- An **oblique collision** is one where the initial velocity vectors of the two colliding bodies are not aligned (in contrast to a ‘head-on collision’)
- We define an inertial frame $\mathcal{I} = (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ such that the particle velocities immediately prior to the collision lie in the $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ plane
- The collision frame $\mathcal{C} = (\hat{\mathbf{e}}_n, \hat{\mathbf{e}}_t, \hat{\mathbf{e}}_3)$ is defined such that $\hat{\mathbf{e}}_n \triangleq \mathcal{I}\hat{\mathbf{v}}_{P/Q}(t_1)$

Oblique Collisions Continued



- We assume constant velocities immediately prior to the collision and define:

$$\mathcal{I}\mathbf{v}_{P/O}(t < t_1) \triangleq \dot{x}_P \hat{\mathbf{e}}_1 + \dot{y}_P \hat{\mathbf{e}}_2$$

$$\mathcal{I}\mathbf{v}_{Q/O}(t < t_1) \triangleq \dot{x}_Q \hat{\mathbf{e}}_1 + \dot{y}_Q \hat{\mathbf{e}}_2$$

$$\hat{\mathbf{e}}_n = \frac{\mathcal{I}\mathbf{v}_{P/Q}(t_1)}{\|\mathcal{I}\mathbf{v}_{P/Q}(t_1)\|} = \frac{\dot{x}_{P/Q} \hat{\mathbf{e}}_1 + \dot{y}_{P/Q} \hat{\mathbf{e}}_2}{\sqrt{\dot{x}_{P/Q}^2 + \dot{y}_{P/Q}^2}}$$

$$\text{where } \dot{x}_{P/Q} \triangleq \dot{x}_{P/O} - \dot{x}_{Q/O},$$

$$\dot{y}_{P/Q} \triangleq \dot{y}_{P/O} - \dot{y}_{Q/O}$$

$$\hat{\mathbf{e}}_t = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_n = \frac{\dot{x}_{P/Q} \hat{\mathbf{e}}_2 - \dot{y}_{P/Q} \hat{\mathbf{e}}_1}{\dot{s}} \quad \text{where } \dot{s} \triangleq \sqrt{\dot{x}_{P/Q}^2 + \dot{y}_{P/Q}^2}$$

$$\hat{\mathbf{e}}_1 = \frac{1}{\dot{s}} (\dot{x}_{P/Q} \hat{\mathbf{e}}_n - \dot{y}_{P/Q} \hat{\mathbf{e}}_t) \quad \hat{\mathbf{e}}_2 = \frac{1}{\dot{s}} (\dot{y}_{P/Q} \hat{\mathbf{e}}_n + \dot{x}_{P/Q} \hat{\mathbf{e}}_t)$$

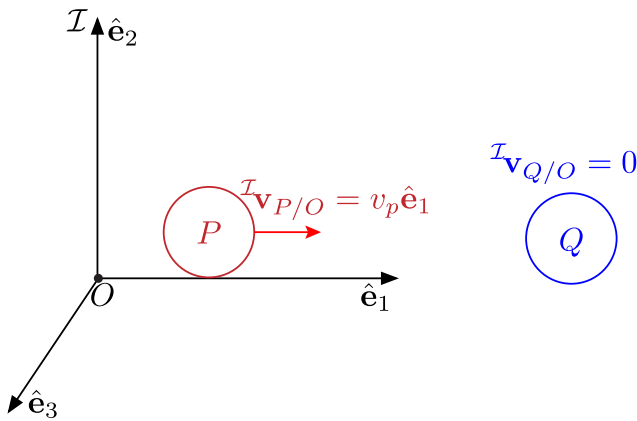
Oblique Collision Velocities

- We can now express the particle velocities at t_1 in terms of our initial conditions and in collision frame components:

$$\begin{aligned}\mathcal{I}\mathbf{v}_{P/O}(t_1) &= \underbrace{\frac{(\dot{x}_{P/Q}\dot{x}_P + \dot{y}_{P/Q}\dot{y}_P)}{\dot{s}}}_{v_P(t_1)} \hat{\mathbf{e}}_n + \underbrace{\frac{(\dot{x}_{P/Q}\dot{y}_P - \dot{y}_{P/Q}\dot{x}_P)}{\dot{s}}}_{u_P(t_1)} \hat{\mathbf{e}}_t \\ \mathcal{I}\mathbf{v}_{Q/O}(t_1) &= \underbrace{\frac{(\dot{x}_{P/Q}\dot{x}_Q + \dot{y}_{P/Q}\dot{y}_Q)}{\dot{s}}}_{v_Q(t_1)} \hat{\mathbf{e}}_n + \underbrace{\frac{(\dot{x}_{P/Q}\dot{y}_Q - \dot{y}_{P/Q}\dot{x}_Q)}{\dot{s}}}_{u_Q(t_1)} \hat{\mathbf{e}}_t\end{aligned}$$

- Recall that the $\hat{\mathbf{e}}_t$ components will not change: the collision impacts only the $\hat{\mathbf{e}}_n$ components of velocity
- Given a value for the coefficient of restitution, we now have all of the information we need to compute the final velocities

Head-on Collision with Stationary Particle



- Particle P travels at a constant velocity $v_P(t_1)\hat{\mathbf{e}}_1$ and hits particle Q , which is stationary
- The inertial and collision frames are aligned: $\hat{\mathbf{e}}_n \equiv \hat{\mathbf{e}}_1$
- There is no motion at all in $\hat{\mathbf{e}}_t$

$$\mathcal{I}\mathbf{v}_{P/O}(t_2) = \frac{m_P - em_Q}{m_P + m_Q} v_P(t_1) \hat{\mathbf{e}}_n$$

$$\mathcal{I}\mathbf{v}_{Q/O}(t_2) = \frac{m_P(1+e)}{m_P + m_Q} v_P(t_1) \hat{\mathbf{e}}_n$$

If $m_P = m_Q \Rightarrow$

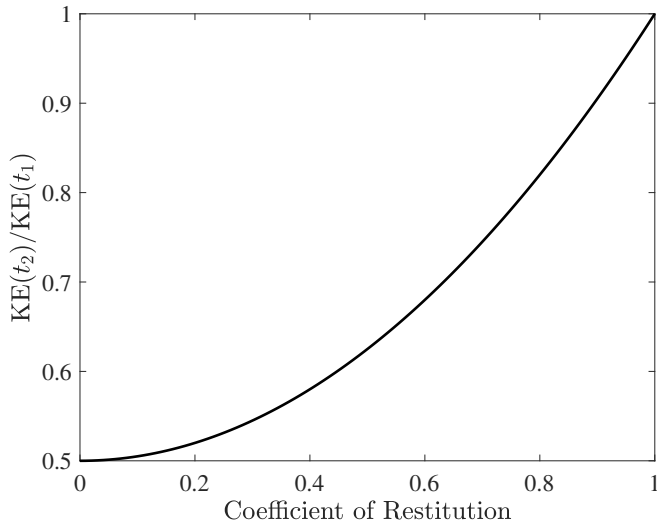
$$\mathcal{I}\mathbf{v}_{P/O}(t_2) = \frac{1-e}{2} v_P(t_1) \hat{\mathbf{e}}_n$$

$$\mathcal{I}\mathbf{v}_{Q/O}(t_2) = \frac{1+e}{2} v_P(t_1) \hat{\mathbf{e}}_n$$

KE of Head-on Collision with Stationary Particle

For $m_P = m_Q$: $\text{KE}(t_1) = \frac{m_P}{2} v_P^2(t_1)$,

$$\text{KE}(t_2) = \left[\left(\frac{1-e}{2} \right)^2 + \left(\frac{1+e}{2} \right)^2 \right] \frac{m_P}{2} v_P^2(t_1)$$

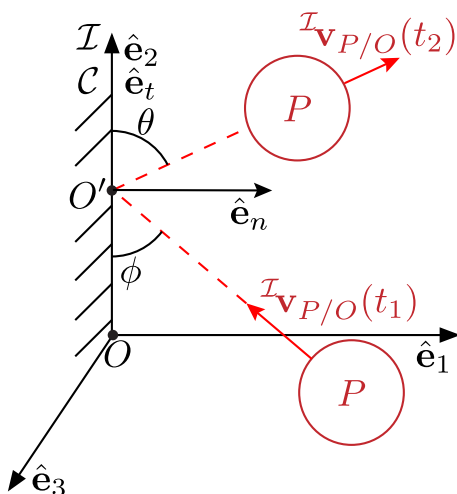


- When $e = 1$, there is a total energy transfer from P to Q (P stops and Q moves)
- When $e = 0$, the particles stick together and half the energy is lost:

$$\begin{aligned} \mathcal{I}_{\mathbf{v}_{PQ/O}}(t_2) &= \frac{m_P \mathcal{I}_{\mathbf{v}_{P/O}}(t_1) + m_Q \mathcal{I}_{\mathbf{v}_{Q/O}}(t_1)}{m_P + m_Q} \\ &= \frac{v_P(t_1)}{2} \hat{\mathbf{e}}_n \end{aligned}$$

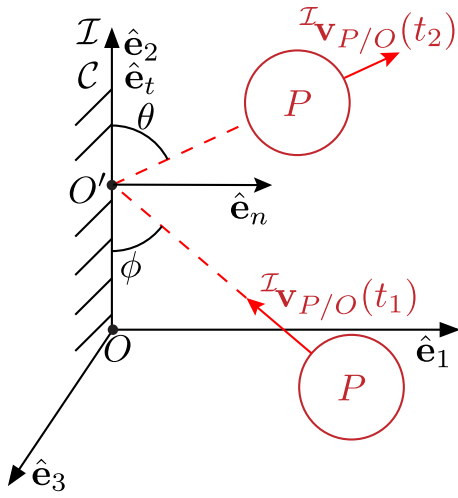
Modeling Collisions with Surfaces

Fundamental question: Given the velocity of the particle before a collision, what will be its velocity afterwards?



- In collisions with a surface, we have an external force due to the surface, and need a slightly different model:
 - 1 The collision occurs over a very short time interval (instantaneous velocity changes)
 - 2 No external forces on particle during collision, other than from surface
 - 3 Infinitesimal particles can 'collide' (collision is overlap in position)
 - 4 No frictional forces between particle and surface
 - 5 The surface is smooth, rigid and fixed (constant normal direction & external force holding surface in place)

Collision Frame for Surfaces



- $\hat{\mathbf{e}}_n$ is always normal to the surface at the impact point (O')
- The wall imparts an impulse on the particle of the form $\bar{\mathbf{F}}_P(t_1, t_2) = \bar{F}_P(t_1, t_2)\hat{\mathbf{e}}_n$:

$$m_P \mathcal{I}_{\mathbf{v}_{P/O}}(t_2) = m_P \mathcal{I}_{\mathbf{v}_{P/O}}(t_1) + \bar{F}_P(t_1, t_2)\hat{\mathbf{e}}_n$$

- Let $\mathcal{I}_{\mathbf{v}_{P/O}}(t_1) = v_P \hat{\mathbf{e}}_n + u_P \hat{\mathbf{e}}_t$

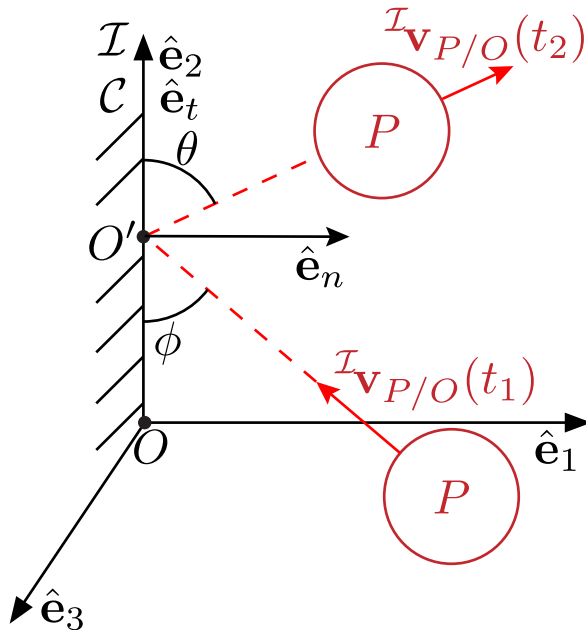
$$m_P u_P(t_2) = m_P u_P(t_1)$$

$$m_P v_P(t_2) = m_P v_P(t_1) + \bar{F}_P(t_1, t_2)$$

- At the time of collision $v_P(t_c) = 0$

- Define $e \triangleq \frac{\bar{F}_P(t_c, t_2)}{\bar{F}_P(t_1, t_c)} = -\frac{v_P(t_2)}{v_P(t_1)}$

Rebounds



- $\mathcal{I}_{\mathbf{v}_{P/O}}(t_1) = -v_P \hat{\mathbf{e}}_n + u_P \hat{\mathbf{e}}_t$
- $\phi = \tan^{-1} \left(\frac{v_P(t_1)}{u_P(t_1)} \right)$
- $\mathcal{I}_{\mathbf{v}_{P/O}}(t_2) = e v_P \hat{\mathbf{e}}_n + u_P \hat{\mathbf{e}}_t$
- $\theta = \tan^{-1} \left(\frac{e v_P(t_1)}{u_P(t_1)} \right)$
- $\theta = \phi$ only when $e = 1$

Week 7

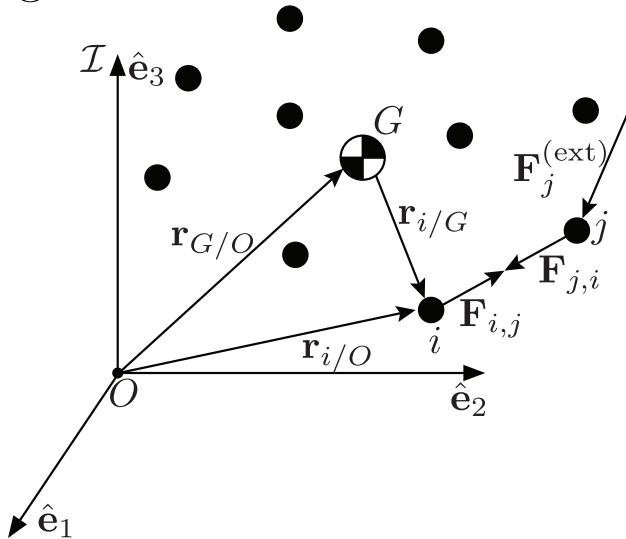
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Cornell University

MAE 2030, Spring 2026

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Angular Momentum of Multiparticle Systems



- Just as with linear momentum, we can extend the concept of angular momentum to multiparticle systems
- The angular momentum of particle i about point O is:

$$\mathcal{I}\mathbf{h}_{i/O} = \mathbf{r}_{i/O} \times m_i \mathcal{I}\mathbf{v}_{i/O}$$

- We define the **total angular momentum** about point O as:

$$\mathcal{I}\mathbf{h}_O \triangleq \sum_{i=1}^N \mathcal{I}\mathbf{h}_{i/O} = \sum_{i=1}^N \mathbf{r}_{i/O} \times m_i \mathcal{I}\mathbf{v}_{i/O}$$

Inertial Derivative of the Total Angular Momentum

- We can see how the total angular momentum changes in time by taking its inertial time derivative:

$$\begin{aligned}\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt}\mathbf{h}_O &= \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \left(\sum_{i=1}^N \mathbf{r}_{i/O} \times m_i {}^{\mathcal{I}}\mathbf{v}_{i/O} \right) &&= {}^{\mathcal{I}}\mathbf{a}_{i/O} \\ &= \sum_{i=1}^N \underbrace{\left(\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \mathbf{r}_{i/O} \times m_i {}^{\mathcal{I}}\mathbf{v}_{i/O} \right)}_{=0} + \sum_{i=1}^N \underbrace{\left(\mathbf{r}_{i/O} \times m_i \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{v}_{i/O} \right)}_{= \mathbf{r}_{i/O} \times \mathbf{F}_i}\end{aligned}$$

- As we did with linear momentum, we can split forces on each particle into internal and external forces:

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt}\mathbf{h}_O = \sum_{i=1}^N \left(\mathbf{r}_{i/O} \times \mathbf{F}_i^{(\text{ext})} \right) + \sum_{i=1}^N \left(\mathbf{r}_{i/O} \times \sum_{j=1}^N \mathbf{F}_{i,j} \right)$$

Multiparticle System Moments

- Recall the definition of moments (torques):

$$\begin{aligned}\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt}\mathbf{h}_O &= \sum_{i=1}^N \underbrace{\left(\mathbf{r}_{i/O} \times \mathbf{F}_i^{(\text{ext})} \right)}_{\triangleq \mathbf{M}_{i/O}^{(\text{ext})}} + \underbrace{\sum_{i=1}^N \left(\mathbf{r}_{i/O} \times \sum_{j=1}^N \mathbf{F}_{i,j} \right)}_{\equiv \sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/O} \times \mathbf{F}_{i,j})}\end{aligned}$$

- Note that the order of summation in the second term is arbitrary: we can switch i and j and the resulting expression is still true:

$$\sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/O} \times \mathbf{F}_{i,j}) = \sum_{j=1}^N \sum_{i=1}^N (\mathbf{r}_{j/O} \times \mathbf{F}_{j,i})$$

Multiparticle System Moments Continued

- If we add both forms of this summation, we'll have twice the original quantity:

$$\sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/O} \times \mathbf{F}_{i,j}) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/O} \times \mathbf{F}_{i,j} - \mathbf{r}_{j/O} \times \mathbf{F}_{i,j})$$

where we also used the fact that $\mathbf{F}_{j,i} = -\mathbf{F}_{i,j}$

- As $\mathbf{r}_{i/O} - \mathbf{r}_{j/O} = \mathbf{r}_{i/j}$, the internal moment term becomes:

$$\sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/O} \times \mathbf{F}_{i,j}) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/j} \times \mathbf{F}_{i,j})$$

Internal Moment Assumption

- We have now found that:

$$\mathcal{I} \frac{d}{dt} \mathcal{I} \mathbf{h}_O = \sum_{i=1}^N \mathbf{M}_{i/O}^{(\text{ext})} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/j} \times \mathbf{F}_{i,j})$$

- In the case of linear momentum, the internal force summation cancelled out due to Newton's 3rd law. The same is *not* true for internal moments
- However, for many multiparticle systems, this summation does equal zero. We call this the **internal moment assumption**:

$$\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/j} \times \mathbf{F}_{i,j}) = \mathbf{0}$$

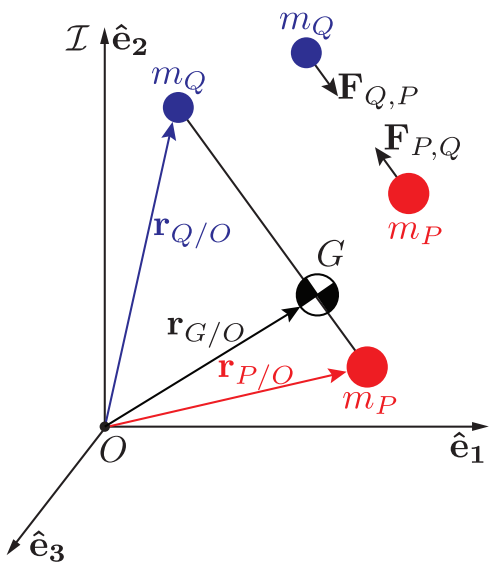
- When the assumption holds:

$$\mathcal{I} \frac{d}{dt} \mathcal{I} \mathbf{h}_O = \sum_{i=1}^N \mathbf{M}_{i/O}^{(\text{ext})} \triangleq \mathbf{M}_O^{(\text{ext})}$$

Angular Motion of the COM

- We now have (using the internal moment assumption) an angular momentum statement of Newton's 2nd law for collections of particles
- However, this form $\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \mathbf{h}_O = \mathbf{M}_O^{(\text{ext})}$ is less useful than its linear momentum counterpart, as it doesn't tell us about the motion of the COM
- In general: $\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \mathbf{h}_O \neq \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \mathbf{h}_{G/O}$ and $\mathbf{M}_O^{(\text{ext})} \neq \mathbf{M}_{G/O}^{(\text{ext})}$

Two Particle System



- Consider a system of two tethered particles

$$\begin{aligned} \mathbf{r}_{P/O} &= x_P \hat{e}_1 + y_P \hat{e}_2 & \mathbf{r}_{Q/O} &= x_Q \hat{e}_1 + y_Q \hat{e}_2 \\ {}^{\mathcal{I}}\mathbf{v}_{P/O} &= \dot{x}_P \hat{e}_1 + \dot{y}_P \hat{e}_2 & {}^{\mathcal{I}}\mathbf{v}_{Q/O} &= \dot{x}_Q \hat{e}_1 + \dot{y}_Q \hat{e}_2 \end{aligned}$$

- The total angular momentum about O:

$$\begin{aligned} \mathbf{h}_O &= \mathbf{r}_{P/O} \times m_P {}^{\mathcal{I}}\mathbf{v}_{P/O} + \mathbf{r}_{Q/O} \times m_Q {}^{\mathcal{I}}\mathbf{v}_{Q/O} \\ &= m_P (x_P \dot{y}_P - y_P \dot{x}_P) \hat{e}_3 \\ &\quad + m_Q (x_Q \dot{y}_Q - y_Q \dot{x}_Q) \hat{e}_3 \end{aligned}$$

- There are no external forces and the internal moment assumption holds

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \mathbf{h}_O = [m_P (x_P \ddot{y}_P - y_P \ddot{x}_P) + m_Q (x_Q \ddot{y}_Q - y_Q \ddot{x}_Q)] \hat{e}_3 = \mathbf{0}$$

Separation Principle

- Recall that we can always write

$$\mathbf{r}_{i/O} = \mathbf{r}_{G/O} + \mathbf{r}_{i/G} \Rightarrow \mathcal{I}_{\mathbf{v}_{i/O}} = \mathcal{I}_{\mathbf{v}_{G/O}} + \mathcal{I}_{\mathbf{v}_{i/G}}$$

$$\begin{aligned} \mathcal{I}_{\mathbf{h}_O} &= \sum_{i=1}^N \mathbf{r}_{i/O} \times m_i \mathcal{I}_{\mathbf{v}_{i/O}} = \sum_{i=1}^N m_i ((\mathbf{r}_{G/O} + \mathbf{r}_{i/G}) \times (\mathcal{I}_{\mathbf{v}_{G/O}} + \mathcal{I}_{\mathbf{v}_{i/G}})) = \\ &= \underbrace{\left(\sum_{i=1}^N m_i \right) \mathbf{r}_{G/O} \times \mathcal{I}_{\mathbf{v}_{G/O}}}_{m_G} + \underbrace{\mathbf{r}_{G/O} \times \sum_{i=1}^N m_i \mathcal{I}_{\mathbf{v}_{i/G}}}_0 + \underbrace{\sum_{i=1}^N m_i \mathbf{r}_{i/G} \times \mathcal{I}_{\mathbf{v}_{G/O}}}_0 + \sum_{i=1}^N (m_i \mathbf{r}_{i/G} \times \mathcal{I}_{\mathbf{v}_{i/G}}) \end{aligned}$$

$$\boxed{\mathcal{I}_{\mathbf{h}_O} = \underbrace{m_G \mathbf{r}_{G/O} \times \mathcal{I}_{\mathbf{v}_{G/O}}}_{\triangleq \mathcal{I}_{\mathbf{h}_{G/O}}} + \underbrace{\sum_{i=1}^N m_i \mathbf{r}_{i/G} \times \mathcal{I}_{\mathbf{v}_{i/G}}}_{\triangleq \mathcal{I}_{\mathbf{h}_G}}}$$

- The total angular momentum about O equals the angular momentum about O of the COM plus the total angular momentum of all particles about the COM

Differentiating the COM's Angular momentum about O

$$\begin{aligned} \frac{\mathcal{I}}{dt} \mathcal{I}_{\mathbf{h}_{G/O}} &= \frac{\mathcal{I}}{dt} (m_G \mathbf{r}_{G/O} \times \mathcal{I}_{\mathbf{v}_{G/O}}) = m_G \overbrace{\left(\frac{\mathcal{I}}{dt} \mathbf{r}_{G/O} \times \mathcal{I}_{\mathbf{v}_{G/O}} \right)}^0 + \underbrace{\mathbf{r}_{G/O} \times m_G \frac{\mathcal{I}}{dt} \mathcal{I}_{\mathbf{v}_{G/O}}}_{\triangleq \mathbf{M}_{G/O}^{(\text{ext})}} \end{aligned}$$

- $\mathcal{I}_{\mathbf{h}_{G/O}}$ describes the motion of a particle of mass m_G located at the COM

Differentiating the Angular momentum about the COM

$$\begin{aligned}
 \frac{\mathcal{I}d}{dt} \mathcal{I} \mathbf{h}_G &= \frac{\mathcal{I}d}{dt} \left(\sum_{i=1}^N m_i \mathbf{r}_{i/G} \times \mathcal{I} \mathbf{v}_{i/G} \right) = \sum_{i=1}^N \overbrace{\left(m_i \frac{\mathcal{I}d}{dt} \mathbf{r}_{i/G} \times \mathcal{I} \mathbf{v}_{i/G} \right)}^{\mathbf{0}} + \sum_{i=1}^N \left(m_i \mathbf{r}_{i/G} \times \frac{\mathcal{I}d}{dt} \mathcal{I} \mathbf{v}_{i/G} \right) \\
 &= \sum_{i=1}^N m_i \mathbf{r}_{i/G} \times \frac{\mathcal{I}d}{dt} \underbrace{(\mathcal{I} \mathbf{v}_{i/O} - \mathcal{I} \mathbf{v}_{G/O})}_{= \mathcal{I} \mathbf{v}_{i/G}} = \sum_{i=1}^N \left(\mathbf{r}_{i/G} \times m_i \frac{\mathcal{I}d}{dt} \mathcal{I} \mathbf{v}_{i/O} \right) - \sum_{i=1}^N \underbrace{(m_i \mathbf{r}_{i/G}) \times \frac{\mathcal{I}d}{dt} \mathcal{I} \mathbf{v}_{G/O}}_{\mathbf{0}} \\
 &= m_i \mathcal{I} \mathbf{a}_{i/O} = \mathbf{F}_i^{(\text{ext})} + \sum_{j=1}^N \mathbf{F}_{i,j} \\
 &= \sum_{i=1}^N \underbrace{(\mathbf{r}_{i/G} \times \mathbf{F}_i^{(\text{ext})})}_{\triangleq \mathbf{M}_{i/G}^{(\text{ext})}} + \sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/G} \times \mathbf{F}_{i,j})
 \end{aligned}$$

- The second term is zero when the internal moment assumption holds true

Angular Momentum Separation Evolution

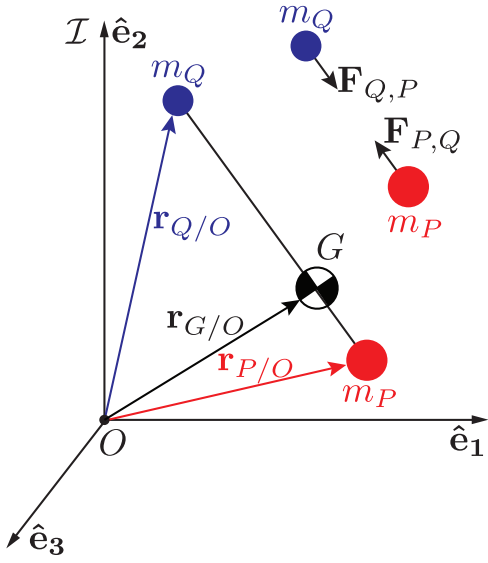
$$\frac{\mathcal{I}d}{dt} \mathcal{I} \mathbf{h}_{G/O} = \mathbf{r}_{G/O} \times \mathbf{F}_G^{(\text{ext})} \triangleq \mathbf{M}_{G/O}^{(\text{ext})}$$

- If the internal moment assumption applies to the collection:

$$\frac{\mathcal{I}d}{dt} \mathcal{I} \mathbf{h}_G = \sum_{i=1}^N (\mathbf{r}_{i/G} \times \mathbf{F}_i^{(\text{ext})}) = \sum_{i=1}^N \mathbf{M}_{i/G}^{(\text{ext})} \triangleq \mathbf{M}_G^{(\text{ext})}$$

- These two expressions are entirely separate from one another
- Conservation of one quantity does *not* imply conservation of the other

Back to the Two Particle System



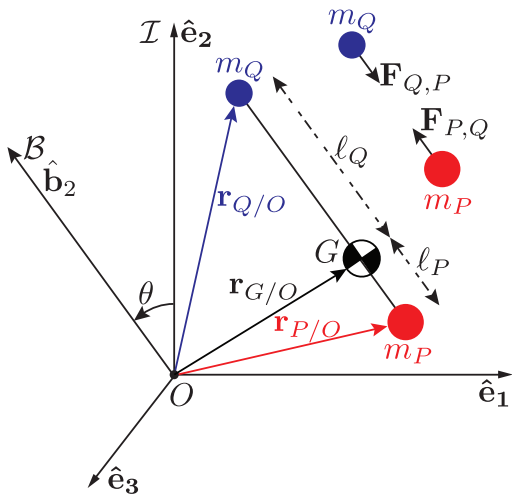
$$\begin{aligned}\mathbf{r}_{G/O} &= \frac{m_P \mathbf{r}_{P/O} + m_Q \mathbf{r}_{Q/O}}{m_P + m_Q} \\ &= \underbrace{\frac{(m_P x_P + m_Q x_Q)}{m_P + m_Q}}_{\triangleq x_G} \hat{\mathbf{e}}_1 + \underbrace{\frac{(m_P y_P + m_Q y_Q)}{m_P + m_Q}}_{\triangleq y_G} \hat{\mathbf{e}}_2\end{aligned}$$

$$\mathcal{I} \mathbf{v}_{G/O} = \dot{x}_G \hat{\mathbf{e}}_1 + \dot{y}_G \hat{\mathbf{e}}_2$$

$$\begin{aligned}\mathcal{I} \mathbf{h}_{G/O} &= \underbrace{(m_P + m_Q)}_{m_G} \mathbf{r}_{G/O} \times \mathcal{I} \mathbf{v}_{G/O} \\ &= m_G (x_G \dot{y}_G - y_G \dot{x}_G) \hat{\mathbf{e}}_3\end{aligned}$$

No external moments so $\mathcal{I} \mathbf{h}_{G/O}$ is conserved: $m_G (x_G \dot{y}_G - y_G \dot{x}_G)$ is constant

Back to the Two Particle System (Continued)



$$\mathcal{B} \triangleq (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3), \hat{\mathbf{b}}_2 \equiv \hat{\mathbf{r}}_{Q/P}, \hat{\mathbf{b}}_3 \equiv \hat{\mathbf{e}}_3, \mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta} \hat{\mathbf{e}}_3$$

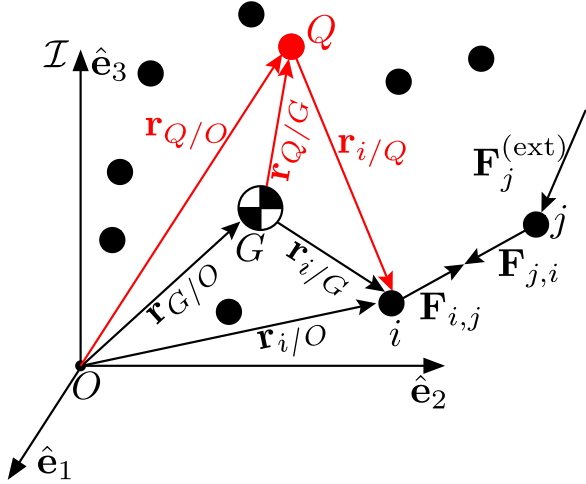
$$\mathbf{r}_{Q/G} = \ell_Q \hat{\mathbf{b}}_2 \quad \mathbf{r}_{P/G} = -\ell_P \hat{\mathbf{b}}_2$$

$$\mathcal{I} \mathbf{v}_{Q/G} = -\ell_Q \dot{\theta} \hat{\mathbf{b}}_1 \quad \mathcal{I} \mathbf{v}_{P/G} = \ell_P \dot{\theta} \hat{\mathbf{b}}_1$$

$$\begin{aligned}\mathcal{I} \mathbf{h}_G &= \mathbf{r}_{Q/G} \times m_Q \mathcal{I} \mathbf{v}_{Q/G} + \mathbf{r}_{P/G} \times m_P \mathcal{I} \mathbf{v}_{P/G} \\ &= (m_Q \ell_Q^2 + m_P \ell_P^2) \dot{\theta} \hat{\mathbf{e}}_3\end{aligned}$$

No external moments so $\mathcal{I} \mathbf{h}_G$ is conserved: $(m_Q \ell_Q^2 + m_P \ell_P^2) \dot{\theta}$ is constant

Total Angular Momentum About Arbitrary Points



- The total angular momentum about any arbitrary point Q is:

$$\mathcal{I}\mathbf{h}_{i/Q} = \mathbf{r}_{i/Q} \times m_i \mathcal{I}\mathbf{v}_{i/Q}$$

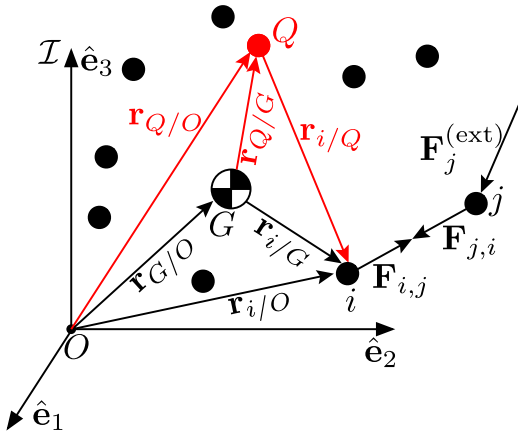
$$\mathcal{I}\mathbf{h}_Q \triangleq \sum_{i=1}^N \mathbf{r}_{i/Q} \times m_i \mathcal{I}\mathbf{v}_{i/Q}$$

- As before: $\mathbf{r}_{i/G} = \mathbf{r}_{i/Q} + \mathbf{r}_{Q/G}$ so:

$$\mathbf{r}_{i/Q} = \mathbf{r}_{i/G} - \mathbf{r}_{Q/G} \Rightarrow \mathcal{I}\mathbf{v}_{i/Q} = \mathcal{I}\mathbf{v}_{i/G} - \mathcal{I}\mathbf{v}_{Q/G}$$

$$\mathcal{I}\mathbf{h}_Q = \underbrace{\sum_{i=1}^N (\mathbf{r}_{i/G} \times m_i \mathcal{I}\mathbf{v}_{i/G})}_{\mathcal{I}\mathbf{h}_G} + (\mathbf{r}_{Q/G} \times \mathcal{I}\mathbf{v}_{Q/G}) \underbrace{\sum_{i=1}^N m_i}_{m_G}$$

Differentiating $\mathcal{I}\mathbf{h}_Q$



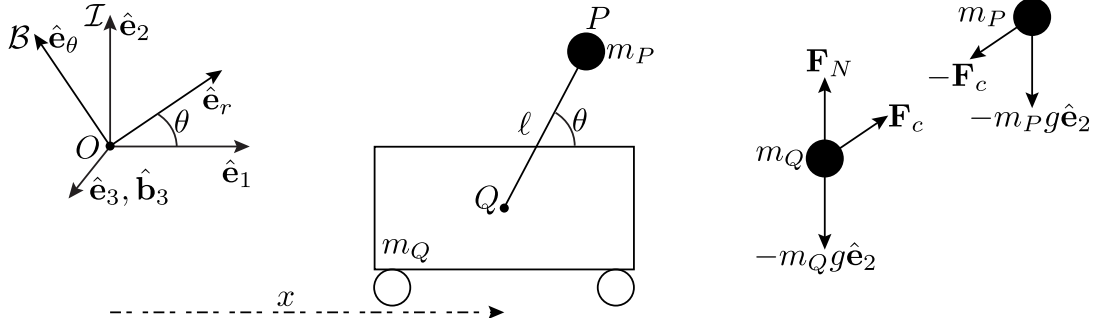
$$\frac{\mathcal{I} d}{dt} \mathcal{I}\mathbf{h}_Q = \underbrace{\frac{\mathcal{I} d}{dt} \mathcal{I}\mathbf{h}_G}_{\mathbf{M}_G^{(\text{ext})}} + m_G \underbrace{\frac{\mathcal{I} d}{dt} (\mathbf{r}_{Q/G} \times \mathcal{I}\mathbf{v}_{Q/G})}_{\mathbf{r}_{Q/G} \times \mathcal{I}\mathbf{a}_{Q/G}}$$

- $\mathbf{M}_G^{(\text{ext})} = \sum_{i=1}^N (\mathbf{r}_{i/G} \times \mathbf{F}_i^{(\text{ext})})$

- $\mathcal{I}\mathbf{a}_{Q/G} = \mathcal{I}\mathbf{a}_{Q/O} - \mathcal{I}\mathbf{a}_{G/O}$

$$\frac{\mathcal{I} d}{dt} \mathcal{I}\mathbf{h}_Q = \underbrace{\sum_{i=1}^N (\mathbf{r}_{i/Q} \times \mathbf{F}_i^{(\text{ext})})}_{\triangleq \mathbf{M}_Q^{(\text{ext})}} + \mathbf{r}_{Q/G} \times m_G \mathcal{I}\mathbf{a}_{Q/O}$$

Recall the Pendulum on a Cart



$\mathcal{I} \triangleq (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ is inertial; $\mathcal{B} \triangleq (\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$ is non-inertial, with $\hat{\mathbf{e}}_r \equiv \hat{\mathbf{r}}_{P/Q}$, $\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\hat{\mathbf{e}}_3$

- The cart has no angular momentum about Q , so the total angular momentum about Q is due solely to m_P :

$$\mathcal{I}\mathbf{h}_Q = m_P \mathbf{r}_{P/Q} \times \mathcal{I}\mathbf{v}_{P/Q} = m_P \ell \hat{\mathbf{e}}_r \times \ell \dot{\theta} \hat{\mathbf{e}}_\theta = m_P \ell^2 \dot{\theta} \hat{\mathbf{e}}_3 \Rightarrow \frac{\mathcal{I}d}{dt} \mathcal{I}\mathbf{h}_Q = m_P \ell^2 \ddot{\theta} \hat{\mathbf{e}}_3$$

$$\mathbf{M}_Q = \mathbf{r}_{P/Q} \times \sum \mathbf{F}_P = \ell \hat{\mathbf{e}}_r \times (-m_P g \hat{\mathbf{e}}_2 - F_c \hat{\mathbf{e}}_r) = -\ell m_P g \cos(\theta) \hat{\mathbf{e}}_3$$

$$\begin{aligned} \frac{\mathcal{I}d}{dt} \mathcal{I}\mathbf{h}_Q &= \mathbf{M}_Q^{(\text{ext})} + \mathbf{r}_{Q/G} \times m_G \mathcal{I}\mathbf{a}_{Q/O} \implies \\ m_P \ell^2 \ddot{\theta} \hat{\mathbf{e}}_3 &= -\ell m_P g \cos(\theta) \hat{\mathbf{e}}_3 + (\mathbf{r}_{Q/O} - \mathbf{r}_{G/O}) \times m_G \mathcal{I}\mathbf{a}_{Q/O} \\ \mathbf{r}_{Q/O} = x \hat{\mathbf{e}}_1 &\Rightarrow \mathcal{I}\mathbf{a}_{Q/O} = \ddot{x} \hat{\mathbf{e}}_1 \quad \mathbf{r}_{G/O} = \frac{m_Q x \hat{\mathbf{e}}_1 + m_P (x \hat{\mathbf{e}}_1 + \ell \hat{\mathbf{e}}_r)}{m_P + m_Q} = x \hat{\mathbf{e}}_1 + \frac{m_P}{m_G} \ell \hat{\mathbf{e}}_r \\ m_P \ell^2 \ddot{\theta} \hat{\mathbf{e}}_3 &= -\ell m_P g \cos(\theta) \hat{\mathbf{e}}_3 - \frac{m_P}{m_G} \ell \hat{\mathbf{e}}_r \times m_G \ddot{x} \hat{\mathbf{e}}_1 = -\ell m_P g \cos(\theta) \hat{\mathbf{e}}_3 - m_P \ell (-\sin(\theta)) \ddot{x} \hat{\mathbf{e}}_3 \end{aligned}$$

Pendulum on a Cart Continued

- We have found that $\ell\ddot{\theta} = (\ddot{x} \sin(\theta) - g \cos(\theta))$, which matches our previously found equations for this system.
- We have one equation, but two unknowns $(\ddot{x}, \ddot{\theta})$. For the second equation, let's look at the linear momentum balance of the COM:

$$\mathbf{r}_{G/O} = x\hat{\mathbf{e}}_1 + \frac{m_P}{m_G}\ell\hat{\mathbf{e}}_r \implies \mathcal{I}_{\mathbf{a}_{G/O}} = \ddot{x}\hat{\mathbf{e}}_1 + \frac{m_P}{m_G}(\ell\ddot{\theta}\hat{\mathbf{e}}_\theta - \ell\dot{\theta}^2\hat{\mathbf{e}}_r)$$

$$m_G\ddot{x}\hat{\mathbf{e}}_1 + m_P(\ell\ddot{\theta}\hat{\mathbf{e}}_\theta - \ell\dot{\theta}^2\hat{\mathbf{e}}_r) = (F_N - m_Gg)\hat{\mathbf{e}}_2$$

- We only need to look at the $\hat{\mathbf{e}}_1$ component:

$$m_G\ddot{x} + m_P\ell\ddot{\theta}(-\sin(\theta)) - m_P\ell\dot{\theta}^2\cos(\theta) = 0 \implies \ddot{x} = \frac{m_P}{m_G}\ell\ddot{\theta}\sin(\theta) + \frac{m_P}{m_G}\ell\dot{\theta}^2\cos(\theta)$$

- Plugging into the $\ddot{\theta}$ expression produces the exact same EOMs as we previously found using linear momentum balance only

Total Kinetic Energy of Multiparticle Systems

- Recall that while the kinetic and potential energy are scalar values, they are computed based on vector values, and all derivatives involved must be inertial
- For particle P and inertially fixed point O : $T_{P/O} \triangleq \frac{1}{2}m_p(\mathcal{I}_{\mathbf{v}_{P/O}} \cdot \mathcal{I}_{\mathbf{v}_{P/O}})$
- For a collection of particles, define **total kinetic energy** as:

$$T_O \triangleq \sum_{i=1}^N T_{i/O} = \frac{1}{2} \sum_{i=1}^N m_i \mathcal{I}_{\mathbf{v}_{i/O}} \cdot \mathcal{I}_{\mathbf{v}_{i/O}} = \frac{1}{2} \sum_{i=1}^N m_i \|\mathcal{I}_{\mathbf{v}_{i/O}}\|^2$$

- Since $\mathcal{I}_{\mathbf{v}_{i/O}} = \mathcal{I}_{\mathbf{v}_{G/O}} + \mathcal{I}_{\mathbf{v}_{i/G}}$:

$$T_O = \frac{1}{2} \sum_{i=1}^N m_i \left(\mathcal{I}_{\mathbf{v}_{G/O}} \cdot \mathcal{I}_{\mathbf{v}_{G/O}} + \underbrace{2\mathcal{I}_{\mathbf{v}_{G/O}} \cdot \mathcal{I}_{\mathbf{v}_{i/G}}}_N + \mathcal{I}_{\mathbf{v}_{i/G}} \cdot \mathcal{I}_{\mathbf{v}_{i/G}} \right)$$

$$\mathcal{I}_{\mathbf{v}_{G/O}} \cdot \sum_{i=1}^N m_i \mathcal{I}_{\mathbf{v}_{i/G}} = 0$$

Kinetic Energy Separation

$$T_O = \underbrace{\frac{1}{2} \sum_{i=1}^N m_i \|\mathbf{v}_{G/O}\|^2}_{\triangleq T_{G/O}} + \underbrace{\frac{1}{2} \sum_{i=1}^N m_i \|\mathbf{v}_{i/G}\|^2}_{\triangleq T_G}$$

- Just as with angular momentum, the total kinetic energy separates into the KE of the COM ($T_{G/O}$) and the KE of all particles relative to the COM (T_G)
- $T_{G/O}$ is the KE of a particle with the total mass of the collection located at the COM
- T_G is the sum of kinetic energies of all particles based on their *inertial* velocities with respect to the COM

Total Work of Multiparticle Systems

- Recall that the total work on a particle i is the change in kinetic energy:

$$W_i^{(\text{tot})}(\mathbf{r}_{i/O}(t); \gamma_i) = \underbrace{T_{i/O}(t_2) - T_{i/O}(t_1)}_{\triangleq \Delta T_{i/O}}$$

- The **total work** of a multiparticle system is the sum of the work done on all particles:

$$W^{(\text{tot})}(\{\mathbf{r}_{i/O}(t); \gamma_i\}_{i=1}^N) = \sum_{i=1}^N W_i^{(\text{tot})}(\mathbf{r}_{i/O}(t); \gamma_i) = \sum_{i=1}^N \Delta T_{i/O} \triangleq \Delta T_O$$

Total Work of Multiparticle Systems Continued

- Work is force integrated along a trajectory:

$$W_P^{\mathbf{F}_P}(\mathbf{r}_{P/O}; \gamma_P) \triangleq \int_{\gamma_P} \mathbf{F}_P \cdot \mathcal{I} d\mathbf{r}_{P/O}$$

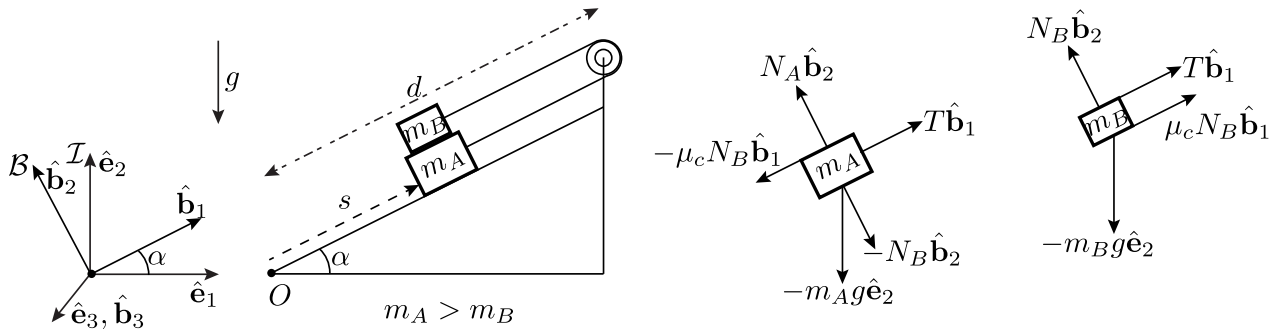
- The total work can be split into external and internal force components:

$$\begin{aligned} W^{(\text{tot})} \left(\{\mathbf{r}_{i/O}; \gamma_i\}_{i=1}^N \right) &= \underbrace{\sum_{i=1}^N \int_{\gamma_i} \mathbf{F}_i^{(\text{ext})} \cdot \mathcal{I} d\mathbf{r}_{i/O}}_{\triangleq W^{(\text{ext})} \left(\{\mathbf{r}_{i/O}; \gamma_i\}_{i=1}^N \right)} + \underbrace{\sum_{i=1}^N \int_{\gamma_i} \sum_{j=1}^N \mathbf{F}_{i,j} \cdot \mathcal{I} d\mathbf{r}_{i/O}}_{\triangleq W^{(\text{int})} \left(\{\mathbf{r}_{i/O}; \gamma_i\}_{i=1}^N \right)} \\ &\triangleq W^{(\text{ext})} \left(\{\mathbf{r}_{i/O}; \gamma_i\}_{i=1}^N \right) \triangleq W^{(\text{int})} \left(\{\mathbf{r}_{i/O}; \gamma_i\}_{i=1}^N \right) \end{aligned}$$

- Internal work can also be written as: $\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \int_{\gamma_i} \mathbf{F}_{i,j} \cdot \mathcal{I} d\mathbf{r}_{i/j}$

$$W^{(\text{tot})} = T_O(t_2) - T_O(t_1) = W^{(\text{ext})} + W^{(\text{int})}$$

Ramp+Pulley with Two Masses



- Ramp is frictionless but there is friction between the two masses.

System is released from rest at t_1

$$\mathbf{r}_{A/O} = s \hat{\mathbf{b}}_1 \quad \mathbf{r}_{B/O} = (d - s) \hat{\mathbf{b}}_1 + h_A \hat{\mathbf{b}}_2$$

$$\mathcal{I} \mathbf{v}_{A/O} = \dot{s} \hat{\mathbf{b}}_1 \quad \mathcal{I} \mathbf{v}_{B/O} = -\dot{s} \hat{\mathbf{b}}_1$$

$$\mathcal{I} \mathbf{a}_{A/O} = \ddot{s} \hat{\mathbf{b}}_1 \quad \mathcal{I} \mathbf{a}_{B/O} = -\ddot{s} \hat{\mathbf{b}}_1$$

$$(T + \mu_c N_B) \hat{\mathbf{b}}_1 + N_B \hat{\mathbf{b}}_2 - m_B g \hat{\mathbf{e}}_2 = -m_B \ddot{s} \hat{\mathbf{b}}_1$$

$$N_B = m_B g \cos(\alpha)$$

Ramp+Pulley with Two Masses Work

- The total work is: $W^{(\text{tot})} = \int_{\gamma_A} \mathbf{F}_A \cdot \mathcal{I} d\mathbf{r}_{A/O} + \int_{\gamma_B} \mathbf{F}_B \cdot \mathcal{I} d\mathbf{r}_{B/O}$

$$\begin{aligned} \int_{\gamma_A} \mathbf{F}_A \cdot \mathcal{I} d\mathbf{r}_{A/O} &= \int_{s(t_1)}^{s(t_2)} \left[(T - \mu_c N_B) \hat{\mathbf{b}}_1 + (N_A - N_B) \hat{\mathbf{b}}_2 - m_A g \hat{\mathbf{e}}_2 \right] \cdot ds \hat{\mathbf{b}}_1 \\ &= \int_{s(t_1)}^{s(t_2)} (T - \mu_c N_B - m_A g \sin(\alpha)) ds = (T - \mu_c N_B - m_A g \sin(\alpha)) (s(t_2) - s(t_1)) \end{aligned}$$

$$\begin{aligned} \int_{\gamma_B} \mathbf{F}_B \cdot \mathcal{I} d\mathbf{r}_{B/O} &= \int_{s(t_1)}^{s(t_2)} \left[(T + \mu_c N_B) \hat{\mathbf{b}}_1 + N_B \hat{\mathbf{b}}_2 - m_B g \hat{\mathbf{e}}_2 \right] \cdot d(d - s) \hat{\mathbf{b}}_1 \\ &= (T + \mu_c N_B - m_B g \sin(\alpha)) (-s(t_2) + s(t_1)) \end{aligned}$$

$$W^{(\text{tot})} = [-2\mu_c N_B + (m_B - m_A)g \sin(\alpha)] (s(t_2) - s(t_1))$$

Ramp+Pulley with Two Masses Continued

- $W^{(\text{tot})} = T_O(t_2) - T_O(t_1) \implies T_O(t_2) = W^{(\text{tot})} + T_O(t_1)$
- @ t_1 : $\mathcal{I}\mathbf{v}_{A/O} = \mathcal{I}\mathbf{v}_{B/O} = \mathbf{0}$ so $T_O(t_1) = 0$
- Therefore, $\forall t > t_1$:

$$\frac{m_A}{2} \dot{s}^2 + \frac{m_B}{2} \dot{s}^2 = [-2\mu_c m_B g \cos(\alpha) + (m_B - m_A)g \sin(\alpha)] (s(t_2) - s(t_1))$$

- We can solve this for $\dot{s}$:

$$\dot{s} = \sqrt{\frac{2(-2\mu_c m_B g \cos(\alpha) + (m_B - m_A)g \sin(\alpha)) (s(t_2) - s(t_1))}{m_A + m_B}}$$

Power of Multiparticle Systems

- Recall the power for one particle is given by:

$$P_P = \frac{d}{dt} W_P^{\mathbf{F}_P}(\mathbf{r}_{P/O}; \gamma_P) = \mathbf{F}_P \cdot \mathcal{I}_{\mathbf{v}_{P/O}}$$

- We define the **total power** of a multiparticle collection as:

$$P^{(\text{tot})} = \sum_{i=1}^N \left(\mathbf{F}_i^{(\text{ext})} \cdot \mathcal{I}_{\mathbf{v}_{i/O}} \right) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \left(\mathbf{F}_{i,j} \cdot \mathcal{I}_{\mathbf{v}_{i/j}} \right)$$

- Substituting $\mathcal{I}_{\mathbf{v}_{i/O}} = \mathcal{I}_{\mathbf{v}_{G/O}} + \mathcal{I}_{\mathbf{v}_{i/G}}$:

$$P^{(\text{tot})} = \underbrace{\sum_{i=1}^N \mathbf{F}_i^{(\text{ext})}}_{\mathbf{F}_G^{(\text{ext})}} \cdot \mathcal{I}_{\mathbf{v}_{G/O}} + \sum_{i=1}^N \left(\mathbf{F}_i^{(\text{ext})} \cdot \mathcal{I}_{\mathbf{v}_{i/G}} \right) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \left(\mathbf{F}_{i,j} \cdot \mathcal{I}_{\mathbf{v}_{i/j}} \right)$$

- The first term is the power associated with a single particle of mass m_G located at the COM

Power of Multiparticle Systems Continued

- We can check whether the COM satisfies our previously found energy conservation laws by differentiating the kinetic energy of the COM:

$$\begin{aligned} \frac{d}{dt} T_{G/O} &= \frac{d}{dt} \left(\frac{m_G}{2} (\mathcal{I}_{\mathbf{v}_{G/O}} \cdot \mathcal{I}_{\mathbf{v}_{G/O}}) \right) \\ &= \frac{m_G}{2} \left(\frac{d}{dt} \mathcal{I}_{\mathbf{v}_{G/O}} \cdot \mathcal{I}_{\mathbf{v}_{G/O}} + \mathcal{I}_{\mathbf{v}_{G/O}} \cdot \frac{d}{dt} \mathcal{I}_{\mathbf{v}_{G/O}} \right) = \underbrace{m_G \frac{d}{dt} \mathcal{I}_{\mathbf{v}_{G/O}} \cdot \mathcal{I}_{\mathbf{v}_{G/O}}}_{\mathbf{F}_G^{(\text{ext})}} \end{aligned}$$

- The power associated with the total force acting on the center of mass is equal to the rate of change of kinetic energy:

$$\frac{d}{dt} T_{G/O} = \mathbf{F}_G^{(\text{ext})} \cdot \mathcal{I}_{\mathbf{v}_{G/O}} \triangleq P_{G/O}$$

- Power splits between a particle of the mass of the whole collection at the COM and the internal interactions of the collection

Total Energy

- Recall that $\mathbf{F}_P^{(\text{cons})} = -\nabla U_{P/O}^{(\mathbf{F}_P^{(\text{cons})})}$
- For a collection of particles:

$$\mathbf{F}_i^{(\text{cons,ext})} = -\nabla U_{i/O}^{(\text{ext})}(\mathbf{r}_{i/O}) \quad \text{external}$$

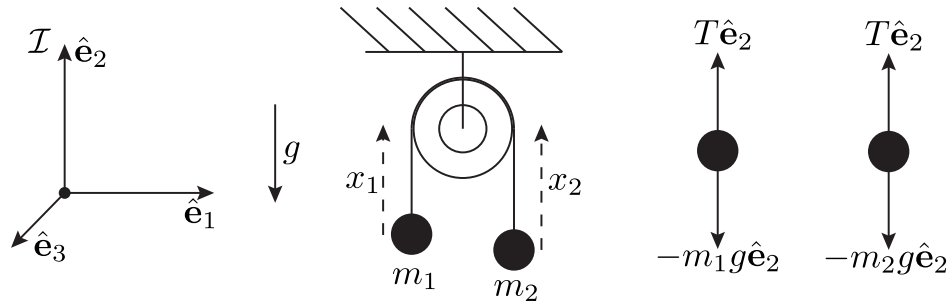
$$\mathbf{F}_{i,j}^{(\text{cons})} = -\nabla U_{i/j}^{(\mathbf{F}_{i,j})}(\mathbf{r}_{i/j}) \quad \text{internal}$$

- We define $U_O^{(\text{ext})}(t) \triangleq \sum_{i=1}^N U_{i/O}^{(\text{ext})}(\mathbf{r}_{i/O}(t))$ and

$$U_O^{(\text{int})}(t) \triangleq \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N U_{i,j}^{(\mathbf{F}_{i,j})}(\mathbf{r}_{i/j}(t))$$

- Thus: $W^{(\text{cons})}(t_1, t_2) = \left(U_O^{(\text{ext})}(t_1) - U_O^{(\text{ext})}(t_2) \right) + \left(U_O^{(\text{int})}(t_1) - U_O^{(\text{int})}(t_2) \right)$
- Total Energy:** $E_O(t) = \underbrace{T_{G/O}(t) + T_G(t)}_{T_O(t)} + U_O^{(\text{ext})}(t) + U_O^{(\text{int})}(t)$
- $E_O(t_2) = E_O(t_1) + W^{(\text{nc,ext})}(t_1, t_2) + W^{(\text{nc,int})}(t_1, t_2)$. Total energy is conserved if and only if the total non-conservative force is zero

Atwood's Machine



- Atwood's machine consists of two masses connected by an inextensible rope, via a massless, frictionless pulley
 - All motion is one-dimensional: $\mathbf{r}_{1/O} = x_1 \hat{\mathbf{e}}_2$, $\mathbf{r}_{2/O} = x_2 \hat{\mathbf{e}}_2$ and $x_1 = -x_2$
- $$\left. \begin{aligned} (T - m_1 g) \hat{\mathbf{e}}_2 &= m_1 \ddot{x}_1 \hat{\mathbf{e}}_2 \\ (T - m_2 g) \hat{\mathbf{e}}_2 &= m_2 \ddot{x}_2 \hat{\mathbf{e}}_2 \end{aligned} \right\} (m_2 - m_1)g = (m_1 \ddot{x}_1 - m_2 \ddot{x}_2) = (m_1 + m_2) \ddot{x}_1$$

$$\ddot{x}_1 = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g \quad T = \frac{2m_1 m_2}{m_1 + m_2} g$$

Atwood's Machine with Work

$$W^{(\text{tot})} = W_1^{(\text{tot})} + W_2^{(\text{tot})} = \int_{\gamma_1} (T - m_1 g) \hat{\mathbf{e}}_2 \cdot \mathcal{I} d \overbrace{\mathbf{r}_{1/O}}^{x_1 \hat{\mathbf{e}}_2} + \int_{\gamma_2} (T - m_2 g) \hat{\mathbf{e}}_2 \cdot \mathcal{I} d \overbrace{\mathbf{r}_{2/O}}^{x_2 \hat{\mathbf{e}}_2}$$

$$x_1 = -x_2 \Rightarrow \gamma_1 = -\gamma_2 \quad \Rightarrow \quad W^{(\text{tot})} = \int_{\gamma_1} (m_2 - m_1) g dx_1$$

- We assume (without loss of generality) that the system moves from rest with $x_1 = x_2 = 0$ to a state where $x_1 = -x_2$ and $\dot{x}_1 = -\dot{x}_2$

$$\underbrace{\int_0^{x_1} (m_2 - m_1) g dx_1}_{W^{(\text{tot})}} = \underbrace{\frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} - 0}_{\Delta \text{KE}} \Rightarrow (m_2 - m_1) g x_1 = \frac{m_1 + m_2}{2} \dot{x}_1^2$$

- Differentiating this expression (assuming $\dot{x}_1 \neq 0$):

$$(m_2 - m_1) g \dot{x}_1 = \frac{m_1 + m_2}{2} 2 \dot{x}_1 \ddot{x}_1 \Rightarrow \ddot{x}_1 = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g$$

Week 8

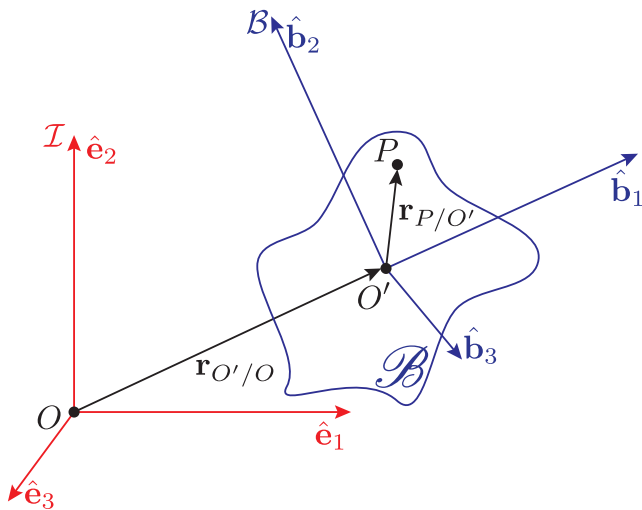
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MAE 2030, Spring 2026

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Rigid Bodies as Reference Frames



- A **rigid body** is one that does not deform—all points in a rigid body are fixed with respect to one another, and can only move together
- A rigid body is therefore equivalent to a reference frame
- We can always attach a **body fixed frame** (or just body frame) to a rigid body, and use the two interchangeably
- Consider a point P fixed to rigid body $\mathcal{B}$:

$$\mathbf{r}_{P/O} = \mathbf{r}_{P/O'} + \mathbf{r}_{O'/O} \Rightarrow \mathcal{I}_{\mathbf{v}_{P/O}} = \mathcal{I}_{\mathbf{v}_{P/O'}} + \mathcal{I}_{\mathbf{v}_{O'/O}}$$

Rigid Bodies as Reference Frames Continued

- Let $\mathbf{r}_{P/O'} = b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3$. As P is fixed in $\mathcal{B}$ and $\mathcal{B}$ is a body-fixed frame, b_1, b_2, b_3 are all constant

$$\begin{aligned} \frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O'} &= \frac{{}^{\mathcal{I}}d}{dt} (b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3) = b_1 \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_1 + b_2 \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_2 + b_3 \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_3 \\ &= b_1 \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_1 \right) + b_2 \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_2 \right) + b_3 \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_3 \right) = {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'} \end{aligned}$$

- Remember: we have already shown that $\frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}} = {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}$ for any unit vector $\hat{\mathbf{b}}$ fixed in $\mathcal{B}$. So for any points O' and P fixed to a rigid body:

$$\mathbf{r}_{P/O'} = \|\mathbf{r}_{P/O'}\| \hat{\mathbf{r}}_{P/O'} \quad \Rightarrow \quad \frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O'} = \|\mathbf{r}_{P/O'}\| \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{r}}_{P/O'} \right) = {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}$$

- Thus for any point P fixed in $\mathcal{B}$: ${}^{\mathcal{I}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{O'/O} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}$

What if P is not fixed in $\mathcal{B}$?

- If P is not fixed in the body/body frame, $\frac{{}^{\mathcal{B}}d}{dt} \mathbf{r}_{P/O'} \neq \mathbf{0}$:

$$\begin{aligned} \frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O'} &= \frac{{}^{\mathcal{I}}d}{dt} (b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3) \\ &= \dot{b}_1 \hat{\mathbf{b}}_1 + b_1 \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_1 + \dot{b}_2 \hat{\mathbf{b}}_2 + b_2 \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_2 + \dot{b}_3 \hat{\mathbf{b}}_3 + b_3 \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_3 \\ &= \underbrace{\dot{b}_1 \hat{\mathbf{b}}_1 + \dot{b}_2 \hat{\mathbf{b}}_2 + \dot{b}_3 \hat{\mathbf{b}}_3}_{\frac{{}^{\mathcal{B}}d}{dt} \mathbf{r}_{P/O'}} + \underbrace{b_1 \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_1 \right) + b_2 \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_2 \right) + b_3 \left({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_3 \right)}_{{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}} \end{aligned}$$

- Thus: $\frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O'} = \frac{{}^{\mathcal{B}}d}{dt} \mathbf{r}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}$
- If ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \mathbf{0}$ then $\frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O'} = \frac{{}^{\mathcal{B}}d}{dt} \mathbf{r}_{P/O'}$

The Transport Equation

$$\frac{{}^{\mathcal{A}}d}{dt}\mathbf{c} = \frac{{}^{\mathcal{B}}d}{dt}\mathbf{c} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{c}$$

- Based on the preceding derivation, we can make a completely general statement: The time derivative of *any* vector $\mathbf{c}$ with respect to *any* frame $\mathcal{A}$, is equal to the time derivative of $\mathbf{c}$ with respect to *any* frame $\mathcal{B}$ plus the cross product of the angular velocity of $\mathcal{B}$ in $\mathcal{A}$ with $\mathbf{c}$
- Other names: Basic Kinematic Equation, three-term vector derivative equation, transport theorem, Q-dot formula
- Not to be confused with Reynolds transport theorem

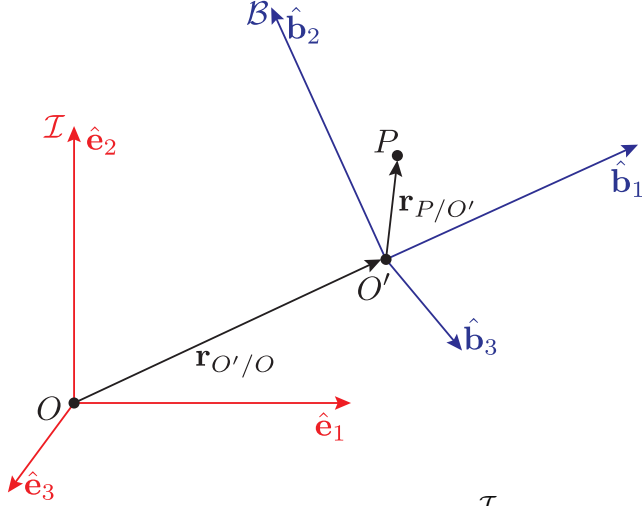
Adding Angular Velocities

- Consider the case where we define 3 reference frames: $\mathcal{I}$, $\mathcal{A}$ and $\mathcal{B}$
- We can write three different angular velocities between pairs of frames: ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$, ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{A}}$, and ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}$
- By the transport equation, for any vector $\mathbf{r}$:

$$\begin{aligned} {}^{\mathcal{I}}\frac{d}{dt}\mathbf{r} &= \underbrace{{}^{\mathcal{A}}\frac{d}{dt}\mathbf{r}}_{{}^{\mathcal{B}}\frac{d}{dt}\mathbf{r} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = \frac{{}^{\mathcal{B}}d}{dt}\mathbf{r} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} \\ {}^{\mathcal{B}}\frac{d}{dt}\mathbf{r} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} &= \frac{{}^{\mathcal{B}}d}{dt}\mathbf{r} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} \end{aligned}$$

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$$

Centripetal and Coriolis Accelerations



- We can now derive completely general forms for the kinematic acceleration terms associated with rotating frames
- Consider a translating *and* rotating frame $\mathcal{B}$ and an inertial frame $\mathcal{I}$

$$\begin{aligned}\mathbf{r}_{P/O} &= \mathbf{r}_{O'/O} + \mathbf{r}_{P/O'} \\ \mathcal{I}\mathbf{v}_{P/O} &= \mathcal{I}\mathbf{v}_{O'/O} + \underbrace{\mathcal{I}\mathbf{v}_{P/O'}}_{\mathcal{B}\frac{d}{dt}\mathbf{r}_{P/O'} + \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}}\end{aligned}$$

$$\mathcal{I}\mathbf{a}_{P/O} = \frac{\mathcal{I}d}{dt}\mathcal{I}\mathbf{v}_{P/O} = \underbrace{\frac{\mathcal{I}d}{dt}\mathcal{I}\mathbf{v}_{O'/O}}_{\mathcal{I}\mathbf{a}_{O'/O}} + \frac{\mathcal{I}d}{dt}\left(\mathcal{B}\frac{d}{dt}\mathbf{r}_{P/O'}\right) + \frac{\mathcal{I}d}{dt}(\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'})$$

Centripetal and Coriolis Accelerations Continued

$$\mathcal{I}\mathbf{a}_{P/O} = \mathcal{I}\mathbf{a}_{O'/O} + \frac{\mathcal{I}d}{dt}\left(\mathcal{B}\frac{d}{dt}\mathbf{r}_{P/O'}\right) + \frac{\mathcal{I}d}{dt}(\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'})$$

$$\frac{\mathcal{I}d}{dt}\left(\mathcal{B}\frac{d}{dt}\mathbf{r}_{P/O'}\right) = \underbrace{\mathcal{B}\frac{d}{dt}\left(\mathcal{B}\frac{d}{dt}\mathbf{r}_{P/O'}\right)}_{\mathcal{B}\mathbf{a}_{P/O'}} + \underbrace{\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathcal{B}\frac{d}{dt}\mathbf{r}_{P/O'}}_{\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathcal{B}\mathbf{v}_{P/O'}}$$

$$\frac{\mathcal{I}d}{dt}(\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}) = \underbrace{\frac{\mathcal{I}d}{dt}\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}}_{\triangleq \mathcal{I}\boldsymbol{\alpha}^{\mathcal{B}}} + \underbrace{\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \frac{\mathcal{I}d}{dt}\mathbf{r}_{P/O'}}_{\mathcal{B}\frac{d}{dt}\mathbf{r}_{P/O'} + \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}}$$

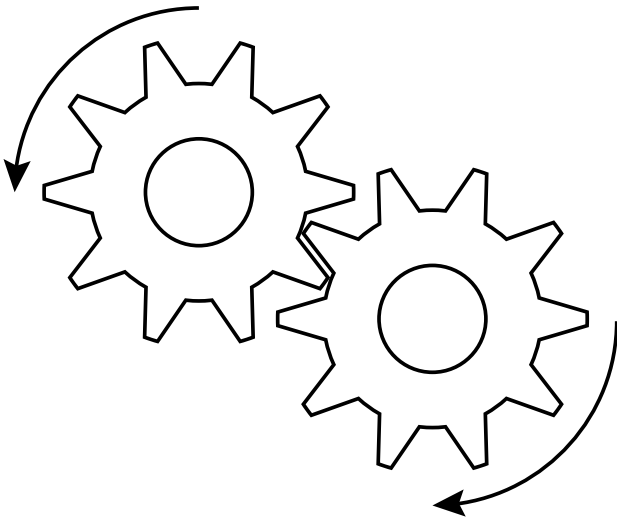
$$\mathcal{I}\mathbf{a}_{P/O} = \mathcal{I}\mathbf{a}_{O'/O} + \mathcal{B}\mathbf{a}_{P/O'} + \mathcal{I}\boldsymbol{\alpha}^{\mathcal{B}} \times \mathbf{r}_{P/O'} + \underbrace{2\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathcal{B}\mathbf{v}_{P/O'}}_{\text{Coriolis}} + \underbrace{\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times (\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'})}_{\text{Centripetal}}$$

Consistency Check

- Let $\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$, $\mathcal{B} \triangleq (O', \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$, and ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\hat{\mathbf{e}}_3 \Rightarrow {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{B}} = \ddot{\theta}\hat{\mathbf{e}}_3$
- $\mathbf{r}_{P/O'} = r\hat{\mathbf{e}}_r$ and $\mathbf{r}_{O'/O} = \mathbf{0}$

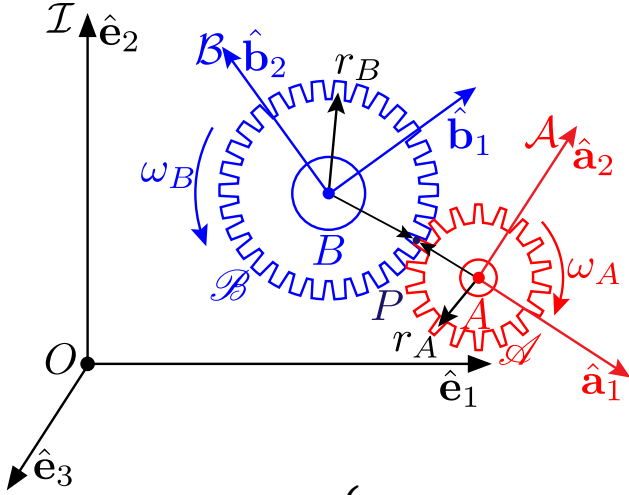
$$\begin{aligned}
 {}^{\mathcal{I}}\mathbf{a}_{P/O} &= {}^{\mathcal{I}}\mathbf{a}_{O'/O} + {}^{\mathcal{B}}\mathbf{a}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{B}} \times \mathbf{r}_{P/O'} + 2{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{B}}\mathbf{v}_{P/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times ({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}) \\
 &= \mathbf{0} \quad + \ddot{r}\hat{\mathbf{e}}_r \quad + \ddot{\theta}\hat{\mathbf{e}}_3 \times r\hat{\mathbf{e}}_r \quad + 2\dot{\theta}\hat{\mathbf{e}}_3 \times \dot{r}\hat{\mathbf{e}}_r \quad + \dot{\theta}\hat{\mathbf{e}}_3 \times (\dot{\theta}\hat{\mathbf{e}}_3 \times r\hat{\mathbf{e}}_r) \\
 &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{\theta}\dot{r})\hat{\mathbf{e}}_\theta
 \end{aligned}$$

Gears



- **Gears** are a type of simple machine for transferring rotational motion
- Gears can alter angular velocity (and therefore torque) in both magnitude and direction and can create mechanical advantage
- Multiple gears form a **gear train** or **transmission**
- The instantaneous point of contact on both gears must have the same inertial velocity

Gear Train Analysis



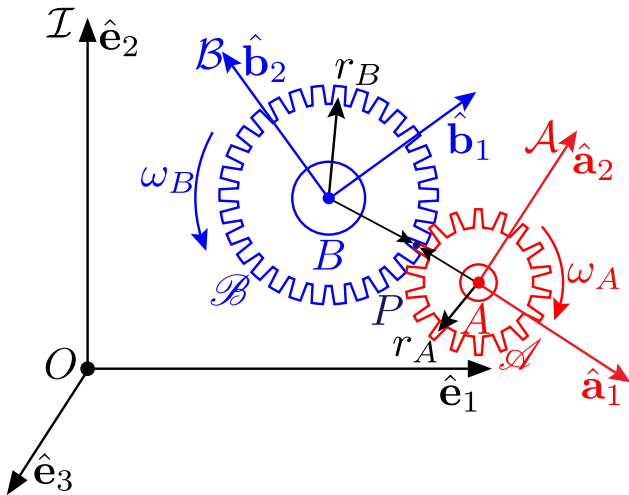
$$\mathcal{I}\mathbf{v}_{P/O} = \begin{cases} \mathcal{I}\mathbf{v}_{P/A} + \mathcal{I}\mathbf{v}_{A/O} = \mathcal{I}\mathbf{v}_{A/O} + \mathcal{A}\mathbf{v}_{P/A} + \mathcal{I}\boldsymbol{\omega}^A \times \mathbf{r}_{P/A} \\ \mathcal{I}\mathbf{v}_{P/B} + \mathcal{I}\mathbf{v}_{B/O} = \mathcal{I}\mathbf{v}_{B/O} + \mathcal{B}\mathbf{v}_{P/B} + \mathcal{I}\boldsymbol{\omega}^B \times \mathbf{r}_{P/B} \end{cases}$$

- In order to maintain contact, the centers of the gears must stay at a fixed distance from one another, thus $\mathcal{I}\mathbf{v}_{A/O} \equiv \mathcal{I}\mathbf{v}_{B/O}$

- Consider a train composed of 2 gears, $\mathcal{A}$ and $\mathcal{B}$, centered at points A and B , with radii r_A and r_B , respectively
- The gears rotate with angular rates ω_A and ω_B , respectively. We attach frames $\mathcal{A}$ and $\mathcal{B}$ to each gear such that:
 $\mathcal{I}\boldsymbol{\omega}^A = \omega_A \hat{\mathbf{e}}_3, \mathcal{I}\boldsymbol{\omega}^B = \omega_B \hat{\mathbf{e}}_3$
- The gears mesh at point P

Gear Train Analysis Continued

$$\mathcal{I}\mathbf{v}_{A/O} + \mathcal{A}\mathbf{v}_{P/A} + \mathcal{I}\boldsymbol{\omega}^A \times \mathbf{r}_{P/A} = \mathcal{I}\mathbf{v}_{B/O} + \mathcal{B}\mathbf{v}_{P/B} + \mathcal{I}\boldsymbol{\omega}^B \times \mathbf{r}_{P/B}$$



- The contact point P is fixed in each gear, so: $\mathcal{A}\mathbf{v}_{P/A} = \mathcal{B}\mathbf{v}_{P/B} = \mathbf{0}$
- $\mathbf{r}_{P/A}$ and $\mathbf{r}_{P/B}$ must point in opposite directions ($\hat{\mathbf{r}}_{P/A} = -\hat{\mathbf{r}}_{P/B}$) so:

$$\mathcal{I}\boldsymbol{\omega}^A \times \mathbf{r}_{P/A} = \omega_A \hat{\mathbf{e}}_3 \times (r_A \hat{\mathbf{r}}_{P/A})$$

$$\mathcal{I}\boldsymbol{\omega}^B \times \mathbf{r}_{P/B} = \omega_B \hat{\mathbf{e}}_3 \times (-r_B \hat{\mathbf{r}}_{P/A})$$

- Putting all this together gives us the **gear equation**:

$$\omega_B = -\frac{r_A}{r_B} \omega_A$$

More on Gears

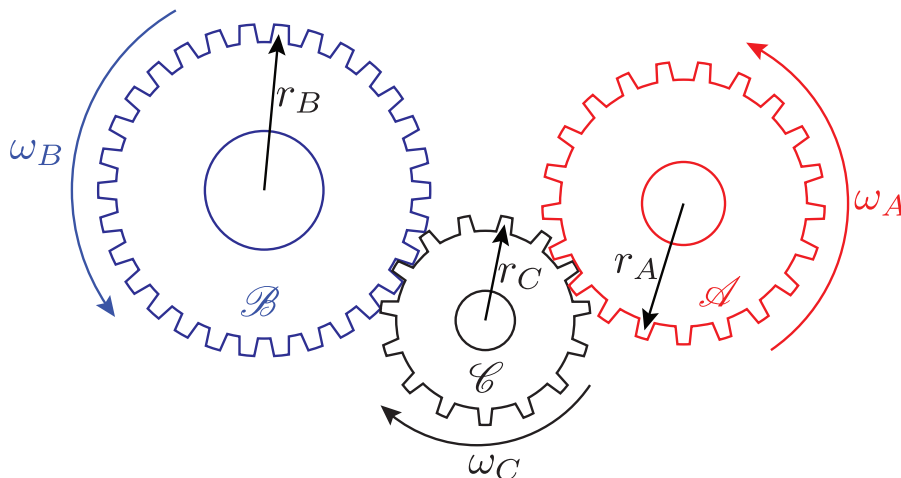
- In order to mesh, teeth on both gears must be the same size and spaced equally, so the number of teeth is proportional to the gear radius:

$$\frac{r_A}{r_B} = \frac{N_A}{N_B} = -\frac{\omega_B}{\omega_A}$$

- These fractions all represent the **gear ratio**, which is also the ratio of input to output torque and therefore the mechanical advantage of the gear train
- The gear constraint is a **motion** constraint (the contact point has the same inertial velocity on both gears)
 - This is a constraint on velocities rather than coordinates, but can still reduce the DOF and the number of required coordinates

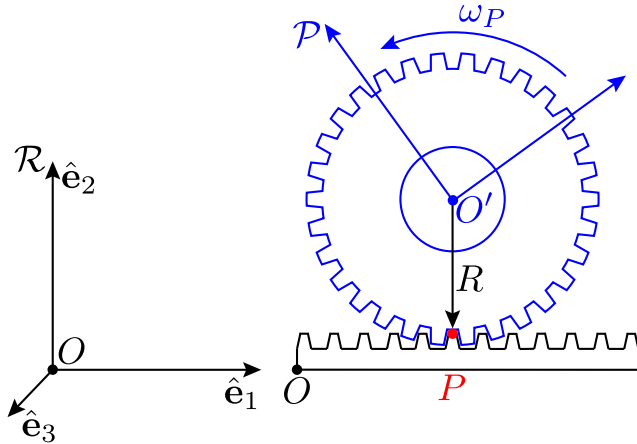
Idler Gears

- Note the negative sign in the gear equation: the direction of rotation is reversed by each gear
- Frequently gear trains will include additional gears that don't affect the overall gear ratio, but change the rotation direction. These are called **idler gears**



$$\begin{aligned}\omega_C &= -\frac{r_A}{r_C}\omega_A \\ \omega_B &= -\frac{r_C}{r_B}\omega_C = \frac{r_A}{r_B}\omega_A\end{aligned}$$

Rack and Pinion



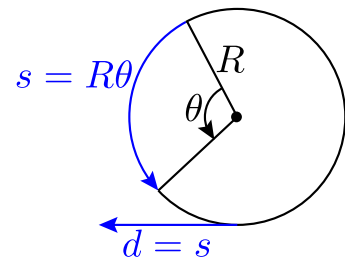
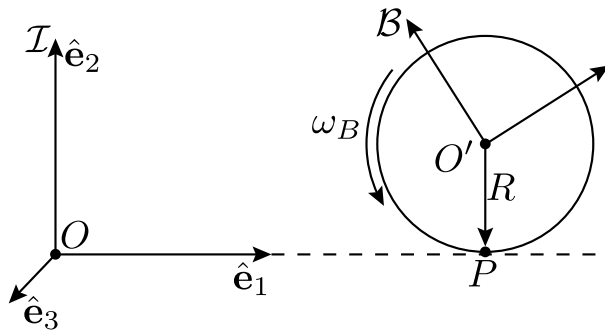
- Gears don't actually have to be circular. Consider a **rack and pinion** system
- $\mathcal{R}$ is fixed to the rack and $\mathcal{P}$ is fixed to the pinion
- How fast is the pinion moving along the rack (i.e., what is ${}^{\mathcal{R}}\mathbf{v}_{O'/O}$)?

$${}^{\mathcal{R}}\mathbf{v}_{O'/O} = \underbrace{{}^{\mathcal{R}}\mathbf{v}_{P/O}}_0 + {}^{\mathcal{R}}\mathbf{v}_{O'/P} = \underbrace{{}^{\mathcal{P}}\mathbf{v}_{O'/P}}_0 + {}^{\mathcal{R}}\boldsymbol{\omega}^{\mathcal{P}} \times \mathbf{r}_{O'/P} = \omega_P \hat{\mathbf{e}}_3 \times R \hat{\mathbf{e}}_2 = -\omega_P R \hat{\mathbf{e}}_1$$

- This is direct conversion of rotation into translation

Roll Without Slip

- Consider a rolling wheel where the movement of the wheel's center (O') is due only to rolling (*not* sliding)



- $d = R\theta$ so $\dot{d} = R\dot{\theta}$ and ${}^{\mathcal{I}}\mathbf{v}_{O'/O} = -R\dot{\theta}\hat{\mathbf{e}}_1 \equiv -{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}$

$${}^{\mathcal{I}}\mathbf{v}_{O'/O} = {}^{\mathcal{I}}\mathbf{v}_{P/O} - {}^{\mathcal{I}}\mathbf{v}_{P/O'} = {}^{\mathcal{I}}\mathbf{v}_{P/O} - \underbrace{{}^{\mathcal{B}}\mathbf{v}_{P/O'}}_0 - {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'}$$

- The expressions for ${}^{\mathcal{I}}\mathbf{v}_{O'/O}$ must be equal, meaning that ${}^{\mathcal{I}}\mathbf{v}_{P/O}$ must equal zero

No-Slip Condition

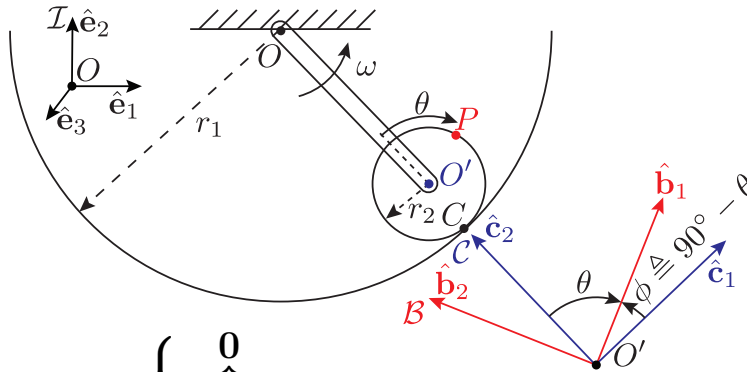
- What we've just found ($\mathcal{I}\mathbf{v}_{P/O} = \mathbf{0}$) is called the **no-slip condition**: in rolling without slip, the contact point on the wheel has the same inertial velocity as the surface the wheel is rolling on (zero for inertially fixed surfaces)
- This is again a velocity constraint that removes a degree of freedom:

$$\mathcal{I}\boldsymbol{\omega}^B = \dot{\theta}\hat{\mathbf{e}}_3 \quad \Rightarrow \quad \underbrace{\mathbf{r}_{O'/O} \cdot \hat{\mathbf{e}}_1}_{\triangleq x} = \int_{t_0}^t -R\dot{\theta} dt \quad \Rightarrow \quad x = -R[\theta(t) - \theta(t_0)] + x(t_0)$$

- NB: There must be some force acting on the rolling object—typically we model this as friction
- This is a static friction force that *does no work*

Wheel on Circular Track

A wheel (centered at O') of radius r_2 is being rotated about a hub (O) at a constant angular speed ω and rolls without slip on a circular track of radius r_1 . What is the inertial acceleration of a point P on the wheel's edge?



$$\begin{aligned} \hat{\mathbf{c}}_2 &\equiv \hat{\mathbf{r}}_{O/O'} & \hat{\mathbf{b}}_1 &\equiv \hat{\mathbf{r}}_{P/O'} \\ \hat{\mathbf{b}}_3 &\equiv \hat{\mathbf{c}}_3 \equiv \hat{\mathbf{e}}_3 \\ \mathcal{I}\boldsymbol{\omega}^C &= \omega\hat{\mathbf{e}}_3 \\ {}^C\boldsymbol{\omega}^B &= \dot{\phi}\hat{\mathbf{e}}_3 = -\dot{\theta}\hat{\mathbf{e}}_3 \\ \mathcal{I}\boldsymbol{\omega}^B &= \mathcal{I}\boldsymbol{\omega}^C + {}^C\boldsymbol{\omega}^B \\ &= (\omega - \dot{\theta})\hat{\mathbf{e}}_3 \end{aligned}$$

$$\mathcal{I}\mathbf{v}_{O'/O} = \begin{cases} \underbrace{0} \\ {}^C\mathbf{v}_{O'/O} + \mathcal{I}\boldsymbol{\omega}^C \times \mathbf{r}_{O'/O} = \omega\hat{\mathbf{e}}_3 \times -(r_1 - r_2)\hat{\mathbf{c}}_2 = \omega(r_1 - r_2)\hat{\mathbf{c}}_1 \\ \underbrace{\mathcal{I}\mathbf{v}_{O'/C}}_0 + \underbrace{\mathcal{I}\mathbf{v}_{C/O}}_0 = \underbrace{{}^B\mathbf{v}_{O'/C}}_0 + \mathcal{I}\boldsymbol{\omega}^B \times \mathbf{r}_{O'/C} = (\omega - \dot{\theta})\hat{\mathbf{e}}_3 \times r_2\hat{\mathbf{c}}_2 = -(\omega - \dot{\theta})r_2\hat{\mathbf{c}}_1 \end{cases}$$

Wheel on Circular Track Continued

- The two velocity expressions must match so:

$$\omega(r_1 - r_2)\hat{\mathbf{c}}_1 = -(\omega - \dot{\theta})r_2\hat{\mathbf{c}}_1 \quad \Rightarrow \quad \dot{\theta} = \frac{r_1}{r_2}\omega \quad \Rightarrow \quad \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} = \omega \left(1 - \frac{r_1}{r_2}\right) \hat{\mathbf{e}}_3$$

- We can now find the velocity and acceleration of P with respect to O :

$$\begin{aligned} \mathcal{I}\mathbf{v}_{P/O} &= \mathcal{I}\mathbf{v}_{O'/O} + \mathcal{I}\mathbf{v}_{P/O'} = \omega(r_1 - r_2)\hat{\mathbf{c}}_1 + \overbrace{\mathcal{B}\mathbf{v}_{P/O'}}^{\mathbf{0}} + \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \overbrace{\mathbf{r}_{P/O'}}^{r_2\hat{\mathbf{b}}_1} \\ &= \omega(r_1 - r_2)\hat{\mathbf{c}}_1 + \omega(r_2 - r_1)\hat{\mathbf{b}}_2 \\ \mathcal{I}\mathbf{a}_{P/O} &= \frac{d}{dt}\mathcal{I}\mathbf{v}_{P/O} = \omega(r_1 - r_2) \underbrace{(\mathcal{I}\boldsymbol{\omega}^{\mathcal{C}} \times \hat{\mathbf{c}}_1)}_{\omega\hat{\mathbf{e}}_3 \times \hat{\mathbf{c}}_1 = \omega\hat{\mathbf{c}}_2} + \omega(r_2 - r_1) \underbrace{(\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_2)}_{-\omega\left(1 - \frac{r_1}{r_2}\right)\hat{\mathbf{b}}_1} \end{aligned}$$

Wheel on Circular Track Concluded

- Putting it all together:

$$\begin{aligned} \mathcal{I}\mathbf{a}_{P/O} &= \omega^2(r_1 - r_2)\hat{\mathbf{c}}_2 - \omega^2 \frac{(r_2 - r_1)^2}{r_2} \underbrace{\hat{\mathbf{b}}_1}_{\sin(\theta)\hat{\mathbf{c}}_1 + \cos(\theta)\hat{\mathbf{c}}_2} \\ &= \omega^2 \left[-\frac{(r_2 - r_1)^2}{r_2} \sin\theta\hat{\mathbf{c}}_1 + \left(r_1 - r_2 - \frac{(r_2 - r_1)^2}{r_2} \cos\theta \right) \hat{\mathbf{c}}_2 \right] \end{aligned}$$

- Key takeaways:

- ① Things get complex *really* fast
- ② Consistent notation can help prevent you from getting lost
- ③ The transport equation is one of your most important tools
- ④ This was not actually a dynamics problem: all we did was kinematics

Week 9

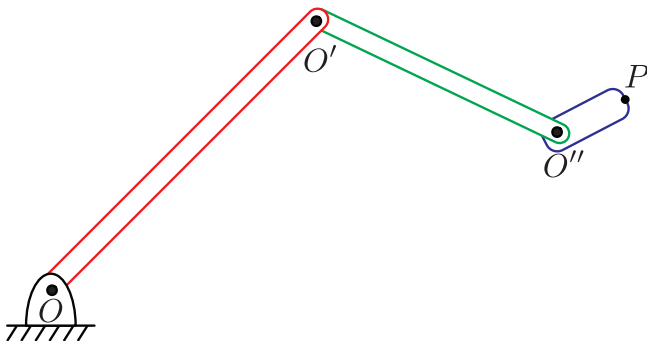
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MAE 2030, Spring 2026

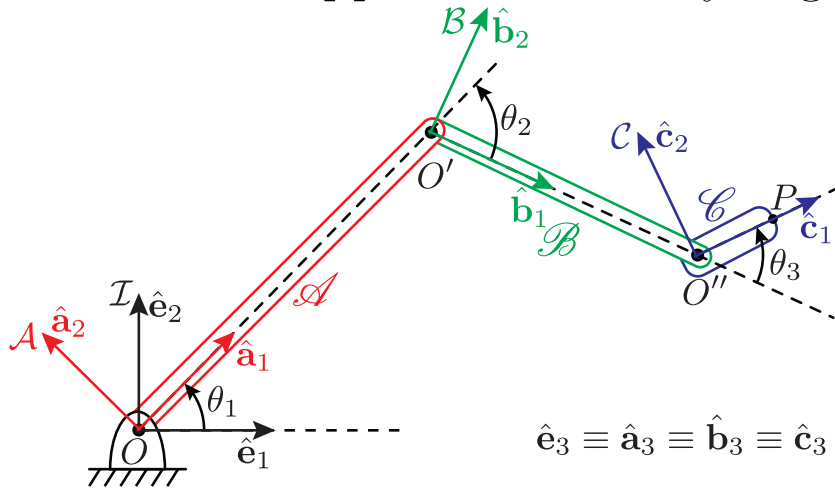
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Linkages



- Mechanical **linkages** are assemblies of connected elements used to transfer forces and motions
 - Can provide mechanical advantage
 - Can convert rotational motion to translational motion or transfer rotational motion
-
- Linkages modeled as rigid links attached by ideal joints are called **kinematic chains**
 - A kinematic chain with one stationary link is a **mechanism**
 - A stationary linkage is a **structure**

Fundamental Approach to Analyzing Linkages

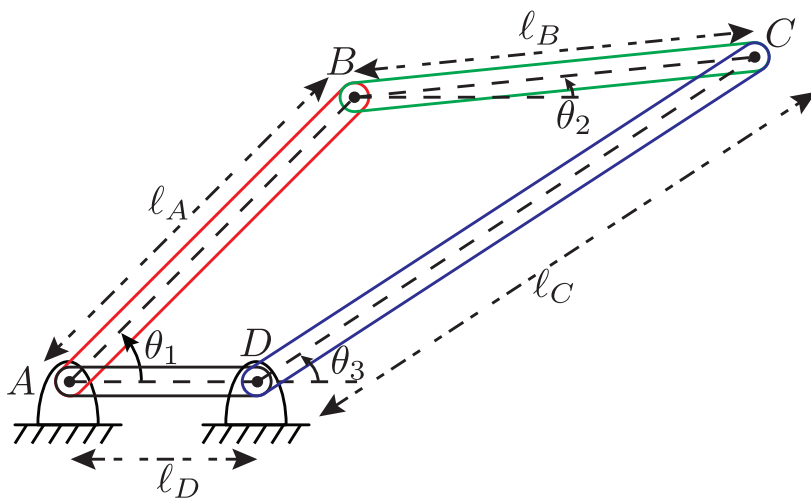


- Attach one reference frame to each link
- For planar systems, all reference frames can have a common unit direction

The relative velocity of any two points attached to a link with respect to the reference frame attached to the link will always be zero

$$\begin{aligned} {}^{\mathcal{A}}\mathbf{v}_{O'/O} &= \mathbf{0} & {}^{\mathcal{B}}\mathbf{v}_{O''/O'} &= \mathbf{0} & {}^{\mathcal{C}}\mathbf{v}_{P/O''} &= \mathbf{0} \\ {}^{\mathcal{I}}\mathbf{v}_{P/O} &= {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{O'/O} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{O''/O'} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{C}} \times \mathbf{r}_{P/O''} \end{aligned}$$

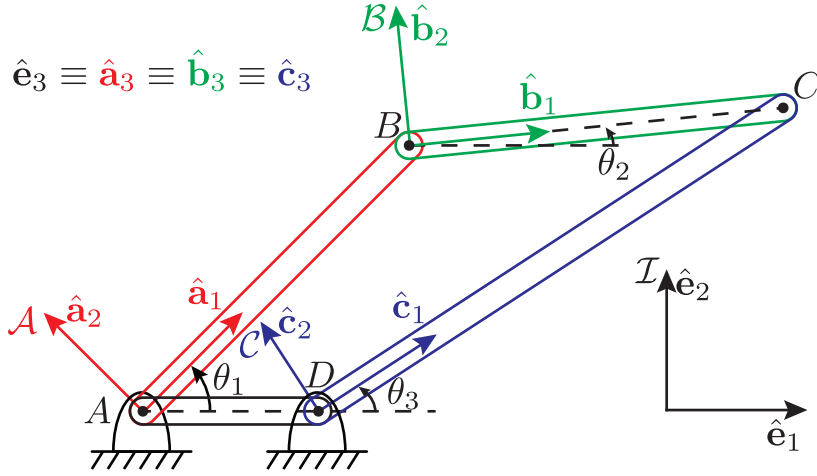
Four-Bar Linkage



- Consider the four-bar linkage formed by quadrilateral $\square ABCD$
- Points A and D are inertially fixed
- Link $\overline{AB}$ is driven, such that θ_1 is known for all time

$$\begin{aligned} \ell_D &= \ell_A \cos(\theta_1) + \ell_B \cos(\theta_2) - \ell_C \cos(\theta_3) \\ \ell_A \sin(\theta_1) &= \ell_C \sin(\theta_3) - \ell_B \sin(\theta_2) \end{aligned}$$

Four-Bar Linkage Analysis



$$\begin{aligned}\mathcal{I}\omega^A &= \dot{\theta}_1 \hat{\mathbf{a}}_3 & \mathcal{I}\omega^B &= \dot{\theta}_2 \hat{\mathbf{b}}_3 \\ \mathcal{I}\omega^C &= \dot{\theta}_3 \hat{\mathbf{c}}_3 \\ {}^A C^{\mathcal{I}} &= C_3(\theta_1) & {}^B C^{\mathcal{I}} &= C_3(\theta_2) \\ {}^C C^{\mathcal{I}} &= C_3(\theta_3)\end{aligned}$$

$$\begin{aligned}\mathcal{I}\mathbf{v}_{D/C} + \mathcal{I}\mathbf{v}_{C/B} + \mathcal{I}\mathbf{v}_{B/A} \\ = \mathcal{I}\mathbf{v}_{D/A} = \mathbf{0}\end{aligned}$$

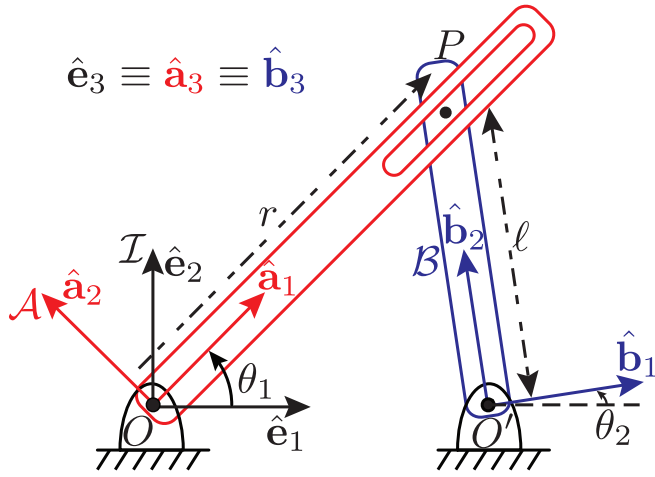
$$\text{Let } \dot{\theta}_1 = \omega \text{ (constant)}$$

$$\begin{aligned}\mathcal{I}\mathbf{v}_{B/A} &= {}^A \mathbf{v}_{B/A} + \mathcal{I}\omega^A \times \mathbf{r}_{B/A} = \ell_A \dot{\theta}_1 \hat{\mathbf{a}}_2 \\ \mathcal{I}\mathbf{v}_{C/B} &= {}^B \mathbf{v}_{C/B} + \mathcal{I}\omega^B \times \mathbf{r}_{C/B} = \ell_B \dot{\theta}_2 \hat{\mathbf{b}}_2 \\ \mathcal{I}\mathbf{v}_{D/C} &= {}^C \mathbf{v}_{D/C} + \mathcal{I}\omega^C \times \mathbf{r}_{D/C} = -\ell_C \dot{\theta}_3 \hat{\mathbf{c}}_2\end{aligned}$$

$$\begin{aligned}\left(-\ell_A \dot{\theta}_1 \sin(\theta_1) - \ell_B \dot{\theta}_2 \sin(\theta_2) + \ell_C \dot{\theta}_3 \sin(\theta_3) \right) \hat{\mathbf{e}}_1 \\ \left(\ell_A \dot{\theta}_1 \cos(\theta_1) + \ell_B \dot{\theta}_2 \cos(\theta_2) - \ell_C \dot{\theta}_3 \cos(\theta_3) \right) \hat{\mathbf{e}}_2 = \mathbf{0}\end{aligned}$$

$$\dot{\theta}_2 = -\frac{\ell_A \omega \sin(\theta_1 - \theta_3)}{\ell_B \sin(\theta_2 - \theta_3)} \quad \dot{\theta}_3 = -\frac{\ell_A \omega \sin(\theta_1 - \theta_2)}{\ell_C \sin(\theta_2 - \theta_3)}$$

Linkages with Sliding Contacts



$$\hat{\mathbf{e}}_3 \equiv \hat{\mathbf{a}}_3 \equiv \hat{\mathbf{b}}_3$$

- Some linkages may have connections that can travel some distance *inside* one of the link elements - these are called **sliding contacts**
- For example: P is fixed in $\mathcal{B}$, but can move in $\mathcal{A}$ (only in $\hat{\mathbf{a}}_1$ direction)

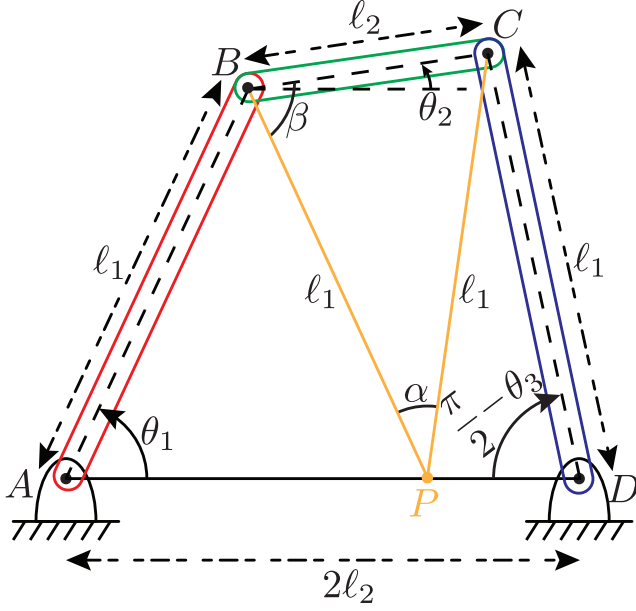
$$\begin{aligned} \mathcal{I}\boldsymbol{\omega}^{\mathcal{A}} &= \dot{\theta}_1 \hat{\mathbf{a}}_3 & \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} &= \dot{\theta}_2 \hat{\mathbf{b}}_3 \\ {}^{\mathcal{A}}C^{\mathcal{I}} &= C_3(\theta_1) & {}^{\mathcal{B}}C^{\mathcal{I}} &= C_3(\theta_2) \\ \mathbf{r}_{P/O} &= r \hat{\mathbf{a}}_1 & \mathbf{r}_{P/O'} &= \ell \hat{\mathbf{b}}_2 \end{aligned}$$

$$\mathcal{I}\mathbf{v}_{P/O} = \begin{cases} {}^{\mathcal{A}}\mathbf{v}_{P/O} + \mathcal{I}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{P/O} & = \dot{r} \hat{\mathbf{a}}_1 + \dot{\theta}_1 r \hat{\mathbf{a}}_2 \\ \mathcal{I}\mathbf{v}_{P/O'} + \mathcal{I}\mathbf{v}_{O'/O} = {}^{\mathcal{B}}\mathbf{v}_{P/O'} + \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O'} & = -\dot{\theta}_2 \ell \hat{\mathbf{b}}_1 \end{cases}$$

$$\begin{aligned} & \left(-r \dot{\theta}_1 \sin(\theta_1) + \dot{r} \cos(\theta_1) \right) \hat{\mathbf{e}}_1 + \left(r \dot{\theta}_1 \cos(\theta_1) + \dot{r} \sin(\theta_1) \right) \hat{\mathbf{e}}_2 \\ & = -\ell \dot{\theta}_2 \cos(\theta_2) \hat{\mathbf{e}}_1 - \ell \dot{\theta}_2 \sin(\theta_2) \hat{\mathbf{e}}_2 \end{aligned}$$

$$\text{For } \dot{\theta}_1 = \omega: \quad \dot{r} = -\frac{\omega r}{\tan(\theta_1 - \theta_2)} \quad \dot{\theta}_2 = \frac{\omega r}{\ell \sin(\theta_1 - \theta_2)}$$

Approximating Straight-Line Motion



$$2\ell_2 = \ell_1 (\cos(\theta_1) + \sin(\theta_3)) + \ell_2 \cos(\theta_2)$$

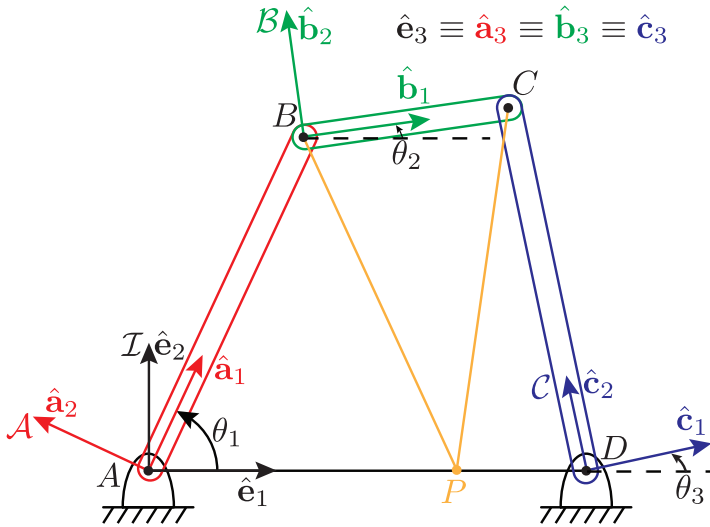
- A **Roberts linkage** is a four-bar linkage where $\overline{AB} = \overline{CD} = \ell_1$ and $\overline{AD} = 2\overline{BC} = 2\ell_2$
- Point P is vertex of isosceles $\triangle PCB$ with $\overline{PB} = \overline{AB}$.
- As link $\overline{AB}$ is driven, P moves approximately along $\overline{AD}$

$$\alpha = \cos^{-1} \left(1 - \frac{1}{2} \left(\frac{\ell_2}{\ell_1} \right)^2 \right)$$

$$\beta = \sin^{-1} \left(\sin(\alpha) \frac{\ell_1}{\ell_2} \right)$$

$$\ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = \ell_1 \cos(\theta_3)$$

Roberts Linkage Analysis



$$\mathcal{I}_{\omega}^A = \dot{\theta}_1 \hat{\mathbf{a}}_3 \quad \mathcal{I}_{\omega}^B = \dot{\theta}_2 \hat{\mathbf{b}}_3 \quad \mathcal{I}_{\omega}^C = \dot{\theta}_3 \hat{\mathbf{c}}_3$$

$${}^A C^{\mathcal{I}} = C_3(\theta_1) \quad {}^B C^{\mathcal{I}} = C_3(\theta_2)$$

$${}^C C^{\mathcal{I}} = C_3(\theta_3)$$

$$\mathcal{I}_{\mathbf{v}_{B/A}} = {}^A \mathbf{v}_{B/A} + \mathcal{I}_{\omega}^A \times \mathbf{r}_{B/A} = \ell_1 \dot{\theta}_1 \hat{\mathbf{a}}_2$$

$$\mathcal{I}_{\mathbf{v}_{C/B}} = {}^B \mathbf{v}_{C/B} + \mathcal{I}_{\omega}^B \times \mathbf{r}_{C/B} = \ell_2 \dot{\theta}_2 \hat{\mathbf{b}}_2$$

$$\mathcal{I}_{\mathbf{v}_{D/C}} = {}^C \mathbf{v}_{D/C} + \mathcal{I}_{\omega}^C \times \mathbf{r}_{D/C} = \ell_1 \dot{\theta}_3 \hat{\mathbf{c}}_1$$

$$\mathcal{I}_{\mathbf{v}_{D/C}} + \mathcal{I}_{\mathbf{v}_{C/B}} + \mathcal{I}_{\mathbf{v}_{B/A}} = \mathcal{I}_{\mathbf{v}_{D/A}} = \mathbf{0}$$

Roberts Linkage Analysis

$$\begin{pmatrix} -\ell_1 \dot{\theta}_1 \sin(\theta_1) + \ell_1 \dot{\theta}_3 \cos(\theta_3) - \ell_2 \dot{\theta}_2 \sin(\theta_2) \\ \ell_1 \dot{\theta}_1 \cos(\theta_1) + \ell_1 \dot{\theta}_3 \sin(\theta_3) + \ell_2 \dot{\theta}_2 \cos(\theta_2) \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \end{pmatrix} = \mathbf{0}$$

Assuming $\theta_1, \dot{\theta}_1$ known:

$$\dot{\theta}_2 = -\frac{\ell_1 \dot{\theta}_1 \cos(\theta_1 - \theta_3)}{\ell_2 \cos(\theta_2 - \theta_3)} \quad \dot{\theta}_3 = \frac{\dot{\theta}_1 \sin(\theta_1 - \theta_2)}{\cos(\theta_2 - \theta_3)}$$

Recall the Simple Harmonic Oscillator

- Any system governed by EOM $\ddot{z} = -\omega_0^2 z$ is a **simple harmonic oscillator** (SHO) with a **natural frequency** of ω_0
- Assume a solution of the form $z(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)$
- $\dot{z} = -A\omega_0 \sin(\omega_0 t) + B\omega_0 \cos(\omega_0 t) \Rightarrow \ddot{z} = -A\omega_0^2 \cos(\omega_0 t) - B\omega_0^2 \sin(\omega_0 t) = -\omega_0^2 z$
- @ $t = 0$: $z(0) = A \cdot 1 + B \cdot 0 \Rightarrow A = z(0)$
- @ $t = 0$: $\dot{z}(0) = -A\omega_0 \cdot 0 + B\omega_0 \cdot 1 \Rightarrow B = \frac{\dot{z}(0)}{\omega_0}$

$$z(t) = z(0) \cos(\omega_0 t) + \frac{\dot{z}(0)}{\omega_0} \sin(\omega_0 t)$$

SHOs as Linear Systems

- We can write the EOM of an SHO in first-order state-space form as:

$$\underbrace{\begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix}}_{\triangleq \dot{Z}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}}_{\triangleq A} \underbrace{\begin{bmatrix} z \\ \dot{z} \end{bmatrix}}_{\triangleq Z}$$

- Assume a solution of the form:

$$Z = \begin{bmatrix} z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} \quad \text{where } a_1, a_2, \lambda \text{ are constants}$$

- Differentiating the assumed solution:

$$\dot{Z} = \begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} \implies \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} = A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t}$$

This is an Eigenvalue Problem!

- Assuming $a_1, a_2 \neq 0$: $(\lambda I - A) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} = 0 \implies (\lambda I - A) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$
- In general: for $n \times n$ matrix A , $n \times 1$ **eigenvector** $\mathbf{v}$, and scalar **eigenvalue** λ :

$$A\mathbf{v} = \lambda\mathbf{v} \implies (A - \lambda I)\mathbf{v} = 0$$

- If $\mathbf{v} \neq 0$, this implies that $(A - \lambda I)$ is singular (has zero determinant). Thus, the only general solution is when:

$$\det(A - \lambda I) = 0$$

- This is called the **characteristic polynomial** (or equation) of A

SHOs as Linear Systems Continued

- The characteristic equation for our SHO is:

$$\det \left(\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = 0 \implies \lambda^2 + \omega_0^2 = 0 \implies \lambda_{1,2} = \pm \omega_0 i$$

- The eigenvalues of A encode the natural frequencies of the system!
These are the system's **modes**
- Plugging back into the original eigenvalue problem allows us to solve for the corresponding eigenvectors (these will not always be unique). In this case:

$$\mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

- These are the system's **mode shapes**

General Homogeneous Solution

- For n distinct, real eigenvalues λ_i , and eigenvectors $\mathbf{v}_i$, the general, homogeneous solution is:

$$Z = \sum c_i \mathbf{v}_i e^{\lambda_i t}$$

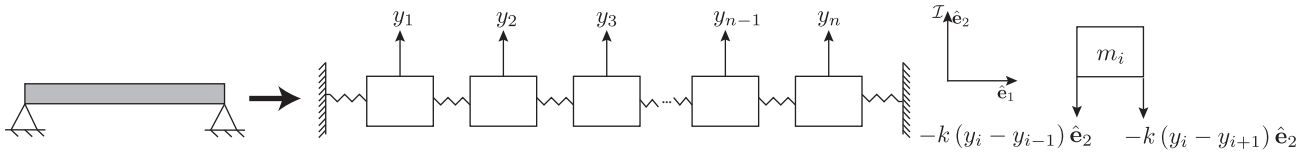
where c_i are constants determined by initial conditions

- For the SHO:

$$Z = \begin{bmatrix} z \\ \dot{z} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} e^{\lambda_2 t}$$

- The motion of the system is the superposition of its modes
- NB: when comparing against our previous solutions to the SHO, remember Euler's formula: $e^{ix} = \cos(x) + i \sin(x)$

Coupled Oscillators



- Consider a beam modeled as n identical masses connected by identical springs of constant k . Assume each mass only moves vertically and neglecting gravity:

$$m\ddot{y}_i = -k(y_i - y_{i+1}) - k(y_i - y_{i-1}) \implies m_i\ddot{y}_i + 2ky_i - k(y_{i+1} + y_{i-1}) = 0$$

- We define the system's state as:

$$Z = [y_1 \quad \dot{y}_1 \quad y_2 \quad \dot{y}_2 \quad \cdots \quad y_{n-1} \quad \dot{y}_{n-1} \quad y_n \quad \dot{y}_n]^T$$

$$\underbrace{\begin{bmatrix} \dot{y}_1 \\ \ddot{y}_1 \\ \dot{y}_2 \\ \ddot{y}_2 \\ \vdots \\ \dot{y}_n \\ \ddot{y}_n \end{bmatrix}}_{\dot{Z}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{2k}{m} & 0 & \frac{k}{m} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \frac{k}{m} & 0 & -\frac{2k}{m} & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{2k}{m} & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \\ \vdots \\ y_n \\ \dot{y}_n \end{bmatrix}}_Z$$

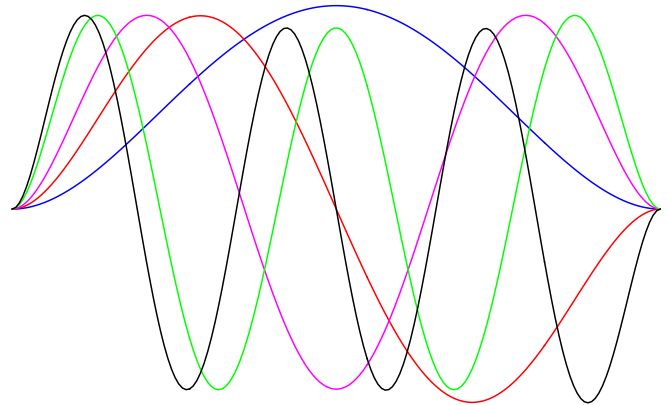
Coupled Oscillators Solution

- We can solve the eigenvalue problem to find this system's natural frequencies and mode shapes:

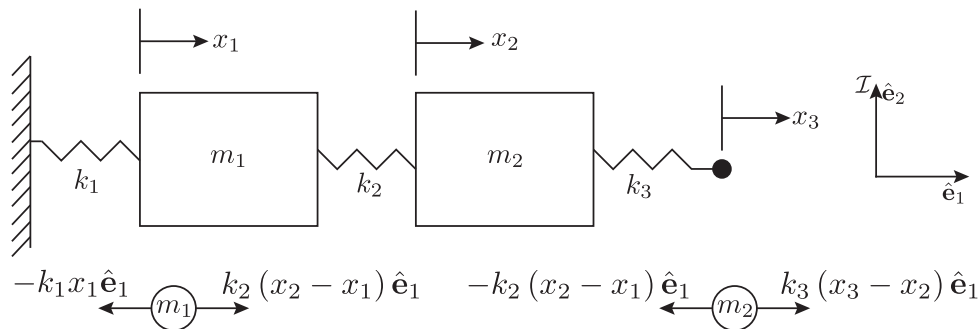
$$\det(A - \lambda I) = 0$$

- Just as with the SHO, the general homogeneous solution is the superposition of all of the modes:

$$Z = \sum c_i \mathbf{v}_i e^{\lambda_i t}$$



2-DOF Driven Coupled Oscillator

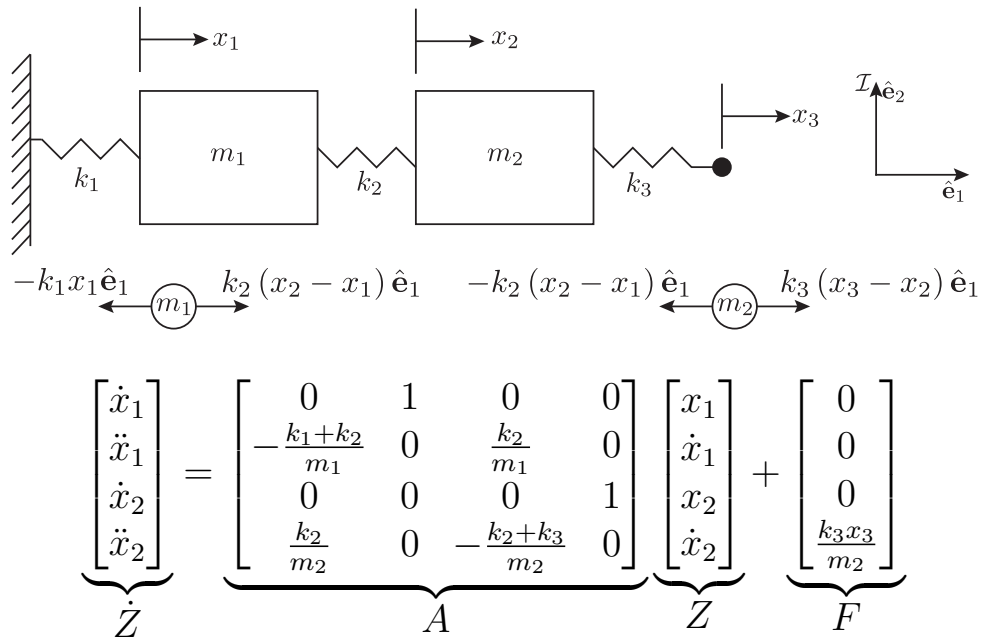


- Consider a system composed of 2 masses and 3 springs, with the masses moving in the horizontal direction only. The final spring is attached to a motorized input such that the position of the end of the spring (x_3) is known for all time.
- x_1, x_2, x_3 are defined from the springs' rest lengths such that:

$$m_1 \ddot{x}_1 = k_2 (x_2 - x_1) - k_1 x_1 \quad m_2 \ddot{x}_2 = k_3 (x_3 - x_2) - k_2 (x_2 - x_1)$$

- We define the system's state as: $Z = [x_1 \ \dot{x}_1 \ x_2 \ \dot{x}_2]^T$
- As x_3 is known, it is not part of the state definition, but acts as a forcing function

2-DOF Driven Coupled Oscillator State Space



2-DOF Driven Coupled Oscillator Homogeneous Solution

- As before, the general solution homogeneous solution is the superposition of the system's modes:

$$Z = \sum c_i \mathbf{v}_i e^{\lambda_i t}$$

- Assuming $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$:

$$\lambda_i = \pm i \sqrt{\frac{k}{m}}, \pm i \sqrt{\frac{3k}{m}} \quad \mathbf{v}_i = \begin{bmatrix} \pm i \sqrt{\frac{m}{k}} \\ 1 \\ \pm i \sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}, \begin{bmatrix} \pm i \sqrt{\frac{m}{3k}} \\ -1 \\ \pm i \sqrt{\frac{m}{3k}} \\ 1 \end{bmatrix}$$

- While there are four distinct eigenvectors produced we can see that they come in pairs. For a 2 DOF system, we expect two independent modes

2-DOF Driven Coupled Oscillator 2nd-Order State Space

- Note that the equations of motion for this system are functions of the x_i coordinates only (and not their derivatives)
- What if we define a second-order state:

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{Z}} = \underbrace{\begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_Z + \underbrace{\begin{bmatrix} 0 \\ \frac{k_3 x_3}{m_2} \end{bmatrix}}_F$$

- Assuming $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$:

$$\lambda_i = -\frac{k}{m}, -\frac{3k}{m} \quad \mathbf{v}_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- We have two distinct modes of motion for two different excitation frequencies—the two masses either move together or move opposite one another

Week 10

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MAE 2030, Spring 2026

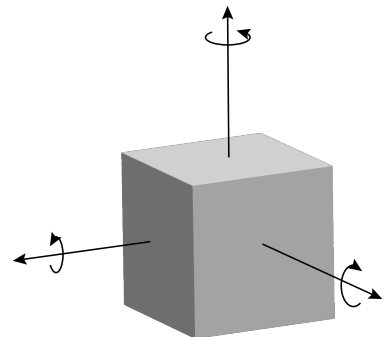
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Degrees of Freedom of Rigid Bodies in 3D

A rigid body

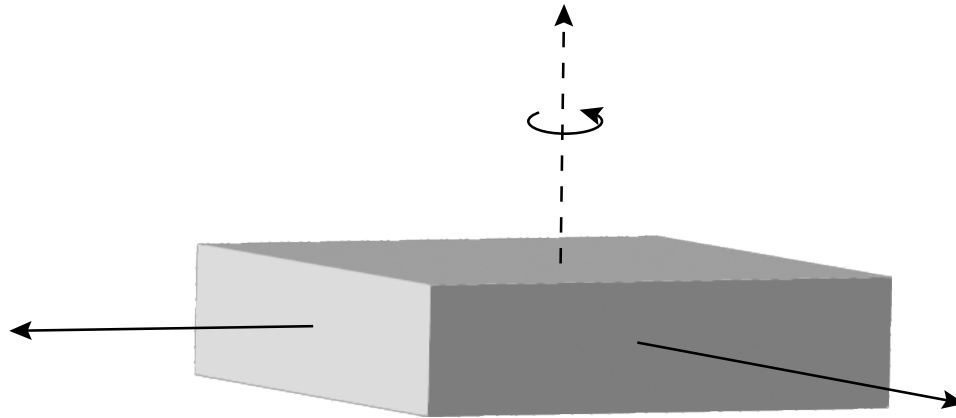
is a collection of ∞ -many infinitesimal particles, constrained to be stationary with respect to one another: $\frac{d}{dt}\|\mathbf{r}_{i/j}\| = 0 \forall i, j$

- A rigid body in 3 dimensions has 6 degrees of freedom
 - 1 We can intuitively split these into three translational and three rotational degrees of freedom
 - 2 We can prove this by induction from our standard definition of $\text{DOF} = 3N - k$

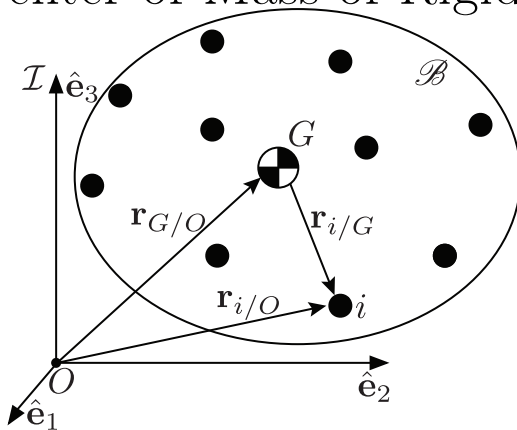


Planar Rigid Bodies

- A **planar** rigid body is one that is constrained to move only in one plane
- This adds three constraints (one translational and two rotational)
- A planar rigid body therefore has 3 DOF



Center of Mass of Rigid Bodies



- Recall our previous definition of the center of mass of a particle collection:

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \sum_{i=1}^N m_i \mathbf{r}_{i/O}$$

- For a body of finite mass, as $N \rightarrow \infty$, $m_i \rightarrow 0$

- A definite integral is a limit of a summation:

$$\int_0^b f(x) dx = \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{b}{N} f\left(\frac{bi}{N}\right)$$

- Therefore, for a rigid body:

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \int_{\mathcal{B}} \mathbf{r}_{dm/O} dm$$

Center of Mass of Rigid Bodies Continued

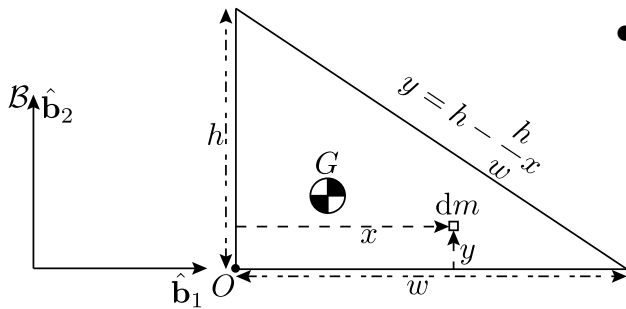
- If the density of our rigid body is given by $\rho(\mathbf{r}_{dm/O})$:

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \int_{\mathcal{B}} \mathbf{r}_{dm/O} \rho(\mathbf{r}_{dm/O}) dV$$

- The center of mass corollary holds for rigid bodies:

$$\int_{\mathcal{B}} \mathbf{r}_{dm/G} \rho(\mathbf{r}_{dm/G}) dV = \mathbf{0}$$

Center of Mass of a Right Triangle



- $\mathbf{r}_{dm/O} = x\hat{\mathbf{b}}_1 + y\hat{\mathbf{b}}_2$
- Assuming constant density:

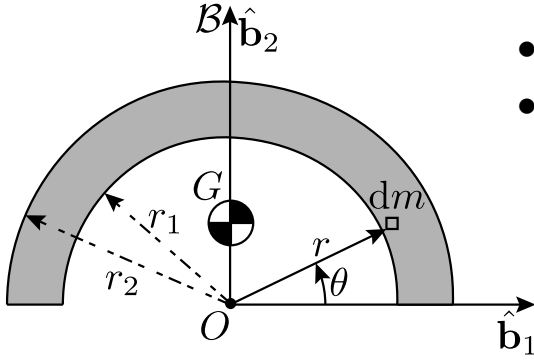
$$\begin{aligned} \mathbf{r}_{G/O} &= \frac{\rho}{m} \int_{\mathcal{B}} \mathbf{r}_{dm/O} dA \\ &= \frac{\rho}{m} \int_0^w \int_0^{h-\frac{h}{w}x} (x\hat{\mathbf{b}}_1 + y\hat{\mathbf{b}}_2) dy dx \end{aligned}$$

$$\mathbf{r}_{G/O} = \frac{\rho}{m} \left[\left(\frac{w^2h}{2} - \frac{hw^3}{3w} \right) \hat{\mathbf{b}}_1 + \left(\frac{h^2w}{2} - \frac{h^2w^2}{2w} + \frac{h^3w^3}{6w^2} \right) \hat{\mathbf{b}}_2 \right] = \frac{\rho}{m} \left(\frac{hw^2}{6} \hat{\mathbf{b}}_1 + \frac{h^2w}{6} \hat{\mathbf{b}}_2 \right)$$

- $m = \rho V$ but for the planar case, we use $m = \rho A$ so $m = \rho \frac{hw}{2}$. Therefore:

$$\mathbf{r}_{G/O} = \frac{w}{3} \hat{\mathbf{b}}_1 + \frac{h}{3} \hat{\mathbf{b}}_2$$

Center of Mass of a Semicircular Annulus



- $\mathbf{r}_{dm/O} = r \cos(\theta) \hat{\mathbf{b}}_1 + r \sin(\theta) \hat{\mathbf{b}}_2$

- Assuming constant density:

$$\mathbf{r}_{G/O} = \frac{\rho}{m} \int_0^\pi \int_{r_1}^{r_2} \left(r \cos(\theta) \hat{\mathbf{b}}_1 + r \sin(\theta) \hat{\mathbf{b}}_2 \right) r \, dr \, d\theta$$

$$\mathbf{r}_{G/O} = \frac{\rho}{m} \left[\frac{r_2^3 - r_1^3}{3} \sin(\theta) \right]_0^\pi \hat{\mathbf{b}}_1 - \frac{r_2^3 - r_1^3}{3} \cos(\theta) \Big|_0^\pi \hat{\mathbf{b}}_2 = \frac{2\rho}{3m} (r_2^3 - r_1^3) \hat{\mathbf{b}}_2$$

$$m = \rho A = \frac{\rho\pi}{2} (r_2^2 - r_1^2) \quad \Rightarrow \quad \mathbf{r}_{G/O} = \frac{4 (r_2^3 - r_1^3)}{3\pi (r_2^2 - r_1^2)} \hat{\mathbf{b}}_2$$

Euler's Laws

I. The product of the inertial acceleration of the center of mass of a rigid body and its total mass is equal to the total external force applied to the body

$$\mathbf{F}_G = m_G \mathcal{I} \mathbf{a}_{G/O}$$

II. The rate of change of the inertial angular momentum of a rigid body about a fixed point O in the inertial frame is equal to the total external moment applied to the body about O

$$\mathcal{I} \frac{d}{dt} \mathbf{h}_O = \mathbf{M}_O$$

Angular Momentum of a Rigid Body

$$\begin{aligned} \mathcal{I}\mathbf{h}_O &= \sum_{i=1}^N m_i \mathbf{r}_{i/O} \times \mathcal{I}\mathbf{v}_{i/O} \xrightarrow[N \rightarrow \infty]{m_i \rightarrow dm} \mathcal{I}\mathbf{h}_O = \int_{\mathcal{B}} \mathbf{r}_{dm/O} \times \mathcal{I}\mathbf{v}_{dm/O} dm \\ \frac{\mathcal{I}d}{dt} (\mathcal{I}\mathbf{h}_O) &= \underbrace{\sum_{i=1}^N \mathbf{r}_{i/O} \times \mathbf{F}_i^{(\text{ext})}}_{\triangleq \mathbf{M}_O^{(\text{ext})}} + \underbrace{\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N (\mathbf{r}_{i/O} - \mathbf{r}_{j/O}) \times \mathbf{F}_{i,j}}_{\text{Equals zero for rigid bodies}} \end{aligned}$$

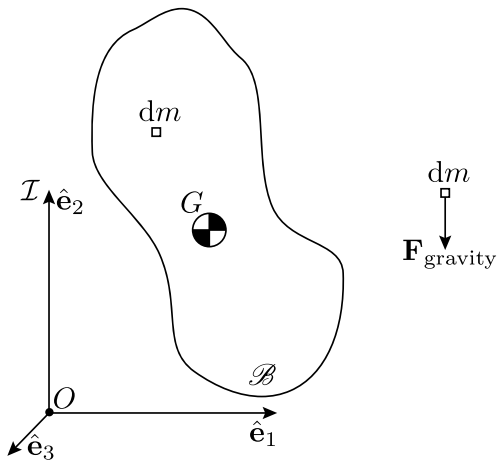
- The internal moment assumption is fundamentally built into Euler's laws (this is how they expand on Newton's laws)
- We typically drop the (ext) superscript as all forces/moments for rigid bodies are going to be external

The Separation Principle

$$\begin{aligned} \mathcal{I}\mathbf{h}_O &= \underbrace{\mathcal{I}\mathbf{h}_{G/O}}_{\text{Angular Momentum of COM about an Inertially Fixed Point}} + \underbrace{\mathcal{I}\mathbf{h}_G}_{\text{Angular Momentum of Body about its COM}} \\ \mathcal{I}\mathbf{h}_{G/O} &\triangleq m_G \mathbf{r}_{G/O} \times \mathcal{I}\mathbf{v}_{G/O} \implies \frac{\mathcal{I}d}{dt} (\mathcal{I}\mathbf{h}_{G/O}) = \mathbf{M}_{G/O} \triangleq \mathbf{r}_{G/O} \times \mathbf{F}_G \\ \mathcal{I}\mathbf{h}_G &\triangleq \begin{cases} \sum_{i=1}^N m_i \mathbf{r}_{i/G} \times \mathcal{I}\mathbf{v}_{i/G} & \text{Particles} \\ \int_{\mathcal{B}} \mathbf{r}_{dm/G} \times \mathcal{I}\mathbf{v}_{dm/G} dm & \text{Continuous Bodies} \end{cases} \\ \frac{\mathcal{I}d}{dt} (\mathcal{I}\mathbf{h}_G) &= \mathbf{M}_G \triangleq \begin{cases} \sum_{i=1}^N \mathbf{r}_{i/G} \times \mathbf{F}_i^{(\text{ext})} & \text{Contact Forces} \\ \int_{\mathcal{B}} \mathbf{r}_{dm/G} \times \mathbf{f}_{dm} dm & \text{Field Forces} \end{cases} \end{aligned}$$

Moment Due to Field Force (Gravity)

- Assuming constant gravitational acceleration:



$$\mathbf{F}_{\text{gravity}} = -mg\hat{\mathbf{e}}_2$$

$$\begin{aligned}\mathbf{M}_G &= \int_{\mathcal{B}} \mathbf{r}_{dm/G} \times \overbrace{\mathbf{f}_{dm}}^{-g\hat{\mathbf{e}}_2} dm \\ &= -g \underbrace{\int_{\mathcal{B}} \mathbf{r}_{dm/G} dm}_{\mathbf{0}} \times \hat{\mathbf{e}}_2\end{aligned}$$

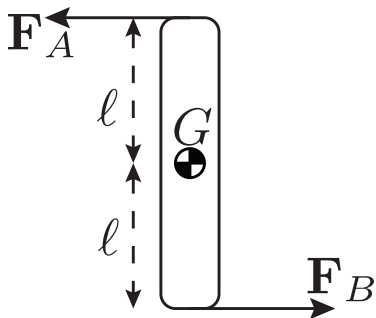
$$\mathbf{F}_{\text{gravity}} = -\frac{Gm_O dm}{\|\mathbf{r}_{dm/O}\|^3} \mathbf{r}_{dm/O}$$

- Assuming Newton's law of gravity:

$$\mathbf{M}_G = - \int_{\mathcal{B}} \frac{Gm_O}{\|\mathbf{r}_{dm/O}\|^3} \mathbf{r}_{dm/G} \times \mathbf{r}_{dm/O} dm$$

- The **center of gravity** is the point where the moment due to gravity is zero

Moment of Force Couples



$$\bullet \text{ If } \mathbf{F}_A = -\mathbf{F}_B \implies \mathbf{F}_G = \mathbf{F}_A + \mathbf{F}_B = \mathbf{0}$$

$$\bullet \mathbf{M}_G = \mathbf{r}_{A/G} \times \mathbf{F}_A + \mathbf{r}_{B/G} \times \mathbf{F}_B$$

$$\bullet \mathbf{r}_{A/G} = -\mathbf{r}_{B/G} \text{ and } \|\mathbf{r}_{A/G}\| = \|\mathbf{r}_{B/G}\| = \ell \text{ so:}$$

$$\mathbf{M}_G = 2\ell \|\mathbf{F}_A\| (\hat{\mathbf{r}}_{A/G} \times \hat{\mathbf{F}}_A)$$

- In the limit as $\ell \rightarrow 0$, you get a **pure torque**
- NB: Pure torques can be applied anywhere on a rigid body and will produce the exact same moment and zero net force

Computing $\mathcal{I}\mathbf{h}_G$ of Planar Rigid Bodies

- Consider the expression: $\mathcal{I}\mathbf{h}_G = \sum_{i=1}^N m_i \mathbf{r}_{i/G} \times \mathcal{I}\mathbf{v}_{i/G}$
- Apply the transport equation: $\mathcal{I}\mathbf{v}_{i/G} = \underbrace{\mathcal{B}\mathbf{v}_{i/G}}_0 + \mathcal{I}\boldsymbol{\omega}^B \times \mathbf{r}_{i/G}$ and substitute:

$$\begin{aligned}\mathcal{I}\mathbf{h}_G &= \sum_{i=1}^N m_i \mathbf{r}_{i/G} \times (\mathcal{I}\boldsymbol{\omega}^B \times \mathbf{r}_{i/G}) = \sum_{i=1}^N m_i (\mathcal{I}\boldsymbol{\omega}^B (\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G}) - \mathbf{r}_{i/G} (\overbrace{\mathcal{I}\boldsymbol{\omega}^B \cdot \mathbf{r}_{i/G}}^0)) \\ &= \underbrace{\sum_{i=1}^N m_i \|\mathbf{r}_{i/G}\|^2 \mathcal{I}\boldsymbol{\omega}^B}_{\triangleq I_G}\end{aligned}$$

- I_G is the **moment of inertia** about the COM of a planar rigid body

Planar Moment of Inertia

- For planar rigid bodies **only**:

$$\mathcal{I}\mathbf{h}_G = I_G \mathcal{I}\boldsymbol{\omega}^B$$

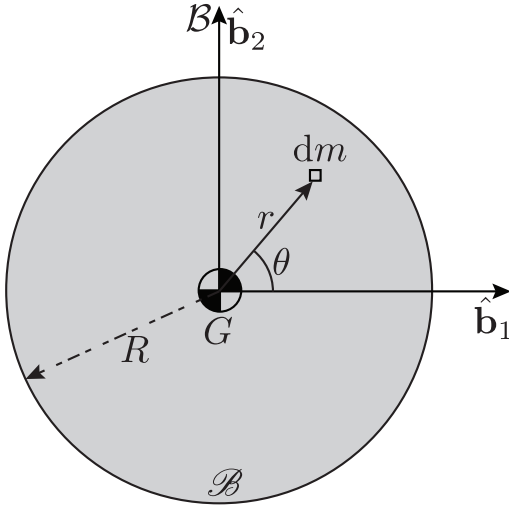
- Since for rigid bodies $\frac{d}{dt} \|\mathbf{r}_{i/j}\| = 0 \forall i, j$ I_G is constant. Therefore:

$$\mathcal{I} \frac{d}{dt} \mathcal{I}\mathbf{h}_G = \mathbf{M}_G = \mathcal{I} \frac{d}{dt} (I_G \mathcal{I}\boldsymbol{\omega}^B) = I_G \mathcal{I}\boldsymbol{\alpha}^B$$

- For a rigid body with density $\rho(\mathbf{r}_{dm/G})$:

$$I_G \triangleq \int_{\mathcal{B}} \|\mathbf{r}_{dm/G}\|^2 dm = \int_{\mathcal{B}} \|\mathbf{r}_{dm/G}\|^2 \rho(\mathbf{r}_{dm/G}) dV$$

Planar Moment of Inertia of a Uniform Disk

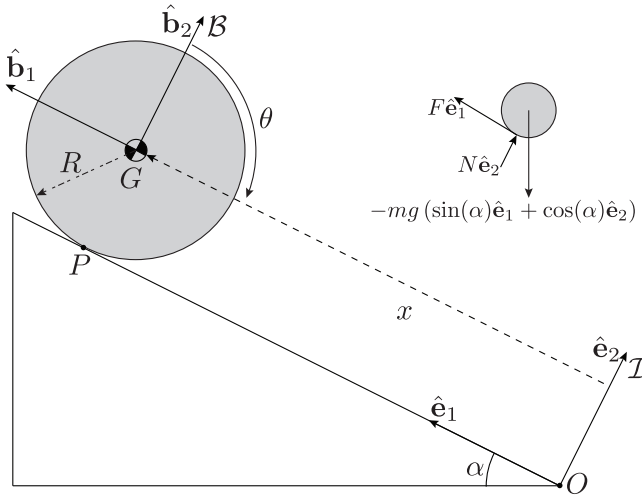


$$\mathbf{r}_{dm/G} = r \cos(\theta) \hat{\mathbf{b}}_1 + r \sin(\theta) \hat{\mathbf{b}}_2$$

$$\begin{aligned} I_G^{\text{disk}} &= \int_{\mathcal{B}} (r^2 \cos^2(\theta) + r^2 \sin^2(\theta)) \rho dV \\ &= \rho \int_0^{2\pi} \int_0^R r^2 r dr d\theta = \underbrace{\rho}_{\frac{m}{\pi R^2}} \left(\frac{\pi}{2} R^4 \right) \end{aligned}$$

$$I_G^{\text{disk}} = \frac{mR^2}{2}$$

Uniform Disk Rolling Without Slip on Ramp



- Note that our FBD now has *extent*
- By the no-slip condition:

$$\dot{x} = -R\dot{\theta} \implies \ddot{x} = -R\ddot{\theta}$$

- $\mathbf{r}_{G/O} = x\hat{\mathbf{e}}_1 + R\hat{\mathbf{e}}_2 \implies \mathcal{I}\mathbf{a}_{G/O} = \ddot{x}\hat{\mathbf{e}}_1$
- Applying Euler's first law:

$$\begin{aligned} m\ddot{x}\hat{\mathbf{e}}_1 &= F\hat{\mathbf{e}}_1 + N\hat{\mathbf{e}}_2 \\ &\quad - mg(\sin(\alpha)\hat{\mathbf{e}}_1 + \cos(\alpha)\hat{\mathbf{e}}_2) \end{aligned}$$

- Dotting with $\hat{\mathbf{e}}_1$:

$$m\ddot{x} \equiv -mR\ddot{\theta} = -mg \sin(\alpha) + F$$

$$\mathbf{M}_G = \mathbf{r}_{P/G} \times F\hat{\mathbf{e}}_1 + \mathbf{r}_{P/G} \times N\hat{\mathbf{e}}_2 = RF\hat{\mathbf{b}}_3$$

$$\mathcal{I} \frac{d}{dt} (I_G \boldsymbol{\omega}^B) = \mathbf{M}_G \implies \ddot{\theta} \frac{mR^2}{2} = RF$$

$$\ddot{\theta} = \frac{2g \sin(\alpha)}{3R}$$

Impulsive Moments

- Recall angular impulse:

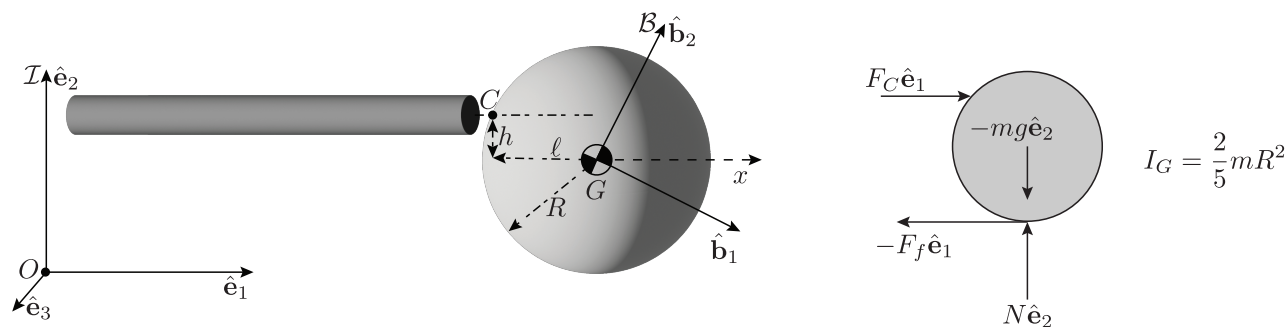
$$\overline{\mathbf{M}}_{P/O}(t_1, t_2) \triangleq \int_{t_1}^{t_2} \mathbf{M}_{P/O} dt \implies \mathcal{I}\mathbf{h}_{P/O}(t_2) = \mathcal{I}\mathbf{h}_{P/O}(t_1) + \overline{\mathbf{M}}_{P/O}(t_1, t_2)$$

- The same holds for $\mathcal{I}\mathbf{h}_G$:
$$\underbrace{\mathcal{I}\mathbf{h}_G(t_2)}_{\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}}(t_2)I_G} = \underbrace{\mathcal{I}\mathbf{h}_G(t_1)}_{\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}}(t_1)I_G} + \overline{\mathbf{M}}_G(t_1, t_2)$$

- Thus, the change in angular velocity due to an angular impulse is given by:

$$\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}}(t_2) - \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}}(t_1) = \frac{\overline{\mathbf{M}}_G(t_1, t_2)}{I_G}$$

A (Slightly) Better Model of Pool



- A pool ball is spherical, but the motion is still planar as we assume that it rotates about one fixed axis and travels in one dimension
- We apply a linear impulse $\overline{\mathbf{F}}_C(t_1, t_2) = \int_{t_1}^{t_2} F_C \hat{\mathbf{e}}_1 dt$ with the ball at rest at t_1 :

$$m \underbrace{\mathcal{I}\mathbf{v}_{G/O}(t_2)}_0 = m \underbrace{\mathcal{I}\mathbf{v}_{G/O}(t_1)}_0 + \overline{\mathbf{F}}_C(t_1, t_2) \implies m \mathcal{I}\mathbf{v}_{G/O}(t_2) = F_C(t_2 - t_1) \hat{\mathbf{e}}_1$$

Pool Shot Continued

- The angular impulse is:

$$\overline{\mathbf{M}}_G(t_1, t_2) = \mathbf{r}_{C/G} \times \overline{\mathbf{F}}_C = (-\ell \hat{\mathbf{e}}_1 + h \hat{\mathbf{e}}_2) \times F_C(t_2 - t_1) \hat{\mathbf{e}}_1 = -h F_C(t_2 - t_1) \hat{\mathbf{e}}_3$$

- Resulting in an angular velocity of:

$$\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}}(t_2) = \underbrace{\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}}(t_1)}_0 + \frac{\overline{\mathbf{M}}_G(t_1, t_2)}{I_G} = -\frac{h F_C(t_2 - t_1)}{2/5 m R^2} \hat{\mathbf{e}}_3 = -\frac{h m \|\mathcal{I}_{\mathbf{v}_{G/O}}(t_2)\|}{2/5 m R^2} \hat{\mathbf{e}}_3$$

- The no-slip condition predicts: $\|\mathcal{I}_{\mathbf{v}_{G/O}}\| = R \|\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}}\|$
- If we assume roll without slip immediately after the impact, we can solve for h :

$$\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}}(t_2) = \frac{h R \|\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}}(t_2)\|}{2/5 R^2} \hat{\mathbf{e}}_3 \implies h = \frac{2}{5} R$$

Pool Shot: Dead Center

- If we instead hit the ball dead center, we expect both sliding and rolling
- After impact:

$$\mathbf{r}_{G/O} = x \hat{\mathbf{e}}_1 + R \hat{\mathbf{e}}_2 \Rightarrow \mathcal{I}_{\mathbf{a}_{G/O}} = \ddot{x} \hat{\mathbf{e}}_1 \Rightarrow m \ddot{x} \hat{\mathbf{e}}_1 = -\mu N \hat{\mathbf{e}}_1 + (N - mg) \hat{\mathbf{e}}_2 \Rightarrow \ddot{x} = -\mu g$$

$$\mathcal{I} \frac{d}{dt} \mathcal{I}_{\mathbf{h}_G} = \mathbf{M}_G \Rightarrow \frac{d}{dt} \left(-I_G \dot{\theta} \hat{\mathbf{e}}_3 \right) = -R \hat{\mathbf{e}}_2 \times (-\mu m g \hat{\mathbf{e}}_1) \Rightarrow \ddot{\theta} = \frac{5}{2} \frac{\mu}{R} g$$

- Initial conditions are: $x, \theta, \dot{\theta} = 0$; $\dot{x} \triangleq v_0$
- Integrating the EOMs:

$$\dot{x} = v_0 - \mu g t \quad \dot{\theta} = \frac{5}{2} \frac{\mu}{R} g t$$

- To get the no-slip condition ($\dot{x} = R \dot{\theta}$):

$$v_0 - \mu g t = R \left(\frac{5}{2} \frac{\mu}{R} g t \right) \Rightarrow t = \frac{2 v_0}{7 \mu g}$$

Week 11

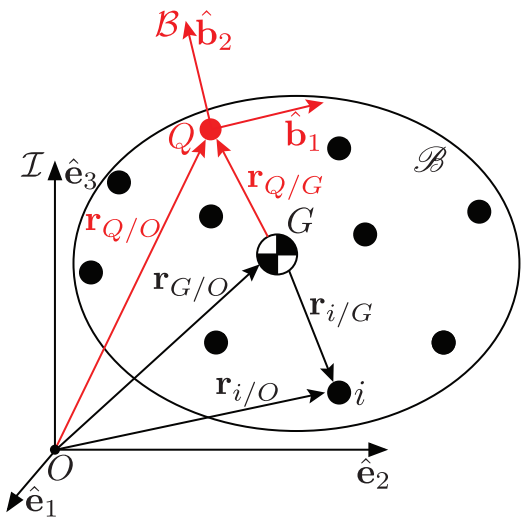
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MAE 2030, Spring 2026

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Rotation About Arbitrary Points



- Just as with collections of particles, we can also define the angular momentum of planar rigid bodies about arbitrary points
- For a point Q fixed within a rigid body, ${}^{\mathcal{B}}\mathbf{v}_{i/Q} = \mathbf{0}$ for any point i in the body:

$$\begin{aligned} \mathcal{I}\mathbf{h}_Q &= \sum_{i=1}^N m_i \mathbf{r}_{i/Q} \times {}^{\mathcal{I}}\mathbf{v}_{i/Q} = \sum_{i=1}^N m_i \mathbf{r}_{i/Q} \times (\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{i/Q}) \\ &= \underbrace{\sum_{i=1}^N m_i \|\mathbf{r}_{i/Q}\|^2}_{\triangleq I_Q} \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \end{aligned}$$

- NB: $\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}}$ will be the same for any body-fixed frame $\mathcal{B}$, regardless of how it is oriented within the body

Rotation About Arbitrary Points Continued

- Our previous result for multiparticle collections was:

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \mathbf{h}_Q = \underbrace{\sum_{i=1}^N \left(\mathbf{r}_{i/Q} \times \mathbf{F}_i^{(\text{ext})} \right)}_{\triangleq \mathbf{M}_Q^{(\text{ext})}} + \mathbf{r}_{Q/G} \times m_G {}^{\mathcal{I}}\mathbf{a}_{Q/O}$$

- This applies exactly to the rigid body case as well:

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \mathbf{h}_Q = I_Q {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{B}} = \mathbf{M}_Q + \mathbf{r}_{Q/G} \times m_G {}^{\mathcal{I}}\mathbf{a}_{Q/O}$$

- This is Euler's 2nd law for rotation about an arbitrary point Q :

Moment Transport Theorem

- Frequently, we may already know the net moment about the center of mass of our body, and want to find the moment about an arbitrary point Q :

$$\begin{aligned} \mathbf{M}_Q &\triangleq \sum_{i=1}^N \left(\underbrace{\mathbf{r}_{i/Q}}_{= \mathbf{r}_{i/G} - \mathbf{r}_{Q/G}} \times \mathbf{F}_i^{(\text{ext})} \right) \\ &= \underbrace{\sum_{i=1}^N \left(\mathbf{r}_{i/G} \times \mathbf{F}_i^{(\text{ext})} \right)}_{\mathbf{M}_G} - \sum_{i=1}^N \left(\mathbf{r}_{Q/G} \times \mathbf{F}_i^{(\text{ext})} \right) \end{aligned}$$

$$\mathbf{M}_Q = \mathbf{M}_G - \mathbf{r}_{Q/G} \times \sum_{i=1}^N \mathbf{F}_i^{(\text{ext})}$$

Parallel Axis Theorem

- Just as with external moments, we can compute the angular momentum about an arbitrary point starting from the angular momentum about the COM
- Our previous result for multiparticle collections was:

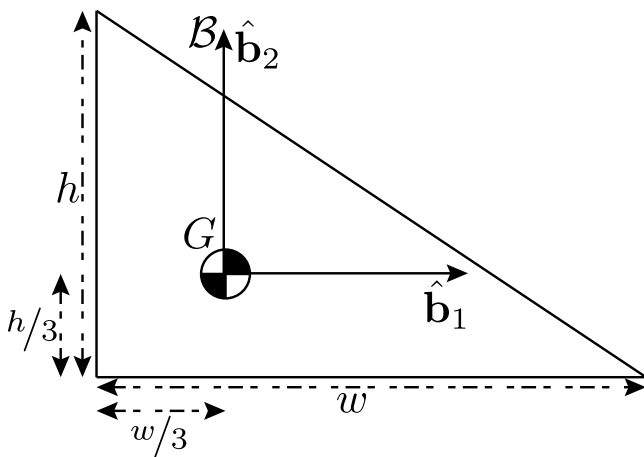
$$\mathcal{I}\mathbf{h}_Q = \mathcal{I}\mathbf{h}_G + m_G \mathbf{r}_{Q/G} \times \underbrace{\mathcal{I}\mathbf{v}_{Q/G}}_{\mathcal{B}\mathbf{v}_{Q/G} + \underbrace{\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{Q/G}}_0} = \mathcal{I}\mathbf{h}_G + m_G \mathbf{r}_{Q/G} \times (\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{Q/G})$$

- $\mathbf{r}_{Q/G} \times (\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{Q/G}) = \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} (\mathbf{r}_{Q/G} \cdot \mathbf{r}_{Q/G}) - \mathbf{r}_{Q/G} (\mathbf{r}_{Q/G} \cdot \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}})$
- In the planar case **only**:

$$\underbrace{\mathcal{I}\mathbf{h}_Q}_{I_Q \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}}} = \underbrace{\mathcal{I}\mathbf{h}_G}_{I_G \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}}} + m_G \|\mathbf{r}_{Q/G}\|^2 \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}}$$

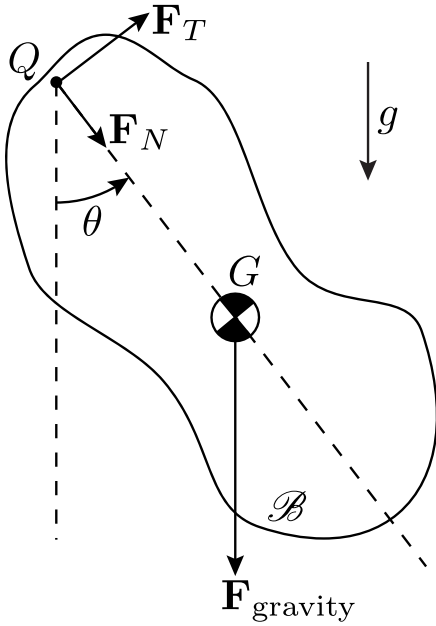
$$I_Q = I_G + m_G \|\mathbf{r}_{Q/G}\|^2$$

Using Pre-Computed Moments of Inertia



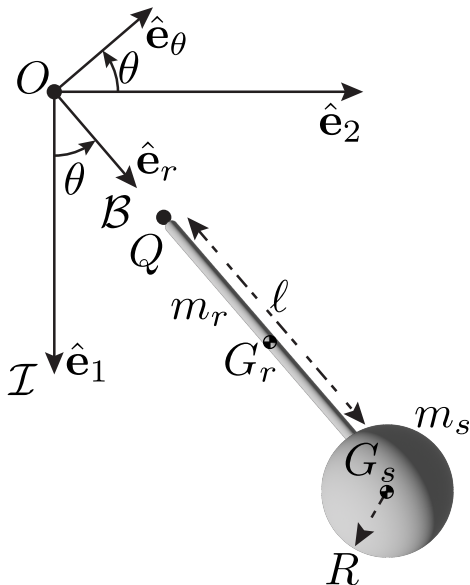
$$[\mathbb{I}_G]_{\mathcal{B}} = \frac{m}{36} \begin{bmatrix} 2h^2 & hw & 0 \\ hw & 2w^2 & 0 \\ 0 & 0 & 2h^2 + 2w^2 \end{bmatrix}_{\mathcal{B}}$$

Compound Pendulums



- A **compound pendulum** is any rigid body pinned at one point and allowed to swing under the force of gravity
- Compound pendulums still have only one degree of freedom (in the planar case) which can be encoded by coordinate θ
- In addition to gravity, we need to model the constraint force at the pin point. We typically split this into two components

Compound Pendulum Example



- Consider a compound pendulum composed of a long, thin rod and a sphere
- The moments of inertia are:

$$I_{G_r}^{\text{rod}} = \frac{m_r \ell^2}{12} \quad I_{G_s}^{\text{sphere}} = \frac{2m_s R^2}{5}$$

- Applying the parallel axis theorem:

$$\begin{aligned} I_Q^{\text{rod}} &= I_{G_r}^{\text{rod}} + m_r \|\mathbf{r}_{Q/G_r}\|^2 \\ &= \frac{m_r \ell^2}{12} + m_r \left\| -\frac{\ell}{2} \hat{\mathbf{e}}_r \right\|^2 \\ I_Q^{\text{sphere}} &= I_{G_s}^{\text{sphere}} + m_s \|\mathbf{r}_{Q/G_s}\|^2 \\ &= \frac{2m_s R^2}{5} + m_s \left\| -(\ell + R) \hat{\mathbf{e}}_r \right\|^2 \end{aligned}$$

- Once both moments of inertia are about the same point, we can add them to find the total moment of inertia: $I_Q^{\text{tot}} = I_Q^{\text{rod}} + I_Q^{\text{sphere}}$

$$I_Q^{\text{tot}} = \frac{m_r \ell^2}{3} + m_s \left(\frac{2R^2}{5} + (\ell + R)^2 \right)$$

- The net angular momentum about Q is ${}^{\mathcal{I}}\mathbf{h}_Q = I_Q {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$ where ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta} \hat{\mathbf{e}}_3$
- Recall that for a constant gravity model, the net moment about the center of mass of a rigid body is zero, so in this case, the moment transport theorem is:

$$\mathbf{M}_Q = \underbrace{\mathbf{M}_G}_{\mathbf{0}} - \mathbf{r}_{Q/G} \times \sum_{i=1}^N \mathbf{F}_i^{(\text{ext})}$$

- The net center of mass can be computed from the individual component COMs:

$$\begin{aligned} \mathbf{r}_{G/Q} &= \frac{1}{m_s + m_r} (m_s \mathbf{r}_{G_s/Q} + m_r \mathbf{r}_{G_r/Q}) \\ &= \frac{1}{m_s + m_r} \left(m_s (\ell + R) \hat{\mathbf{e}}_r + m_r \frac{\ell}{2} \hat{\mathbf{e}}_r \right) \\ \mathbf{M}_Q &= \underbrace{-\mathbf{r}_{Q/G}}_{\mathbf{r}_{G/Q}} \times (m_s + m_r) g \hat{\mathbf{e}}_1 \\ &= -g \left(\frac{\ell m_r}{2} + m_s (R + \ell) \right) \sin(\theta) \hat{\mathbf{e}}_3 \end{aligned}$$

- By Euler's 2nd law:

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{h}_Q = I_Q {}^{\mathcal{I}}\boldsymbol{\alpha}^{\mathcal{B}} = \mathbf{M}_Q + \mathbf{r}_{Q/G} \times m_G \underbrace{\mathbf{a}_{Q/O}}_{\mathbf{0}}$$

$$\ddot{\theta} = -g \sin(\theta) \frac{\left(\frac{\ell m_r}{2} + m_s (R + \ell) \right)}{\frac{\ell^2 m_r}{3} + m_s \left(\frac{2R^2}{5} + (R + \ell)^2 \right)}$$

Radius of Gyration

- Note that our equation of motion for the compound pendulum looks a lot like that of the simple pendulum: $\ddot{\theta} = -\frac{g}{\ell} \sin(\theta)$
- We define the **radius of gyration** as the distance from a point Q on a rigid body at which a particle of mass m_G would produce the same moment of inertia about Q as the rigid body

$$k_Q \triangleq \sqrt{\frac{I_Q}{m_G}}$$

- For *any* compound pendulum where $\|\mathbf{r}_{Q/G}\| = \ell_G$ we find:

$$\ddot{\theta} = -\frac{g}{k_Q^2/\ell_G} \sin(\theta)$$

- This is how we can measure moments of inertia!

Kinetic Energy of Planar Rigid Bodies

- Recall: $T_O = T_{G/O} + T_G$ where $T_{G/O} = \frac{m_G}{2} \|\mathcal{I} \mathbf{v}_{G/O}\|^2$

$$\begin{aligned} T_G &= \frac{1}{2} \sum_{i=1}^N m_i \underbrace{\|\mathcal{I} \mathbf{v}_{i/G}\|^2}_{= \underbrace{\mathcal{B} \mathbf{v}_{i/G}}_{\mathbf{0}} + \mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{i/G}} = \frac{1}{2} \sum_{i=1}^N m_i (\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{i/G}) \cdot (\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{i/G}) \\ &= \frac{1}{2} \sum_{i=1}^N m_i \left((\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} \cdot \mathcal{I} \boldsymbol{\omega}^{\mathcal{B}}) (\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G}) - 2 \underbrace{(\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} \cdot \mathbf{r}_{i/G})}_{\mathbf{0} \text{ for planar motion}} \right) = \frac{1}{2} \|\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}}\|^2 \underbrace{\sum_{i=1}^N m_i \|\mathbf{r}_{i/G}\|^2}_{I_G} \end{aligned}$$

$$T_G = \frac{I_G}{2} \|\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}}\|^2 = \frac{1}{2} \mathcal{I} \mathbf{h}_G \cdot \mathcal{I} \boldsymbol{\omega}^{\mathcal{B}}$$

Total Work of Planar Rigid Bodies

- Total work is still the change in total kinetic energy:

$$W^{(\text{tot})} = T_O(t_2) - T_O(t_1) \triangleq \Delta T_O$$

- We can split it into work due to motion of and motion about the COM:

$$\begin{aligned} W^{(\text{tot})} &= \underbrace{W_{G/O}}_{= \Delta T_{G/O}} + \underbrace{W_G}_{= \Delta T_G} \\ &= \int_{t_1}^{t_2} \mathbf{F}_G \cdot \mathcal{I}_{\mathbf{v}_{G/O}} dt = \int_{t_1}^{t_2} \mathbf{M}_G \cdot \mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}} dt \\ W_G &= \int_{t_1}^{t_2} \sum_{i=1}^N \mathbf{F}_i^{(\text{ext})} \cdot \underbrace{\mathcal{I}_{\mathbf{v}_{i/G}}}_{= \underbrace{\mathcal{B}_{\mathbf{v}_{i/G}}}_0 + \mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}} \times \mathbf{r}_{i/G}} dt = \int_{t_1}^{t_2} \sum_{i=1}^N \underbrace{\left(\mathbf{r}_{i/G} \times \mathbf{F}_i^{(\text{ext})} \right)}_{\mathbf{M}_G} \cdot \mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}} dt \end{aligned}$$

Potential and Total Energy of Planar Rigid Bodies

- The internal potential energy of a rigid body is constant so all changes are due to external conservative forces

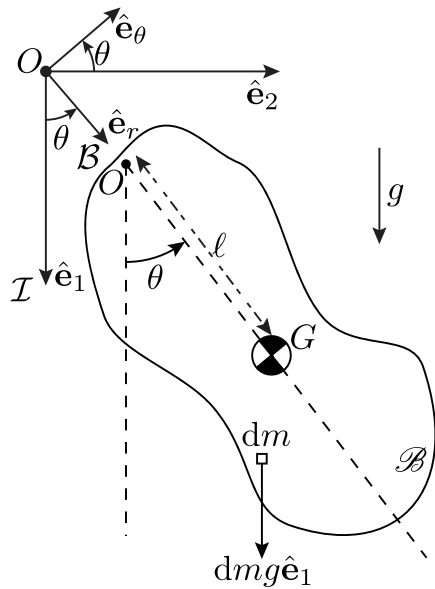
- Contact Forces: $U_O = \sum_{i=1}^N U_{i/O}(\mathbf{r}_{i/O})$

- Field Forces: $U_O = \int_{\mathcal{B}} U_{\text{dm}/O}(\mathbf{r}_{\text{dm}/O}) dm$

- Total energy is: $E_O = T_{G/O} + T_G + U_O$
- Total energy changes only due to non-conservative forces:

$$E_O(t_2) = E_O(t_1) + W_{G/O}^{(\text{nc})} + W_G^{(\text{nc})}$$

Total Energy of a Swinging, Planar Rigid Body

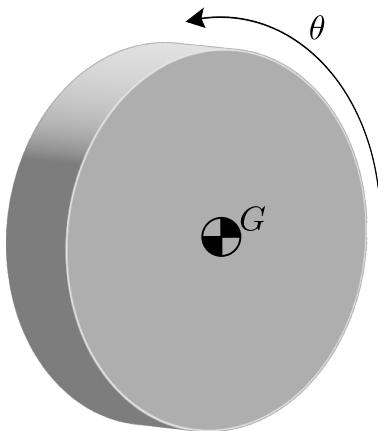


$$\begin{aligned}
 T_O &= T_{G/O} + T_G = \frac{m_G}{2} \underbrace{\|\mathcal{I}_{\mathbf{v}_{G/O}}\|^2}_{\mathbf{0}} + \frac{I_G}{2} \|\mathcal{I}_{\boldsymbol{\omega}^B}\|^2 \\
 &= \frac{1}{2} \underbrace{(m_G \|\mathbf{r}_{G/O}\|^2 + I_G)}_{I_O} \|\mathcal{I}_{\boldsymbol{\omega}^B}\|^2
 \end{aligned}$$

$$\begin{aligned}
 U_O &= \int_{\mathcal{B}} U_{dm/O}(\mathbf{r}_{dm/O}) dm = - \int_{\mathcal{B}} \mathbf{r}_{dm/O} \cdot g \hat{\mathbf{e}}_1 dm \\
 &= -m_G g (\mathbf{r}_{G/O} \cdot \hat{\mathbf{e}}_1) = -m_G g l \cos(\theta)
 \end{aligned}$$

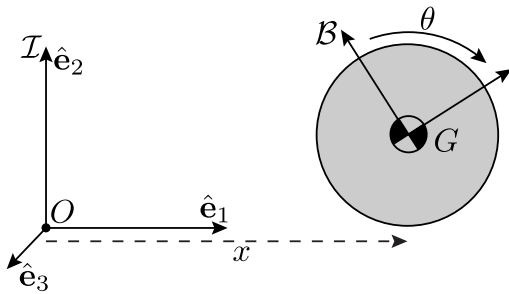
$$E_O = \frac{I_O}{2} \dot{\theta}^2 - m_G g l \cos(\theta)$$

Kinetic Energy of a Spinning, Fixed, Planar Rigid Body



$$\begin{aligned}
 T_O &= T_{G/O} + T_G = \frac{m_G}{2} \underbrace{\|\mathcal{I}_{\mathbf{v}_{G/O}}\|^2}_{\mathbf{0}} + \frac{I_G}{2} \|\mathcal{I}_{\boldsymbol{\omega}^B}\|^2 \\
 &= \frac{1}{2} I_G \dot{\theta}^2
 \end{aligned}$$

Kinetic Energy of a Rolling, Planar Rigid Body



$$\begin{aligned}
 T_O &= T_{G/O} + T_G = \frac{m_G}{2} \|\mathcal{I}\mathbf{v}_{G/O}\|^2 + \frac{I_G}{2} \|\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}}\|^2 \\
 &= \frac{m_G}{2} (\dot{x}\hat{\mathbf{e}}_1 \cdot \dot{x}\hat{\mathbf{e}}_1) + \frac{1}{2} I_G (-\dot{\theta}\hat{\mathbf{e}}_3 \cdot -\dot{\theta}\hat{\mathbf{e}}_3) \\
 &= \frac{m_G}{2} \dot{x}^2 + \frac{1}{2} I_G \dot{\theta}^2
 \end{aligned}$$

Week 12

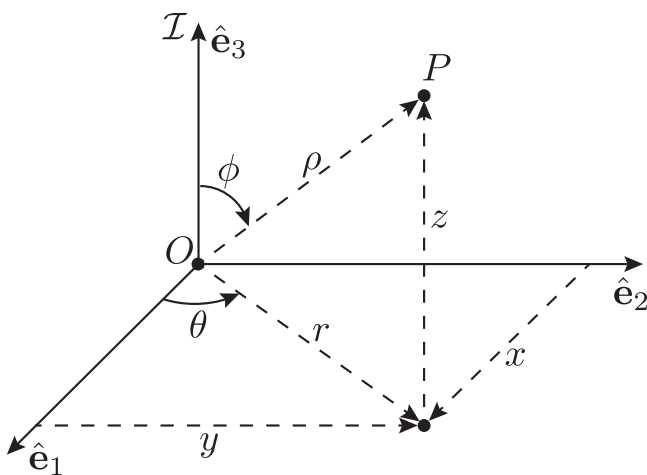
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Cornell University

MAE 2030, Spring 2026

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Cartesian and Cylindrical Coordinates



θ is the **azimuth** (or **azimuthal**)
angle

ϕ is the **zenith** (or **polar**) angle

$90^\circ - \phi$ is the **elevation** (or **altitude**)

Cartesian Coordinates

$$[\mathbf{r}_{P/O}]_{\mathcal{I}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{I}}$$

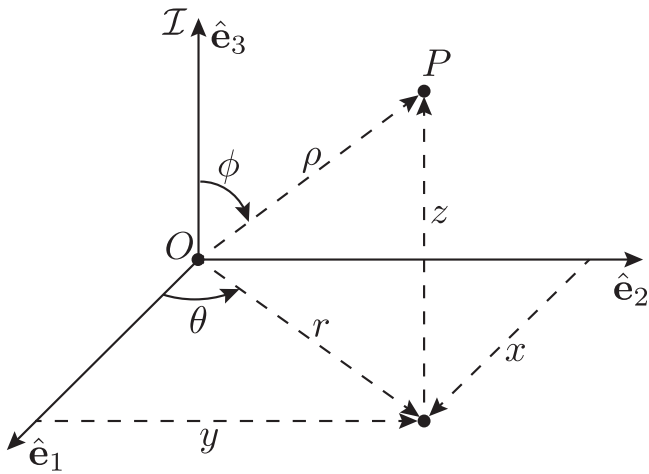
$$\|\mathbf{r}_{P/O}\| = \sqrt{x^2 + y^2 + z^2}$$

Polar/Cylindrical Coordinates

$$[\mathbf{r}_{P/O}]_{\mathcal{I}} = \begin{bmatrix} r \cos \theta \\ r \sin \theta \\ z \end{bmatrix}_{\mathcal{I}}$$

$$\|\mathbf{r}_{P/O}\| = \sqrt{r^2 + z^2}$$

Spherical Coordinates



θ is the **azimuth** (or **azimuthal**) angle

ϕ is the **zenith** (or **polar**) angle

$90^\circ - \phi$ is the **elevation** (or **altitude**)

Spherical Coordinates

$$[\mathbf{r}_{P/O}]_{\mathcal{I}} = \rho \begin{bmatrix} \cos \theta \sin \phi \\ \sin \theta \sin \phi \\ \cos \phi \end{bmatrix}_{\mathcal{I}}$$

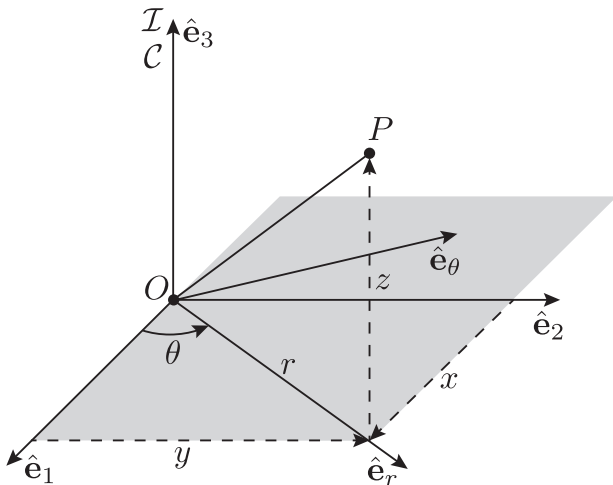
$$\|\mathbf{r}_{P/O}\| = \rho$$

NB: θ and ϕ definitions are frequently reversed and some texts will use elevation instead of polar angles. Many texts will also use r and ρ for both cylindrical and spherical coordinates.

Polar/Cylindrical Rotating Reference Frames

$$\mathcal{C} \triangleq (O, \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$$

$${}^{\mathcal{C}}C^{\mathcal{I}} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\underbrace{[\mathbf{r}_{P/O}]_{\mathcal{C}}}_{\begin{bmatrix} r \\ 0 \\ z \end{bmatrix}_{\mathcal{C}}} = {}^{\mathcal{C}}C^{\mathcal{I}} \underbrace{[\mathbf{r}_{P/O}]_{\mathcal{I}}}_{\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{I}}}$$

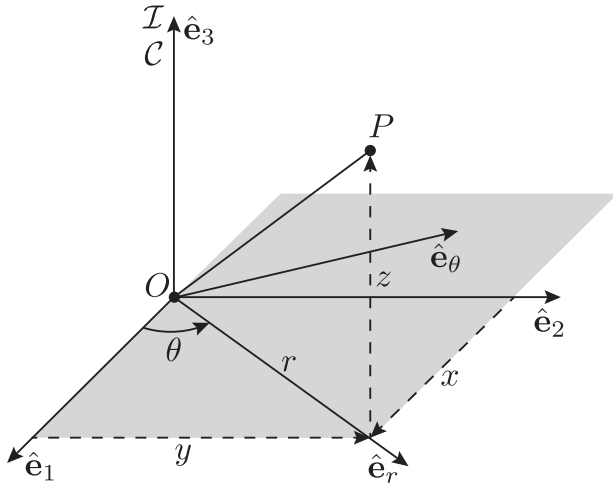
$$\begin{bmatrix} r \\ 0 \\ z \end{bmatrix}_{\mathcal{C}} = {}^{\mathcal{C}}C^{\mathcal{I}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} x \cos(\theta) + y \sin(\theta) \\ -x \sin(\theta) + y \cos(\theta) \\ z \end{bmatrix}_{\mathcal{C}}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{I}} = \underbrace{{}^{\mathcal{I}}C^{\mathcal{C}}}_{({}^{\mathcal{C}}C^{\mathcal{I}})^T} \begin{bmatrix} r \\ 0 \\ z \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} r \cos(\theta) \\ r \sin(\theta) \\ z \end{bmatrix}_{\mathcal{I}}$$

Cylindrical Frame Kinematics

$$\mathcal{C} \triangleq (O, \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$$

$$\mathcal{I}_{\omega}^{\mathcal{C}} = \dot{\theta} \hat{\mathbf{e}}_3$$

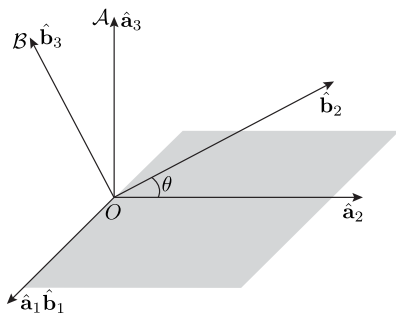


$$\mathbf{r}_{P/O} = r \hat{\mathbf{e}}_r + z \hat{\mathbf{e}}_3$$

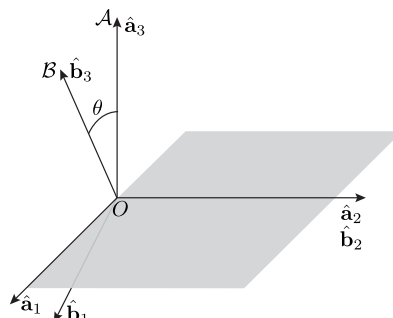
$$\begin{aligned} \mathcal{I}_{\mathbf{v}_{P/O}} &= \dot{r} \hat{\mathbf{e}}_r + r (\mathcal{I}_{\omega}^{\mathcal{C}} \times \hat{\mathbf{e}}_r) + \dot{z} \hat{\mathbf{e}}_3 \\ &= \dot{r} \hat{\mathbf{e}}_r + r \dot{\theta} \hat{\mathbf{e}}_\theta + \dot{z} \hat{\mathbf{e}}_3 \end{aligned}$$

$$\begin{aligned} \mathcal{I}_{\mathbf{a}_{P/O}} &= \ddot{r} \hat{\mathbf{e}}_r + \dot{r} \dot{\theta} \hat{\mathbf{e}}_\theta + \dot{r} \dot{\theta} \hat{\mathbf{e}}_\theta + r \ddot{\theta} \hat{\mathbf{e}}_\theta - r \dot{\theta}^2 \hat{\mathbf{e}}_r + \ddot{z} \hat{\mathbf{e}}_3 \\ &= (\ddot{r} - r \dot{\theta}^2) \hat{\mathbf{e}}_r + (2 \dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\mathbf{e}}_\theta + \ddot{z} \hat{\mathbf{e}}_3 \end{aligned}$$

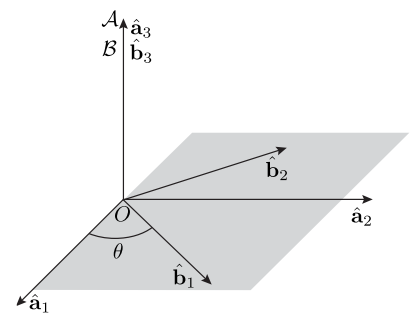
Recall the Simple Direction Cosine Matrices



$$\mathcal{B}_C^{\mathcal{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}}_{\triangleq C_1(\theta)}$$



$$\underbrace{\begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}}_{\triangleq C_2(\theta)}$$



$$\underbrace{\begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\triangleq C_3(\theta)}$$

Multiple Simple Rotations

- Multiple simple rotations in series produce another simple rotation. We express this by left-multiplying by each subsequent direction cosine matrix:

$$c_C^{BB} c_C^{AA} = c_C^{AA}$$

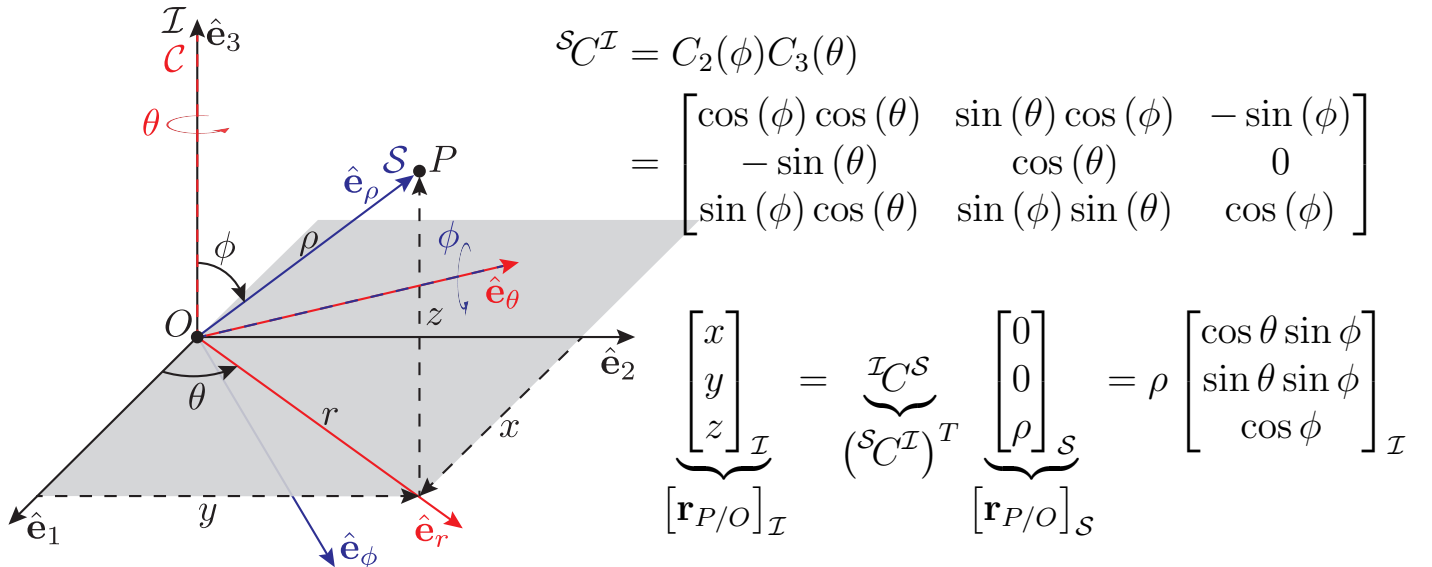
- We can do this for any number of rotations:

$$\mathcal{I}_C^{\mathcal{F}_1 \mathcal{F}_1} c_C^{\mathcal{F}_2 \mathcal{F}_2} c_C^{\mathcal{F}_3 \mathcal{F}_3} \dots c_C^{\mathcal{F}_N \mathcal{F}_N} = \mathcal{I}_C^{\mathcal{F}_N}$$

- Any* simple rotation can be decomposed into three (or fewer) rotations about non-repeating reference frame axes (more on this later)
- The cylindrical frame represents a single rotation (about $\hat{\mathbf{e}}_3$)
- To define a spherical frame where one unit direction follows a point in 3D space, we need two simple rotations

Spherical Reference Frames

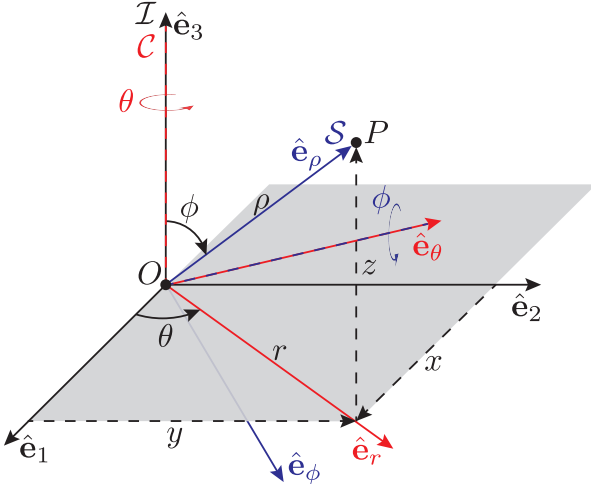
- 1 θ rotation about $\hat{\mathbf{e}}_3$: $\mathcal{I} \rightarrow \mathcal{C} \triangleq (O, \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3)$
- 2 ϕ rotation about $\hat{\mathbf{e}}_\theta$: $\mathcal{C} \rightarrow \mathcal{S} \triangleq (O, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\rho)$



Spherical Frame Kinematics

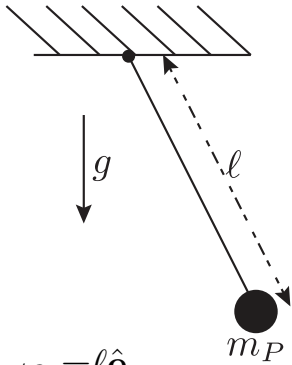
$$\mathcal{S} \triangleq (O, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\rho)$$

$$\begin{aligned}\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{S}}} &= \mathcal{I}_{\boldsymbol{\omega}^{\mathcal{C}}} + \mathcal{C}_{\boldsymbol{\omega}^{\mathcal{S}}} = \dot{\theta} \hat{\mathbf{e}}_3 + \dot{\phi} \hat{\mathbf{e}}_\theta \\ &= -\dot{\theta} \sin(\phi) \hat{\mathbf{e}}_\phi + \dot{\phi} \hat{\mathbf{e}}_\theta + \dot{\theta} \cos(\phi) \hat{\mathbf{e}}_\rho\end{aligned}$$

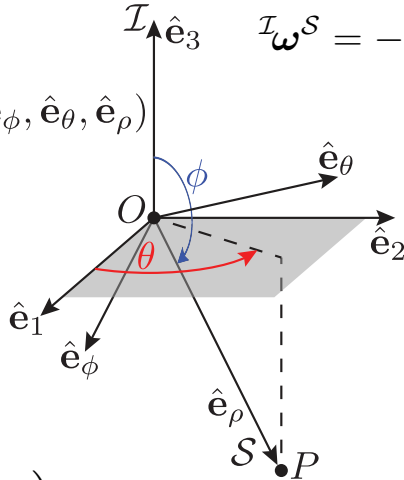


$$\begin{aligned}\mathbf{r}_{P/O} &= \rho \hat{\mathbf{e}}_\rho \\ \mathcal{I}_{\mathbf{v}_{P/O}} &= \dot{\rho} \hat{\mathbf{e}}_\rho + \rho (\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{S}}} \times \hat{\mathbf{e}}_\rho) \\ &= \dot{\phi} \rho \hat{\mathbf{e}}_\phi + \rho \dot{\theta} \sin(\phi) \hat{\mathbf{e}}_\theta + \dot{\rho} \hat{\mathbf{e}}_\rho \\ \mathcal{I}_{\mathbf{a}_{P/O}} &= \frac{d}{dt} \mathcal{I}_{\mathbf{v}_{P/O}} + \mathcal{I}_{\boldsymbol{\omega}^{\mathcal{S}}} \times \mathcal{I}_{\mathbf{v}_{P/O}} \\ &= \left(\ddot{\phi} \rho + 2\dot{\phi} \dot{\rho} - \rho \dot{\theta}^2 \sin(\phi) \cos(\phi) \right) \hat{\mathbf{e}}_\phi \\ &\quad + \left(2\dot{\phi} \rho \dot{\theta} \cos(\phi) + \rho \ddot{\theta} \sin(\phi) + 2\dot{\rho} \dot{\theta} \sin(\phi) \right) \hat{\mathbf{e}}_\theta \\ &\quad + \left(-\dot{\phi}^2 \rho - \rho \dot{\theta}^2 \sin^2(\phi) + \ddot{\rho} \right) \hat{\mathbf{e}}_\rho\end{aligned}$$

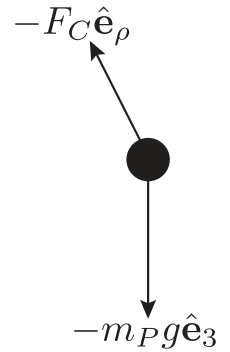
Spherical Simple Pendulum



$$\mathcal{S} \triangleq (O, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\rho)$$



$$\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{S}}} = -\dot{\theta} \sin(\phi) \hat{\mathbf{e}}_\phi + \dot{\phi} \hat{\mathbf{e}}_\theta + \dot{\theta} \cos(\phi) \hat{\mathbf{e}}_\rho$$



$$\begin{aligned}\mathbf{r}_{P/O} &= l \hat{\mathbf{e}}_\rho \\ \mathcal{I}_{\mathbf{v}_{P/O}} &= l \dot{\phi} \hat{\mathbf{e}}_\phi + l \dot{\theta} \sin(\phi) \hat{\mathbf{e}}_\theta \\ \mathcal{I}_{\mathbf{a}_{P/O}} &= \left(l \ddot{\phi} - l \dot{\theta}^2 \sin(\phi) \cos(\phi) \right) \hat{\mathbf{e}}_\phi \\ &\quad + \left(2l \dot{\phi} \dot{\theta} \cos(\phi) + l \ddot{\theta} \sin(\phi) \right) \hat{\mathbf{e}}_\theta \\ &\quad + \left(-l \dot{\phi}^2 - l \dot{\theta}^2 \sin^2(\phi) \right) \hat{\mathbf{e}}_\rho\end{aligned}$$

$$\begin{aligned}\sum \mathbf{F}_P &= -F_C \hat{\mathbf{e}}_\rho - m_P g \underbrace{\hat{\mathbf{e}}_3}_{-\sin(\phi) \hat{\mathbf{e}}_\phi + \cos(\phi) \hat{\mathbf{e}}_\rho}\end{aligned}$$

Spherical Simple Pendulum Continued

$$\begin{aligned}\hat{\mathbf{e}}_\phi : \quad m_P g \sin(\phi) &= m_P \left(\ell \ddot{\phi} - \ell \dot{\theta}^2 \sin(\phi) \cos(\phi) \right) \\ \hat{\mathbf{e}}_\theta : \quad 0 &= m_P \left(2\ell \dot{\phi} \dot{\theta} \cos(\phi) + \ell \ddot{\theta} \sin(\phi) \right) \\ \hat{\mathbf{e}}_\rho : \quad -F_C - m_P g \cos(\phi) &= m_P \left(-\ell \dot{\phi}^2 - \ell \dot{\theta}^2 \sin^2(\phi) \right)\end{aligned}$$

$$\begin{aligned}F_c &= \ell m_P \dot{\phi}^2 + \ell m_P \dot{\theta}^2 \sin^2(\phi) - g m_P \cos(\phi) \\ \ddot{\phi} &= \dot{\theta}^2 \sin(\phi) \cos(\phi) + \frac{g \sin(\phi)}{\ell} \\ \ddot{\theta} &= -\frac{2\dot{\phi} \dot{\theta} \cos(\phi)}{\sin(\phi)}\end{aligned}$$

Spherical Simple Pendulum Energy

- All of our definitions related to work and energy work equivalently in 3 dimensions:

$$\begin{aligned}T_{P/O} &= \frac{m_P}{2} \|\mathcal{I}_{\mathbf{V}_{P/O}}\|^2 = \frac{m_P}{2} \left(\ell^2 \dot{\phi}^2 + \ell^2 \dot{\theta}^2 \sin^2(\phi) \right) \\ U_{P/O} &= m_P g \mathbf{r}_{P/O} \cdot \hat{\mathbf{e}}_3 = m_P g \ell \cos(\phi) \\ E_{P/O} &= T_{P/O} + U_{P/O}\end{aligned}$$

- The only force doing work on the system is conservative, so total energy must be conserved
- This allows us to write a relationship between $\dot{\phi}$ and $\dot{\theta}$:

$$\dot{\phi}^2 = \frac{2E_{P/O}}{\ell^2 m_P} - \dot{\theta}^2 \sin^2(\phi) - \frac{2g \cos(\phi)}{\ell}$$

Encoding Orientation

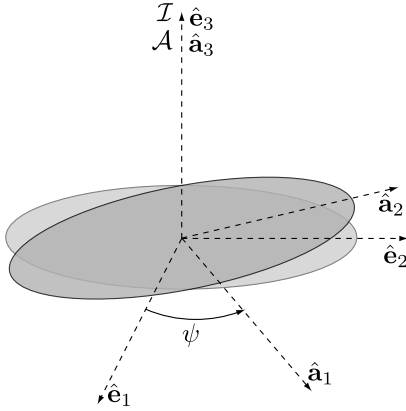
- Recall that a 3D rigid body has 6 degrees of freedom: 3 encoding translation and 3 encoding rotation (**orientation**)
- The translation degrees of freedom can be encoded by Cartesian coordinates of the center of mass (or any coordinates that map to Cartesian coordinates)
- The orientation degrees of freedom can be encoded by *any* coordinate set representing a simple rotation from some inertial frame
- We have already seen several ways of encoding simple rotations:
 - A simple rotation is given by an angle and an axis (one scalar and one unit vector = 3 scalar values)
 - The same information is encoded by direction cosine matrices (9 elements, overdetermined)

Orientation Angles

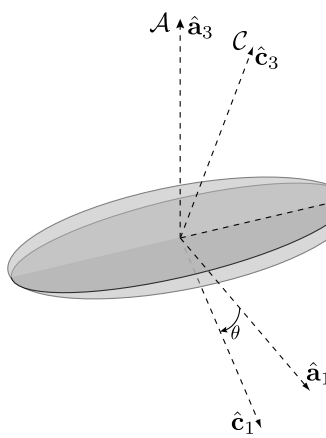
- We have also already introduced the idea that *any* simple rotation can be decomposed into 3 successive rotations about reference frame unit directions
- **Orientation angles** (or **Euler angles**) are an ordered set of angles corresponding to three rotations about specified reference frame directions
- If we use body-fixed frame axes, these are called **body rotations**; if we use inertial frame axes, these are called **space rotations**

- We encode the directions via sets of numbers and can use either 2 or 3 different axes (but cannot repeat rotations about the same axis twice in a row):
 - A Body 1-2-3 $(\psi, \theta, \phi)_{\mathcal{B}}^{\mathcal{I}}$ rotation represents a ψ rotation about the first unit direction, followed by a θ rotation about the second unit direction, followed by a ϕ rotation about the third unit direction, going from frame $\mathcal{I}$ to frame $\mathcal{B}$
 - A Body 3-2-3 $(\psi, \theta, \phi)_{\mathcal{B}}^{\mathcal{I}}$ rotation represents a ψ rotation about the third unit direction, followed by a θ rotation about the second unit direction, followed by a ϕ rotation about the (new) third unit direction, going from frame $\mathcal{I}$ to frame $\mathcal{B}$

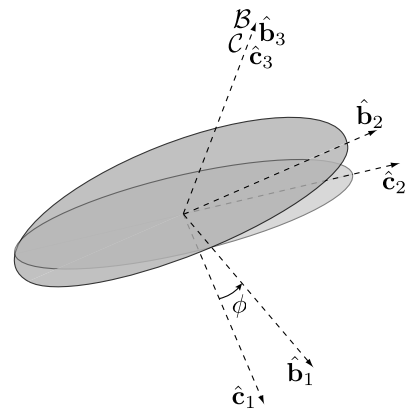
Body 3-2-3 $(\psi, \theta, \phi)_{\mathcal{B}}^{\mathcal{I}}$ Rotation



$$\mathcal{A}_C^{\mathcal{I}} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$${}^c_C{}^A = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

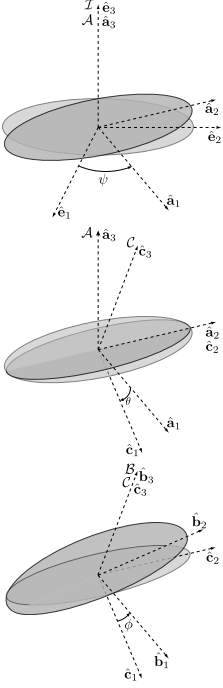


$$\mathcal{B}_C^{\mathcal{C}} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{B}_C^{\mathcal{I}} = \mathcal{B}_C^{\mathcal{C}} \mathcal{C}_C^{\mathcal{A}} \mathcal{A}_C^{\mathcal{I}} =$$

$$\begin{bmatrix} -\sin(\phi) \sin(\psi) + \cos(\phi) \cos(\psi) \cos(\theta) & \sin(\phi) \cos(\psi) + \sin(\psi) \cos(\phi) \cos(\theta) & -\sin(\theta) \cos(\phi) \\ -\sin(\phi) \cos(\psi) \cos(\theta) - \sin(\psi) \cos(\phi) & -\sin(\phi) \sin(\psi) \cos(\theta) + \cos(\phi) \cos(\psi) & \sin(\phi) \sin(\theta) \\ \sin(\theta) \cos(\psi) & \sin(\psi) \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Body 3-2-3 $(\psi, \theta, \phi)_{\mathcal{B}}^{\mathcal{I}}$ Rotation Consistency Check



$$\begin{aligned}\hat{\mathbf{b}}_1 &= \cos(\phi) \underbrace{\hat{\mathbf{a}}_1 - \sin(\theta) \hat{\mathbf{a}}_3}_{\hat{\mathbf{c}}_1} + \sin(\phi) \underbrace{\hat{\mathbf{a}}_2}_{\hat{\mathbf{c}}_2} \\ &= \cos(\phi) \cos(\theta) \underbrace{\hat{\mathbf{a}}_1}_{\cos(\psi) \hat{\mathbf{e}}_1 + \sin(\psi) \hat{\mathbf{e}}_2} + \sin(\phi) \underbrace{\hat{\mathbf{a}}_2}_{-\sin(\psi) \hat{\mathbf{e}}_1 + \cos(\psi) \hat{\mathbf{e}}_2} - \cos(\phi) \sin(\theta) \underbrace{\hat{\mathbf{a}}_3}_{\hat{\mathbf{e}}_3}\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{b}}_1 &= (-\sin(\phi) \sin(\psi) + \cos(\phi) \cos(\psi) \cos(\theta)) \hat{\mathbf{e}}_1 \\ &\quad + (\sin(\phi) \cos(\psi) + \sin(\psi) \cos(\phi) \cos(\theta)) \hat{\mathbf{e}}_2 \\ &\quad - \sin(\theta) \cos(\phi) \hat{\mathbf{e}}_3\end{aligned}$$

Successive Rotations About One Axis

- Consider two successive rotations of θ_1 and θ_2 about the third body-frame axis:

$$\begin{aligned}&\begin{bmatrix} \cos(\theta_2) & \sin(\theta_2) & 0 \\ -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &\begin{bmatrix} -\sin(\theta_1) \sin(\theta_2) + \cos(\theta_1) \cos(\theta_2) & \sin(\theta_1) \cos(\theta_2) + \sin(\theta_2) \cos(\theta_1) & 0 \\ -\sin(\theta_1) \cos(\theta_2) - \sin(\theta_2) \cos(\theta_1) & -\sin(\theta_1) \sin(\theta_2) + \cos(\theta_1) \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) & 0 \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

- This is identical to a $\theta_1 + \theta_2$ rotation about that axis

Euler Angle Sets with Repeated Rotations

- Consider a Body 3-3-1 $(\theta_1, \theta_2, \theta_3)_{\mathcal{B}}^{\mathcal{I}}$ rotation:

$$\begin{aligned} {}^{\mathcal{B}}C^{\mathcal{I}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_3) & \sin(\theta_3) \\ 0 & -\sin(\theta_3) & \cos(\theta_3) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & \sin(\theta_2) & 0 \\ -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) & 0 \\ -\sin(\theta_1 + \theta_2) \cos(\theta_3) & \cos(\theta_3) \cos(\theta_1 + \theta_2) & \sin(\theta_3) \\ \sin(\theta_3) \sin(\theta_1 + \theta_2) & -\sin(\theta_3) \cos(\theta_1 + \theta_2) & \cos(\theta_3) \end{bmatrix} \end{aligned}$$

- The resulting DCM no longer encodes a completely arbitrary orientation in 3D space

Transport Equation in 3D

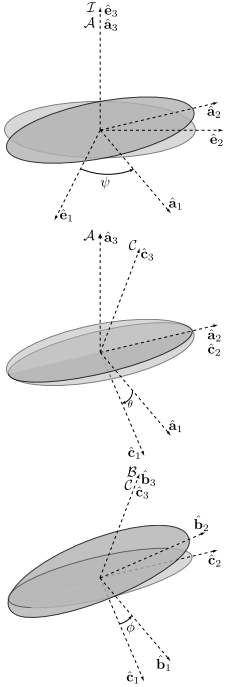
- Consider again the Body 3-2-3 $(\psi, \theta, \phi)_{\mathcal{B}}^{\mathcal{I}}$ rotation with intermediate frames $\mathcal{I} \rightarrow \mathcal{A} \rightarrow \mathcal{C} \rightarrow \mathcal{B}$ and define $\mathbf{r}_{P/O} \triangleq b_1 \hat{\mathbf{b}}_1 + b_2 \hat{\mathbf{b}}_2 + b_3 \hat{\mathbf{b}}_3$

$$\frac{{}^{\mathcal{I}}d}{dt} \mathbf{r}_{P/O} = \underbrace{\dot{b}_1 \hat{\mathbf{b}}_1 + \dot{b}_2 \hat{\mathbf{b}}_2 + \dot{b}_3 \hat{\mathbf{b}}_3}_{\frac{{}^{\mathcal{B}}d}{dt} \mathbf{r}_{P/O}} + \underbrace{b_1 \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_1 + b_2 \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_2 + b_3 \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_3}_{{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O}}$$

- For any of the $\mathcal{B}$ frame unit vectors, we can write:

$$\begin{aligned} \frac{{}^{\mathcal{I}}d}{dt} \hat{\mathbf{b}}_i &= \underbrace{\frac{{}^{\mathcal{A}}d}{dt} \hat{\mathbf{b}}_i + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{A}} \times \hat{\mathbf{b}}_i}_{\frac{{}^{\mathcal{C}}d}{dt} \hat{\mathbf{b}}_i + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{C}} \times \hat{\mathbf{b}}_i} = \underbrace{({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{C}} + {}^{\mathcal{C}}\boldsymbol{\omega}^{\mathcal{B}})}_{{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}} \times \hat{\mathbf{b}}_i \\ &= \underbrace{\frac{{}^{\mathcal{B}}d}{dt} \hat{\mathbf{b}}_i + {}^{\mathcal{C}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_i}_{\frac{{}^{\mathcal{B}}d}{dt} \hat{\mathbf{b}}_i + {}^{\mathcal{C}}\boldsymbol{\omega}^{\mathcal{B}} \times \hat{\mathbf{b}}_i} \end{aligned}$$

Angular Velocity of the Body 3-2-3 $(\psi, \theta, \phi)_{\mathcal{B}}^{\mathcal{I}}$ Rotation



- We have confirmed our previous result of adding angular velocities:

$$\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} = \underbrace{\mathcal{I}\boldsymbol{\omega}^{\mathcal{A}}}_{\dot{\psi}\hat{\mathbf{a}}_3} + \underbrace{\mathcal{A}\boldsymbol{\omega}^{\mathcal{C}}}_{\dot{\theta}\hat{\mathbf{c}}_2} + \underbrace{\mathcal{C}\boldsymbol{\omega}^{\mathcal{B}}}_{\dot{\phi}\hat{\mathbf{b}}_3} \triangleq \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3$$

- From the DCMs:

$$\hat{\mathbf{c}}_2 = \sin(\phi)\hat{\mathbf{b}}_1 + \cos(\phi)\hat{\mathbf{b}}_2$$

$$\hat{\mathbf{a}}_3 = -\sin(\theta)\cos(\phi)\hat{\mathbf{b}}_1 + \sin(\phi)\sin(\theta)\hat{\mathbf{b}}_2 + \cos(\theta)\hat{\mathbf{b}}_3$$

$$\begin{aligned} \mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} = & \overbrace{\left(-\dot{\psi}\sin(\theta)\cos(\phi) + \dot{\theta}\sin(\phi)\right)}^{\omega_1} \hat{\mathbf{b}}_1 \\ & + \underbrace{\left(\dot{\psi}\sin(\phi)\sin(\theta) + \dot{\theta}\cos(\phi)\right)}_{\omega_2} \hat{\mathbf{b}}_2 + \underbrace{\left(\dot{\phi} + \dot{\psi}\cos(\theta)\right)}_{\omega_3} \hat{\mathbf{b}}_3 \end{aligned}$$

Angular Velocity of the Body 3-2-3 $(\psi, \theta, \phi)_{\mathcal{B}}^{\mathcal{I}}$ Rotation

- We can now solve for $\dot{\psi}, \dot{\theta}, \dot{\phi}$ in terms of $\omega_1, \omega_2, \omega_3$:

$$\dot{\psi} = \frac{-\omega_1 \cos(\phi) + \omega_2 \sin(\phi)}{\sin(\theta)}$$

$$\dot{\theta} = \omega_1 \sin(\phi) + \omega_2 \cos(\phi)$$

$$\dot{\phi} = \frac{\omega_1 \cos(\phi)}{\tan(\theta)} - \frac{\omega_2 \sin(\phi)}{\tan(\theta)} + \omega_3$$

- NB: $\omega_i = 0$ can still produce non-zero changes in all Euler angles
- For $\theta = 0$ (or π), $\dot{\psi} \rightarrow \infty$. This state is known as **gimbal lock**: ψ and ϕ are rotating about the same axis

Week 13

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Center of Mass in 3D

- In order to apply Euler's first law, we need to be able to locate the center of mass of a rigid body
- Recall our previous derivation of the COM of a rigid body:

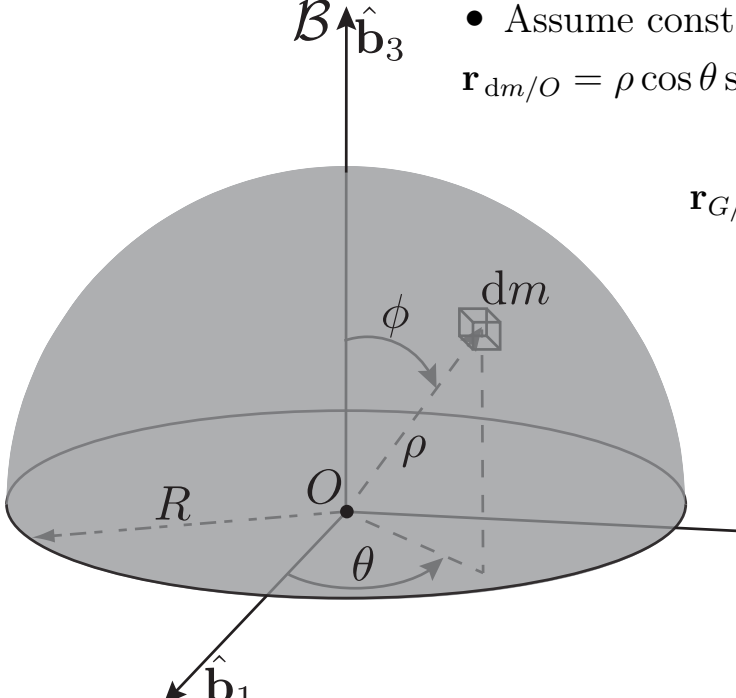
$$\mathbf{r}_{G/O} = \frac{1}{m_G} \int_{\mathcal{B}} \mathbf{r}_{dm/O} dm = \frac{1}{m_G} \int_{\mathcal{B}} \mathbf{r}_{dm/O} \rho(\mathbf{r}_{dm/O}) dV$$

- These expressions are fully valid in 3D—we just need to integrate over the full volume of the rigid body

Example 1: Center of Mass of a Solid Hemisphere

• Assume constant, uniform density ρ_d

$\mathbf{r}_{dm/O} = \rho \cos \theta \sin \phi \hat{\mathbf{b}}_1 + \rho \sin \theta \sin \phi \hat{\mathbf{b}}_2 + \rho \cos \phi \hat{\mathbf{b}}_3$



$$\mathbf{r}_{G/O} = \frac{\rho_d}{m_G} \int_0^{\pi/2} \int_0^{2\pi} \int_0^R \mathbf{r}_{dm/O} \underbrace{dV}_{\rho^2 \sin(\phi) d\rho d\theta d\phi}$$

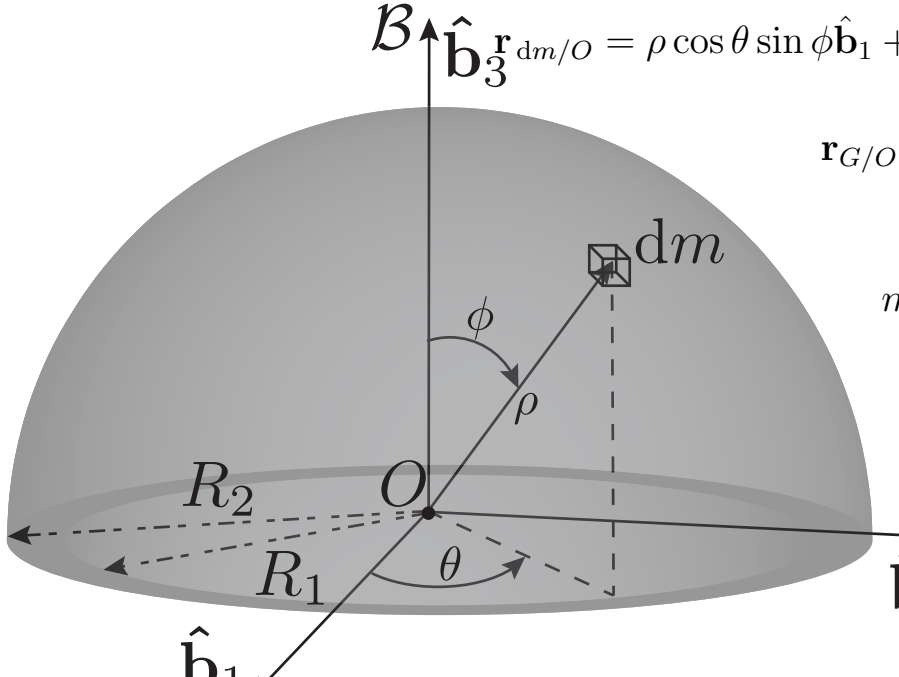
$$= \frac{\rho_d}{m_G} \frac{\pi R^4}{4} \hat{\mathbf{b}}_3$$

$$m_G = \rho_d V = \rho_d \iiint_{\mathcal{B}} dV = \rho_d \frac{2\pi R^3}{3}$$

$$\mathbf{r}_{G/O} = \frac{3}{8} R \hat{\mathbf{b}}_3$$

Example 2: Center of Mass of a Hemispherical Shell

$\mathbf{r}_{dm/O} = \rho \cos \theta \sin \phi \hat{\mathbf{b}}_1 + \rho \sin \theta \sin \phi \hat{\mathbf{b}}_2 + \rho \cos \phi \hat{\mathbf{b}}_3$



$$\mathbf{r}_{G/O} = \frac{\rho_d}{m_G} \int_0^{\pi/2} \int_0^{2\pi} \int_{R_1}^{R_2} \mathbf{r}_{dm/O} dV$$

$$m_G = \rho_d V = \rho_d \iiint_{\mathcal{B}} dV$$

$$= \rho_d \frac{2\pi (R_2^3 - R_1^3)}{3}$$

$$\mathbf{r}_{G/O} = \frac{3 (R_2^4 - R_1^4)}{8 (R_2^3 - R_1^3)} \hat{\mathbf{b}}_3$$

A Brief Introduction to Tensors

- A **tensor** is a generalization of a geometric vector
 - An order (or rank)-0 tensor is a scalar
 - An order-1 tensor is a vector
 - An order-2 tensor is a **dyadic**

Notation

When I typeset tensors, they will always be in blackboard bold and capitalized (e.g. $\mathbb{T}$). When I write tensors, they will be either similarly double-struck or double underlined (e.g. $\underline{\underline{T}}$). *You* can write tensors however you wish, as long as I can always tell the difference between your tensors, vectors and scalars.

- A dyadic tensor has many of the same basic properties as a Euclidean vector
 - Tensor addition is associative and commutative
 - Scalar multiplication works the same as for vectors
 - Zero tensor: $\exists \mathbb{0}$ s.t. $\mathbb{0} + \mathbb{T} = \mathbb{T}$ and additive inverse: $\exists -\mathbb{T}$ s.t. $\mathbb{T} + (-\mathbb{T}) = \mathbb{0}$

Tensor Inner (Dot) Product

- The dot product we've been using with vectors is an example of an **inner product**
- Inner products reduce tensor order
 - The dot product of a vector with a vector is a scalar (order $1 - 1 = 0$)
 - The dot product of a dyadic tensor with a vector is a vector: $\mathbb{T} \cdot \mathbf{a} = \mathbf{b}$ (order $2 - 1 = 1$)
- The dot product between a tensor and vector is distributive over tensor addition and compatible with scalar multiplication, but *not* commutative
- There exists an identity dyadic tensor $\mathbb{U}$ such that $\mathbb{U} \cdot \mathbf{a} = \mathbf{a} \ \forall \mathbf{a}$

Tensors Outer Product

- A dyadic is defined as the **outer product** of two vectors: $\mathbb{T} \triangleq \mathbf{a} \otimes \mathbf{b}$
- NB: The outer product is *not* a cross product. Outer products increase tensor order, while the cross product preserves it
- There is *also* a well-defined cross product between dyadics and Euclidean vectors which produces dyadics
- Outer Product properties:

- $(\mathbf{a} + \mathbf{b}) \otimes \mathbf{c} = \mathbf{a} \otimes \mathbf{c} + \mathbf{b} \otimes \mathbf{c}$
- $\mathbf{c} \otimes (\mathbf{a} + \mathbf{b}) = \mathbf{c} \otimes \mathbf{a} + \mathbf{c} \otimes \mathbf{b}$
- $x(\mathbf{a} \otimes \mathbf{b}) = (x\mathbf{a}) \otimes \mathbf{b} = \mathbf{a} \otimes (x\mathbf{b})$
- $\mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
- $(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$
- $\mathbf{a} \times (\mathbf{b} \otimes \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \otimes \mathbf{c}$
- $(\mathbf{a} \otimes \mathbf{b}) \times \mathbf{c} = \mathbf{a} \otimes (\mathbf{b} \times \mathbf{c})$

Tensor Components

- Just like Euclidean vectors, we can express the components of a tensor with respect to a reference frame

$$\left[\mathbb{T} \triangleq \mathbf{a} \otimes \mathbf{b} \right]_{\mathcal{I}} = [\mathbf{a}]_{\mathcal{I}} [\mathbf{b}]_{\mathcal{I}}^T$$

$$\mathcal{I} \triangleq (O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \quad [\mathbf{a}]_{\mathcal{I}} \triangleq \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{I}} \quad [\mathbf{b}]_{\mathcal{I}} \triangleq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}}$$

$$\left[\mathbb{T} \triangleq \mathbf{a} \otimes \mathbf{b} \right]_{\mathcal{I}} \triangleq \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{I}} \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}} \right)^T = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}_{\mathcal{I}}$$

Tensor Components Continued

$$\mathbf{a} = \sum_{i=1}^3 a_i \hat{\mathbf{e}}_i \implies a_i = \mathbf{a} \cdot \hat{\mathbf{e}}_i \quad \mathbf{b} = \sum_{i=1}^3 b_i \hat{\mathbf{e}}_i \implies b_i = \mathbf{b} \cdot \hat{\mathbf{e}}_i$$

$$\mathbb{T} \triangleq \mathbf{a} \otimes \mathbf{b} = \sum_{i=1}^3 \sum_{j=1}^3 T_{ij} \underbrace{\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j}_{\text{Tensor Basis}} \implies T_{ij} = \hat{\mathbf{e}}_i \cdot \mathbb{T} \cdot \hat{\mathbf{e}}_j = a_i b_j$$

- The components of the identity (or unity) dyadic are the identity matrix

$$[\mathbb{U}]_{\mathcal{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathcal{I}} \triangleq I_{3 \times 3}$$

Tensor Products as Matrix Multiplications

$$\begin{aligned} [\mathbb{T} \triangleq \mathbf{a} \otimes \mathbf{b}]_{\mathcal{I}} &= [\mathbf{a}]_{\mathcal{I}} [\mathbf{b}]_{\mathcal{I}}^T \\ [\mathbf{c} \cdot \mathbb{T}]_{\mathcal{I}} &= ([\mathbf{c}]_{\mathcal{I}}^T [\mathbb{T}]_{\mathcal{I}})^T \\ [\mathbb{T} \cdot \mathbf{c}]_{\mathcal{I}} &= [\mathbb{T}]_{\mathcal{I}} [\mathbf{c}]_{\mathcal{I}} \\ [\mathbf{c} \times \mathbb{T}]_{\mathcal{I}} &= [\mathbf{c} \times]_{\mathcal{I}} [\mathbb{T}]_{\mathcal{I}} \\ [\mathbb{T} \times \mathbf{c}]_{\mathcal{I}} &= [\mathbb{T}]_{\mathcal{I}} [\mathbf{c} \times]_{\mathcal{I}} \end{aligned} \quad [\mathbf{c} \times]_{\mathcal{I}} = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}_{\mathcal{I}}$$

$$[\mathbf{c} \cdot \mathbb{T}]_{\mathcal{I}} = \left(\left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{\mathcal{I}} \right)^T \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{\mathcal{I}} \right)^T = \begin{bmatrix} T_{11}c_1 + T_{21}c_2 + T_{31}c_3 \\ T_{12}c_1 + T_{22}c_2 + T_{32}c_3 \\ T_{13}c_1 + T_{23}c_2 + T_{33}c_3 \end{bmatrix}_{\mathcal{I}}$$

$$[\mathbb{T} \cdot \mathbf{c}]_{\mathcal{I}} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{\mathcal{I}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} T_{11}c_1 + T_{12}c_2 + T_{13}c_3 \\ T_{21}c_1 + T_{22}c_2 + T_{23}c_3 \\ T_{31}c_1 + T_{32}c_2 + T_{33}c_3 \end{bmatrix}_{\mathcal{I}}$$

$$\begin{aligned}
[\mathbf{c} \times \mathbb{T}]_{\mathcal{I}} &= \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}_{\mathcal{I}} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{\mathcal{I}} \\
&= \begin{bmatrix} -T_{21}c_3 + T_{31}c_2 & -T_{22}c_3 + T_{32}c_2 & -T_{23}c_3 + T_{33}c_2 \\ T_{11}c_3 - T_{31}c_1 & T_{12}c_3 - T_{32}c_1 & T_{13}c_3 - T_{33}c_1 \\ -T_{11}c_2 + T_{21}c_1 & -T_{12}c_2 + T_{22}c_1 & -T_{13}c_2 + T_{23}c_1 \end{bmatrix}_{\mathcal{I}} \\
[\mathbb{T} \times \mathbf{c}]_{\mathcal{I}} &= \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{\mathcal{I}} \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}_{\mathcal{I}} \\
&= \begin{bmatrix} T_{12}c_3 - T_{13}c_2 & -T_{11}c_3 + T_{13}c_1 & T_{11}c_2 - T_{12}c_1 \\ T_{22}c_3 - T_{23}c_2 & -T_{21}c_3 + T_{23}c_1 & T_{21}c_2 - T_{22}c_1 \\ T_{32}c_3 - T_{33}c_2 & -T_{31}c_3 + T_{33}c_1 & T_{31}c_2 - T_{32}c_1 \end{bmatrix}_{\mathcal{I}}
\end{aligned}$$

The Moment of Inertia Tensor

- Recall our derivation of the planar moment of inertia:

$$\begin{aligned}
\mathcal{I}_{\mathbf{h}_G} &= \sum_{i=1}^N m_i \mathbf{r}_{i/G} \times \overbrace{\mathcal{B}_{\mathbf{v}_{i/G}} + \mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}} \times \mathbf{r}_{i/G}}^{\mathbf{0}} \\
&= \sum_{i=1}^N m_i \mathbf{r}_{i/G} \times \overbrace{\mathcal{I}_{\mathbf{v}_{i/G}}}^{\mathbf{0}} = \sum_{i=1}^N m_i \mathbf{r}_{i/G} \times (\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}} \times \mathbf{r}_{i/G}) \\
&= \sum_{i=1}^N m_i \left((\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G}) \underbrace{\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}}}_{\mathbb{U} \cdot \mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}}} - \underbrace{(\mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}} \cdot \mathbf{r}_{i/G}) \mathbf{r}_{i/G}}_{(\mathbf{r}_{i/G} \otimes \mathbf{r}_{i/G}) \cdot \mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}}} \right) \\
&= \underbrace{\left(\sum_{i=1}^N m_i [(\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G}) \mathbb{U} - (\mathbf{r}_{i/G} \otimes \mathbf{r}_{i/G})] \right)}_{\triangleq \mathbb{I}_G} \cdot \mathcal{I}_{\boldsymbol{\omega}^{\mathcal{B}}}
\end{aligned}$$

The Moment of Inertia Tensor Continued

$$\mathcal{I}_{\mathbf{h}_G} = \mathbb{I}_G \cdot \mathcal{I}\omega^{\mathcal{B}}$$

- This is a completely general statement and applies for motion in 3D
- This expression *also* applies to planar motion
- Just as with all quantities related to rigid bodies, we can take the limit as the number of particles goes to ∞ :

$$\begin{aligned}\mathbb{I}_G &\triangleq \sum_{i=1}^N m_i [(\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G})\mathbb{U} - (\mathbf{r}_{i/G} \otimes \mathbf{r}_{i/G})] \\ &\triangleq \int_{\mathcal{B}} [(\mathbf{r}_{dm/G} \cdot \mathbf{r}_{dm/G})\mathbb{U} - (\mathbf{r}_{dm/G} \otimes \mathbf{r}_{dm/G})] dm\end{aligned}$$

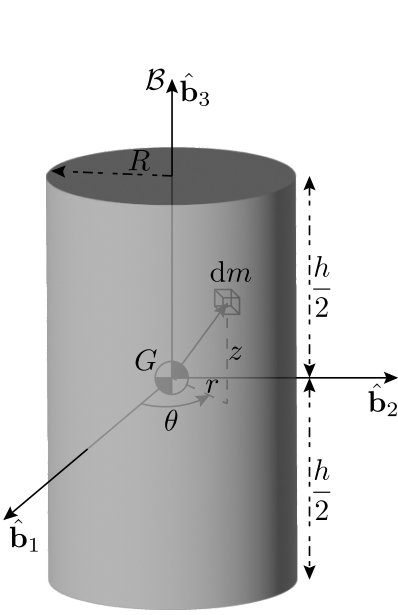
The Moment of Inertia Matrix

- To actually use the moment of inertia tensor to do dynamics, we will need to express it in components with respect to a reference frame—typically a body-fixed frame $\mathcal{B} = (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$

$$\begin{aligned}\mathbb{I}_G &= \sum_{i=1}^3 \sum_{j=1}^3 I_{ij} \hat{\mathbf{b}}_i \otimes \hat{\mathbf{b}}_j & [\mathbb{I}_G]_{\mathcal{B}} &= \sum_{i=1}^N m_i ([\mathbf{r}_{i/G}]_{\mathcal{B}}^T [\mathbf{r}_{i/G}]_{\mathcal{B}} I_{3 \times 3} - [\mathbf{r}_{i/G}]_{\mathcal{B}} [\mathbf{r}_{i/G}]_{\mathcal{B}}^T) \\ & & &= \int_{\mathcal{B}} (\|\mathbf{r}_{dm/G}\|^2 I_{3 \times 3} - [\mathbf{r}_{dm/G}]_{\mathcal{B}} [\mathbf{r}_{dm/G}]_{\mathcal{B}}^T) dm \\ [\mathbb{I}_G]_{\mathcal{B}} &= \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}_{\mathcal{B}} & &= \int_{\mathcal{B}} (\|\mathbf{r}_{dm/G}\|^2 I_{3 \times 3} - [\mathbf{r}_{dm/G}]_{\mathcal{B}} [\mathbf{r}_{dm/G}]_{\mathcal{B}}^T) \rho(\mathbf{r}_{dm/G}) dV\end{aligned}$$

- Moment of inertia matrices are *always* symmetric
- Diagonal entries of the matrix are called **moments of inertia**
- Off-diagonal entries are called **products of inertia**

Example 1: Moment of Inertia of a Solid Cylinder



$$\mathbf{r}_{dm/G} = r \cos \theta \hat{\mathbf{b}}_1 + r \sin \theta \hat{\mathbf{b}}_2 + z \hat{\mathbf{b}}_3$$

$$\|\mathbf{r}_{dm/G}\|^2 = r^2 + z^2$$

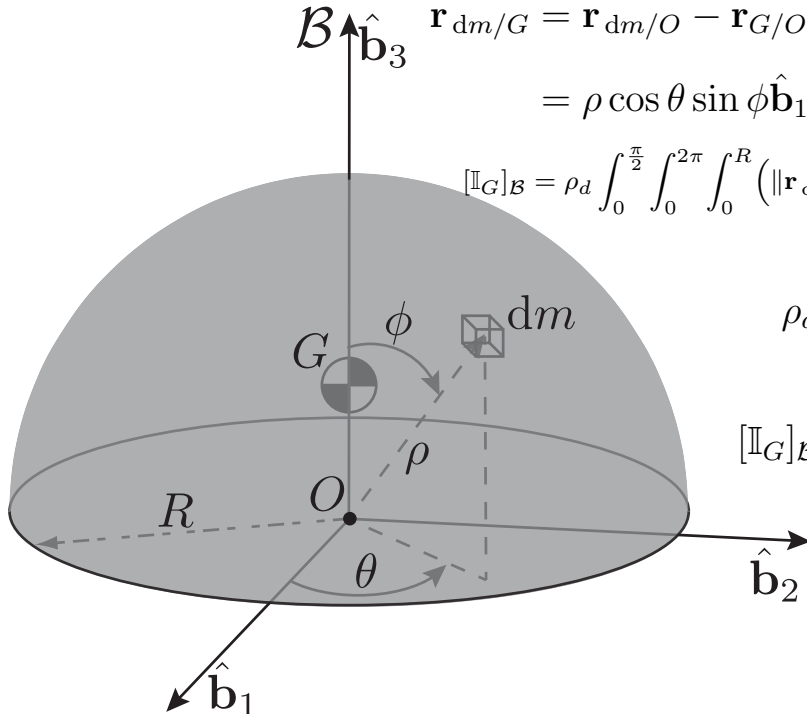
$$[\mathbf{r}_{dm/G}]_{\mathcal{B}} [\mathbf{r}_{dm/G}]_{\mathcal{B}}^T =$$

$$\begin{bmatrix} r^2 \cos^2(\theta) & r^2 \sin(\theta) \cos(\theta) & rz \cos(\theta) \\ r^2 \sin(\theta) \cos(\theta) & r^2 \sin^2(\theta) & rz \sin(\theta) \\ rz \cos(\theta) & rz \sin(\theta) & z^2 \end{bmatrix}_{\mathcal{B}}$$

$$[\mathbb{I}_G]_{\mathcal{B}} = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} \int_0^R (\|\mathbf{r}_{dm/G}\|^2 I_{3 \times 3} - [\mathbf{r}_{dm/G}]_{\mathcal{B}} [\mathbf{r}_{dm/G}]_{\mathcal{B}}^T) r dr d\theta dz$$

$$\rho = \frac{m}{\pi R^2 h} \Rightarrow [\mathbb{I}_G]_{\mathcal{B}} = \frac{m}{12} \begin{bmatrix} 3R^2 + h^2 & 0 & 0 \\ 0 & 3R^2 + h^2 & 0 \\ 0 & 0 & 6R^2 \end{bmatrix}_{\mathcal{B}}$$

Example 2: Moment of Inertia of a Solid Hemisphere



$$\mathbf{r}_{dm/G} = \mathbf{r}_{dm/O} - \mathbf{r}_{G/O}$$

$$= \rho \cos \theta \sin \phi \hat{\mathbf{b}}_1 + \rho \sin \theta \sin \phi \hat{\mathbf{b}}_2 + \left(\rho \cos \phi - \frac{3}{8}R \right) \hat{\mathbf{b}}_3$$

$$[\mathbb{I}_G]_{\mathcal{B}} = \rho_d \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^R (\|\mathbf{r}_{dm/G}\|^2 I_{3 \times 3} - [\mathbf{r}_{dm/G}]_{\mathcal{B}} [\mathbf{r}_{dm/G}]_{\mathcal{B}}^T) \rho^2 \sin(\phi) d\rho d\theta d\phi$$

$$\rho_d = \frac{m}{2\pi R^3/3}$$

$$[\mathbb{I}_G]_{\mathcal{B}} = \frac{mR^2}{320} \begin{bmatrix} 83 & 0 & 0 \\ 0 & 83 & 0 \\ 0 & 0 & 128 \end{bmatrix}_{\mathcal{B}}$$

Euler's 2nd Law in 3D

- Euler's 2nd Law, applied to the center of mass of a rigid body:

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \mathbf{h}_G = \mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}d}{{}^{\mathcal{B}}dt} \boldsymbol{\omega}^{\mathcal{B}} + \boldsymbol{\omega}^{\mathcal{B}} \times (\mathbb{I}_G \cdot \boldsymbol{\omega}^{\mathcal{B}}) = \mathbf{M}_G$$

- With all terms written in components of a body-fixed frame $\mathcal{B}$, this becomes:

$$[\mathbb{I}_G]_{\mathcal{B}} \left[\frac{{}^{\mathcal{B}}d}{{}^{\mathcal{B}}dt} \boldsymbol{\omega}^{\mathcal{B}} \right]_{\mathcal{B}} + [\boldsymbol{\omega}^{\mathcal{B}} \times]_{\mathcal{B}} [\mathbb{I}_G]_{\mathcal{B}} [\boldsymbol{\omega}^{\mathcal{B}}]_{\mathcal{B}} = [\mathbf{M}_G]_{\mathcal{B}}$$

- These equations become much simpler if $[\mathbb{I}_G]_{\mathcal{B}}$ is strictly diagonal
- There *always* exists a body-fixed frame for any rigid body where the moment of inertia matrix is diagonal. We call this the body's **principal axis frame**

Diagonalization

- For any tensor $\mathbb{T}$: $[\mathbb{T}]_{\mathcal{B}} = {}^{\mathcal{B}}C^{\mathcal{A}} [\mathbb{T}]_{\mathcal{A}} {}^{\mathcal{A}}C^{\mathcal{B}} = {}^{\mathcal{B}}C^{\mathcal{A}} [\mathbb{T}]_{\mathcal{A}} ({}^{\mathcal{B}}C^{\mathcal{A}})^T$
- A real, symmetric matrix A has all real eigenvalues and orthogonal eigenvectors:
 $A = A^T \Rightarrow A \mathbf{v}_i = \lambda_i \mathbf{v}_i \Rightarrow \lambda_i \in \mathbb{R} \forall i, \quad \mathbf{v}_i \cdot \mathbf{v}_j = 0 \forall i, j$
- Let P be the matrix of eigenvectors of A ($P \triangleq [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]$) and D be the diagonal matrix of eigenvalues ($D = \text{diag}(\{\lambda_i\})$). Then:
 $A = P D P^{-1} = P D P^T$
- Every real, symmetric matrix (including all moment of inertia matrices) can be diagonalized as $D = P^T A P$
- For any body-fixed frame $\mathcal{B}$, we can find $\mathcal{B}_P$ by diagonalizing $[\mathbb{I}_G]_{\mathcal{B}}$:

$$[\mathbb{I}_G]_{\mathcal{B}_P} = {}^{\mathcal{B}_P}C^{\mathcal{B}} [\mathbb{I}_G]_{\mathcal{B}} {}^{\mathcal{B}}C^{\mathcal{B}_P}$$

Eigenvectors of $[\mathbb{I}_G]_{\mathcal{B}}$

Euler's Equations

$$[\mathbb{I}_G]_{\mathcal{B}_p} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p}$$

$$[\mathbb{I}_G]_{\mathcal{B}_p} \left[\frac{{}^{\mathcal{B}}d}{{}^{\mathcal{B}}dt} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \right]_{\mathcal{B}_p} + [{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times]_{\mathcal{B}_p} [\mathbb{I}_G]_{\mathcal{B}_p} [{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}]_{\mathcal{B}_p} = [\mathbf{M}_G]_{\mathcal{B}_p} \implies$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}_{\mathcal{B}_p} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}_{\mathcal{B}_p} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_{\mathcal{B}_p} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}_{\mathcal{B}_p}$$

Euler's Equations

$$I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = M_1$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_1 \omega_3 = M_2$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 = M_3$$

Angular Momentum and Moments about Arbitrary Points

- Our derivation of the moment transport theorem made no assumptions about planar motion, so it remains exactly the same in the 3D case:

$$\mathbf{M}_Q = \mathbf{M}_G - \mathbf{r}_{Q/G} \times \sum_{i=1}^N \mathbf{F}_i^{(\text{ext})}$$

- However, our derivation of the planar parallel axis theorem did make the assumption of planar motion, so we will need a new version for the 3D case

$$\begin{aligned} {}^{\mathcal{I}}\mathbf{h}_Q &= {}^{\mathcal{I}}\mathbf{h}_G + m_G \mathbf{r}_{Q/G} \times \underbrace{{}^{\mathcal{B}}\mathbf{v}_{Q/G}}_0 + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{Q/G} \\ &= {}^{\mathcal{I}}\mathbf{h}_G + m_G \left((\mathbf{r}_{Q/G} \cdot \mathbf{r}_{Q/G}) \underbrace{{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}}_{\mathbb{U} \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}} - \underbrace{(\mathbf{r}_{Q/G} \otimes \mathbf{r}_{Q/G}) \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}}_{(\mathbf{r}_{Q/G} \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}) \mathbf{r}_{Q/G}} \right) \end{aligned}$$

Parallel Axis Theorem

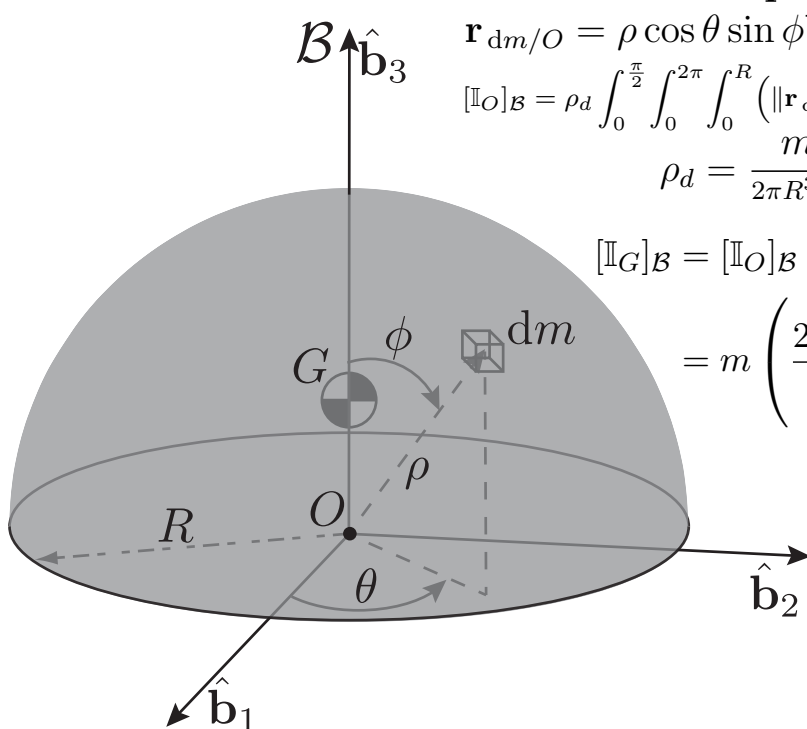
$$\mathbb{I}_Q = \mathbb{I}_G + m_G \left((\mathbf{r}_{Q/G} \cdot \mathbf{r}_{Q/G}) \mathbb{U} - (\mathbf{r}_{Q/G} \otimes \mathbf{r}_{Q/G}) \right)$$

$$[\mathbb{I}_Q]_{\mathcal{B}} = [\mathbb{I}_G]_{\mathcal{B}} + m_G \left(\|\mathbf{r}_{Q/G}\|^2 I_{3 \times 3} - [\mathbf{r}_{Q/G}]_{\mathcal{B}} [\mathbf{r}_{Q/G}]_{\mathcal{B}}^T \right)$$

$${}^{\mathcal{I}}\mathbf{h}_Q = \mathbb{I}_Q \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$$

$${}^{\mathcal{I}}\frac{d}{dt} {}^{\mathcal{I}}\mathbf{h}_Q = {}^{\mathcal{B}}\frac{d}{dt} ({}^{\mathcal{I}}\mathbf{h}_Q) + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_Q = \mathbf{M}_Q + \mathbf{r}_{Q/G} \times m_G {}^{\mathcal{I}}\mathbf{a}_{Q/O}$$

Moment of Inertia of a Solid Hemisphere (Again)



$$\mathbf{r}_{dm/O} = \rho \cos \theta \sin \phi \hat{\mathbf{b}}_1 + \rho \sin \theta \sin \phi \hat{\mathbf{b}}_2 + \rho \cos \phi \hat{\mathbf{b}}_3$$

$$[\mathbb{I}_O]_{\mathcal{B}} = \rho_d \int_0^{\pi/2} \int_0^{2\pi} \int_0^R \left(\|\mathbf{r}_{dm/O}\|^2 I_{3 \times 3} - [\mathbf{r}_{dm/O}]_{\mathcal{B}} [\mathbf{r}_{dm/O}]_{\mathcal{B}}^T \right) \rho^2 \sin(\phi) d\rho d\theta d\phi$$

$$\rho_d = \frac{m}{2\pi R^3/3} \implies [\mathbb{I}_O]_{\mathcal{B}} = \frac{2}{5} m R^2 I_{3 \times 3}$$

$$[\mathbb{I}_G]_{\mathcal{B}} = [\mathbb{I}_O]_{\mathcal{B}} - m \left(\|\mathbf{r}_{O/G}\|^2 I_{3 \times 3} - [\mathbf{r}_{O/G}]_{\mathcal{B}} [\mathbf{r}_{O/G}]_{\mathcal{B}}^T \right)$$

$$= m \left(\frac{2R^2}{5} I_{3 \times 3} - \left(\frac{3R}{8} \right)^2 I_{3 \times 3} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{9R^2}{64} \end{bmatrix}_{\mathcal{B}} \right)$$

$$[\mathbb{I}_G]_{\mathcal{B}} = \frac{mR^2}{320} \begin{bmatrix} 83 & 0 & 0 \\ 0 & 83 & 0 \\ 0 & 0 & 128 \end{bmatrix}_{\mathcal{B}}$$

Week 14

Dmitry Savransky

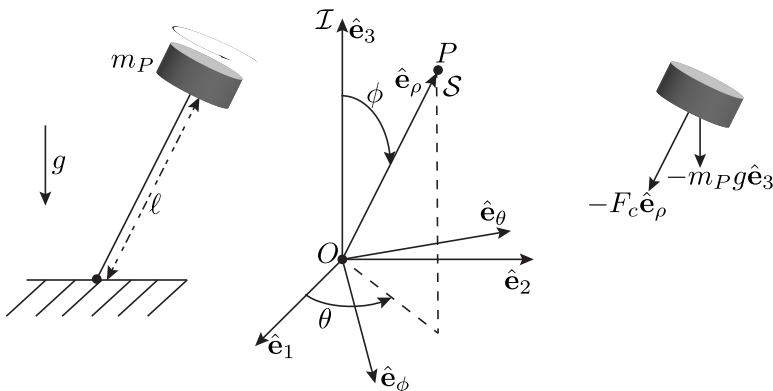
Cornell University

MAE 2030, Spring 2026

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Spherical Gyropendulum

- The spherical gyropendulum is a simple spherical pendulum with the mass replaced by a spinning rotor
- If the rotor is not spinning, this system has exactly the same equations of motion as the simple spherical pendulum



$$\begin{aligned}
 \mathbf{r}_{P/O} &= \ell \hat{\mathbf{e}}_\rho \\
 \mathcal{I} \mathbf{v}_{P/O} &= \ell \dot{\phi} \hat{\mathbf{e}}_\phi + \ell \dot{\theta} \sin \phi \hat{\mathbf{e}}_\theta \\
 \mathcal{I} \mathbf{h}_O &= \mathcal{I} \mathbf{h}_{P/O} + \mathcal{I} \mathbf{h}_P \\
 \mathcal{I} \mathbf{h}_{P/O} &= \mathbf{r}_{P/O} \times m_P \mathcal{I} \mathbf{v}_{P/O} \\
 &= m_P \left(\ell^2 \dot{\phi} \hat{\mathbf{e}}_\theta - \ell^2 \dot{\theta} \sin(\phi) \hat{\mathbf{e}}_\phi \right)
 \end{aligned}$$

- We don't actually need to know the exact form of $\mathcal{I} \mathbf{h}_P$: the rotor always spins about $\hat{\mathbf{e}}_\rho$ so $\mathcal{I} \mathbf{h}_P = h \hat{\mathbf{e}}_\rho$ for some value of h

Spherical Gyropendulum Continued

- The spherical frame angular velocity is:

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{S}} = -\dot{\theta} \sin(\phi) \hat{\mathbf{e}}_{\phi} + \dot{\phi} \hat{\mathbf{e}}_{\theta} + \dot{\theta} \cos(\phi) \hat{\mathbf{e}}_{\rho}$$

$$\begin{aligned} \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{h}_O &= \frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} \left(m_P \left(\ell^2 \dot{\phi} \hat{\mathbf{e}}_{\theta} - \ell^2 \dot{\theta} \sin(\phi) \hat{\mathbf{e}}_{\phi} \right) + h \hat{\mathbf{e}}_{\rho} \right) \\ &= \left(-2\ell^2 m_P \dot{\phi} \dot{\theta} \cos(\phi) - \ell^2 m_P \ddot{\theta} \sin(\phi) + h \dot{\phi} \right) \hat{\mathbf{e}}_{\phi} \\ &\quad + \left(\ell^2 m_P \ddot{\phi} - \ell^2 m_P \dot{\theta}^2 \sin(\phi) \cos(\phi) + h \dot{\theta} \sin(\phi) \right) \hat{\mathbf{e}}_{\theta} \end{aligned}$$

- The only moment about O is due to gravity (as the constraint force acts in $\hat{\mathbf{e}}_{\rho}$):

$$\begin{aligned} \mathbf{M}_O &= \mathbf{r}_{P/O} \times \mathbf{F}_P = \ell \hat{\mathbf{e}}_{\rho} \times (-m_P g \underbrace{\hat{\mathbf{e}}_3}_{-\sin(\phi) \hat{\mathbf{e}}_{\phi} + \cos(\phi) \hat{\mathbf{e}}_{\rho}}) = \ell g m_P \sin(\phi) \hat{\mathbf{e}}_{\theta} \\ &\quad - \sin(\phi) \hat{\mathbf{e}}_{\phi} + \cos(\phi) \hat{\mathbf{e}}_{\rho} \end{aligned}$$

$$\ddot{\phi} = \dot{\theta}^2 \sin(\phi) \cos(\phi) + \frac{g \sin(\phi)}{\ell} - \frac{h \dot{\theta} \sin(\phi)}{\ell^2 m_P}, \quad \ddot{\theta} = -\frac{2 \dot{\phi} \dot{\theta} \cos(\phi)}{\sin(\phi)} + \frac{h \dot{\phi}}{\ell^2 m_P \sin(\phi)}$$

- The EOMs of the spherical gyropendulum are exactly the same as those of the simple spherical pendulum, with one extra term in each
- The magnitude of h mediates the tradeoff in acceleration between θ and ϕ
- In many similar systems, the angular momentum is dominated by the fastest spinning part (the rotor), so we approximate

$${}^{\mathcal{I}}\mathbf{h}_O = {}^{\mathcal{I}}\mathbf{h}_{G/O} + {}^{\mathcal{I}}\mathbf{h}_G \approx {}^{\mathcal{I}}\mathbf{h}_G$$

- We also approximate that the angular momentum is approximately

$$\text{constant in the body-fixed frame } \frac{{}^{\mathcal{B}}d}{{}^{\mathcal{B}}dt} {}^{\mathcal{I}}\mathbf{h}_G \approx 0 \text{ so:}$$

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{h}_G = \frac{{}^{\mathcal{B}}d}{{}^{\mathcal{B}}dt} {}^{\mathcal{I}}\mathbf{h}_G + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_G \approx {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_G$$

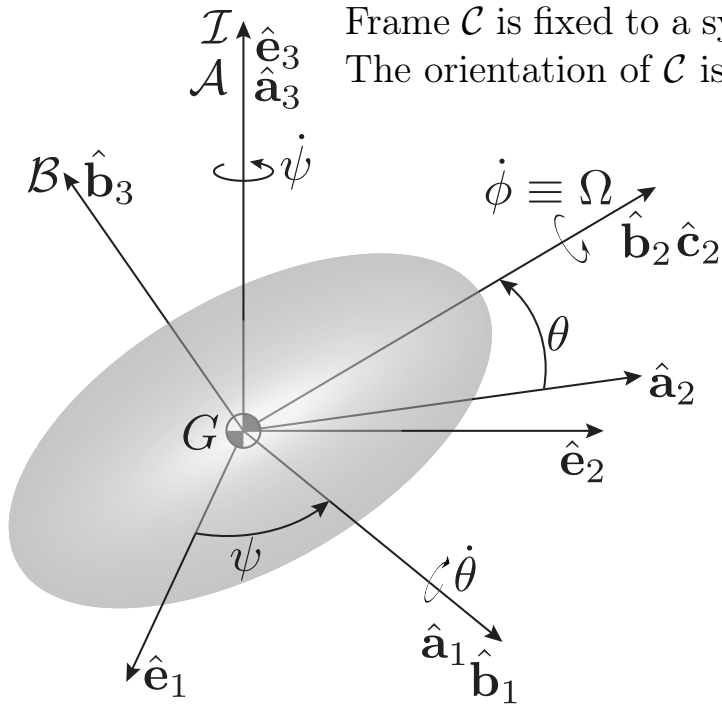
- For the spherical gyropendulum ${}^{\mathcal{I}}\mathbf{h}_O \approx h \hat{\mathbf{e}}_{\rho}$ when $h \gg m_P \ell^2 \dot{\phi}, m_P \ell^2 \dot{\theta} \sin(\phi)$ so

$$\frac{{}^{\mathcal{I}}d}{{}^{\mathcal{I}}dt} {}^{\mathcal{I}}\mathbf{h}_O \approx {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times h \hat{\mathbf{e}}_{\rho} = \mathbf{M}_O = \ell g m_P \sin(\phi) \hat{\mathbf{e}}_{\theta} \quad \Rightarrow \quad {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \approx -\frac{\ell g m_P}{h} \sin(\phi) \hat{\mathbf{e}}_{\phi}$$

- Since $\hat{\mathbf{e}}_3 = -\sin(\phi) \hat{\mathbf{e}}_{\phi} + \cos(\phi) \hat{\mathbf{e}}_{\rho}$, this predicts

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \approx \frac{\ell g m_P}{h} (\hat{\mathbf{e}}_3 - \cos(\phi) \hat{\mathbf{e}}_{\rho}) \text{ CCW rotation about the vertical with some other rotation about } \hat{\mathbf{e}}_{\rho}$$

Spinning Symmetric Rigid Bodies



Frame $\mathcal{C}$ is fixed to a symmetric rigid body with symmetry axis $\hat{\mathbf{c}}_2$. The orientation of $\mathcal{C}$ is given by a Body 3-1-2 $(\psi, \theta, \phi)_{\mathcal{C}}^T$ rotation

Intermediate frame $\mathcal{B}$ is defined with orientation 3-1 $(\psi, \theta)_{\mathcal{B}}^T$ such that $\hat{\mathbf{b}}_2 \equiv \hat{\mathbf{c}}_2$. This is the spin and symmetry axis

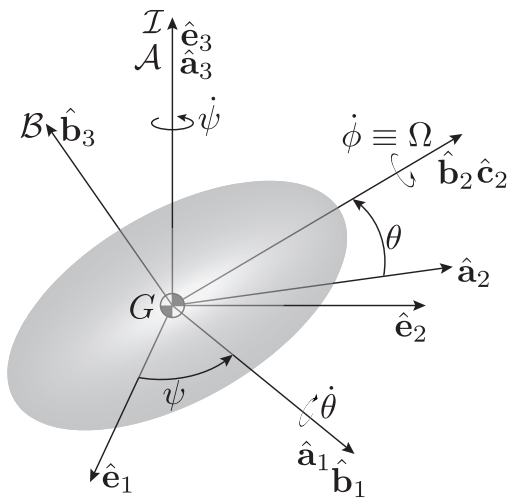
$${}^{\mathcal{B}}\mathcal{C}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{aligned} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} &= \dot{\psi}\hat{\mathbf{a}}_3 + \dot{\theta}\hat{\mathbf{b}}_1 \\ &= \dot{\theta}\hat{\mathbf{b}}_1 + \dot{\psi}\sin\theta\hat{\mathbf{b}}_2 + \dot{\psi}\cos\theta\hat{\mathbf{b}}_3 \end{aligned}$$

$${}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{C}} = \Omega\hat{\mathbf{b}}_2$$

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{C}} = {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{C}}$$

Spinning Symmetric Rigid Bodies Continued



- $\mathcal{C}$ is the principal axis frame for the symmetric body, and because $\hat{\mathbf{c}}_2$ is the symmetry axis, we expect the first and third components along the diagonal of the moment of inertia matrix to be identical:

$$[\mathbb{I}_G]_{\mathcal{C}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_1 \end{bmatrix}_{\mathcal{C}}$$

- Because of this symmetry, and because $\hat{\mathbf{b}}_2 \equiv \hat{\mathbf{c}}_2$, we can also write:

$$[\mathbb{I}_G]_{\mathcal{B}} = [\mathbb{I}_G]_{\mathcal{C}}$$

Spinning Symmetric Rigid Body Dynamics

$$[\mathcal{I}\mathbf{h}_G]_{\mathcal{B}} = [\mathbb{I}_G]_{\mathcal{B}} [\mathcal{I}\boldsymbol{\omega}^c]_{\mathcal{B}} = \begin{bmatrix} I_1 \dot{\theta} \\ I_2 \left(\Omega + \dot{\psi} \sin(\theta) \right) \\ I_1 \dot{\psi} \cos(\theta) \end{bmatrix}_{\mathcal{B}}$$

NB: This is **not** a typo

$$\begin{aligned} \left[\frac{\mathcal{I} d}{dt} \mathcal{I}\mathbf{h}_G \right]_{\mathcal{B}} &= \left[\frac{\mathcal{B} d}{dt} \mathcal{I}\mathbf{h}_G \right]_{\mathcal{B}} + [\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times]_{\mathcal{B}} [\mathcal{I}\mathbf{h}_G]_{\mathcal{B}} \\ &= \begin{bmatrix} I_1 \dot{\psi}^2 \sin(\theta) \cos(\theta) + I_1 \ddot{\theta} - I_2 \dot{\psi} \left(\Omega + \dot{\psi} \sin(\theta) \right) \cos(\theta) \\ I_2 \left(\dot{\Omega} + \ddot{\psi} \sin(\theta) + \dot{\psi} \dot{\theta} \cos(\theta) \right) \\ I_1 \ddot{\psi} \cos(\theta) - 2I_1 \dot{\psi} \dot{\theta} \sin(\theta) + I_2 \dot{\theta} \left(\Omega + \dot{\psi} \sin(\theta) \right) \end{bmatrix}_{\mathcal{B}} \end{aligned}$$

Spinning Symmetric Rigid Body Dynamics Continued

Note: $(\mathcal{I}\boldsymbol{\omega}^{\mathcal{B}} \times \mathcal{I}\mathbf{h}_G) \cdot \hat{\mathbf{b}}_2 = 0$

Therefore, if $\frac{\mathcal{I} d}{dt} \mathcal{I}\mathbf{h}_G \cdot \hat{\mathbf{b}}_2 = 0$ then

$$\frac{\mathcal{B} d}{dt} \mathcal{I}\mathbf{h}_G \cdot \hat{\mathbf{b}}_2 = 0 \implies I_2 \frac{d}{dt} \left(\Omega + \dot{\psi} \sin \theta \right) = 0$$

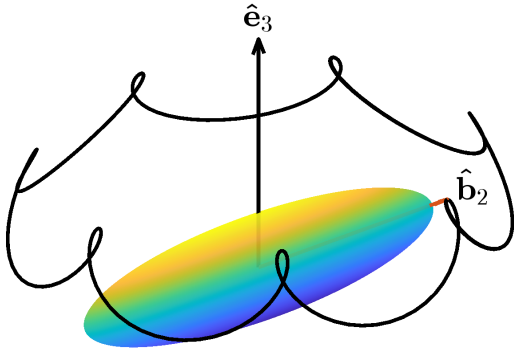
$$\implies \Omega + \dot{\psi} \sin \theta = C \text{ (constant)}$$

Assume: $\mathbf{M}_G = -M_1 \hat{\mathbf{b}}_1$. Then:

$$\begin{aligned} \ddot{\theta} &= \frac{1}{I_1} \left(C I_2 \dot{\psi} \cos(\theta) - \frac{I_1 \dot{\psi}^2}{2} \sin(2\theta) - M_1 \right) \\ \ddot{\psi} &= \frac{\dot{\theta}}{I_1} \left(-\frac{C I_2}{\cos(\theta)} + 2 I_1 \dot{\psi} \tan(\theta) \right) \end{aligned}$$

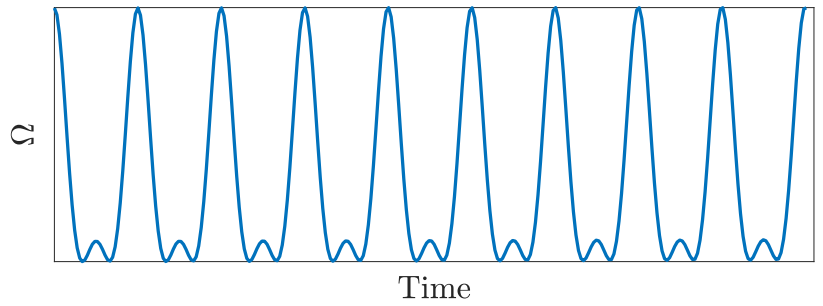
Torqued, Spinning, Symmetric Rigid Body Motion

In general, a torqued spinning, symmetric, rigid body will display two bulk types of motion:



- **Precession:** Rotation of the spin/symmetry axis about the inertially fixed axis orthogonal to the direction of the torque vector ($\hat{\mathbf{e}}_3$ in our example)
- **Nutation:** Secondary rotation of the spin/symmetry axis along its precessing trajectory

Remember that Ω (the spin rate) is not constant in time. Only the quantity $\Omega + \dot{\psi} \sin \theta$ is conserved.



Kinetic Energy of Rigid Bodies

- Recall our derivation of T_G :

$$\begin{aligned}
 T_G &= \frac{1}{2} \sum_{i=1}^N m_i \left\| \underbrace{\mathbf{0}}_{\mathbf{v}_{i/G}} + \mathcal{I} \boldsymbol{\omega}^B \times \mathbf{r}_{i/G} \right\|^2 = \frac{1}{2} \sum_{i=1}^N m_i \underbrace{(\mathcal{I} \boldsymbol{\omega}^B \times \mathbf{r}_{i/G}) \cdot (\mathcal{I} \boldsymbol{\omega}^B \times \mathbf{r}_{i/G})}_{\mathcal{I} \boldsymbol{\omega}^B \cdot (\mathbf{r}_{i/G} \times (\mathcal{I} \boldsymbol{\omega}^B \times \mathbf{r}_{i/G}))} \\
 &= \frac{1}{2} \sum_{i=1}^N m_i \underbrace{(\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G}) \mathcal{I} \boldsymbol{\omega}^B - (\mathcal{I} \boldsymbol{\omega}^B \cdot \mathbf{r}_{i/G}) \mathbf{r}_{i/G}}_{\mathbb{U} \cdot \mathcal{I} \boldsymbol{\omega}^B - (\mathbf{r}_{i/G} \otimes \mathbf{r}_{i/G}) \cdot \mathcal{I} \boldsymbol{\omega}^B} \\
 &= \frac{1}{2} \mathcal{I} \boldsymbol{\omega}^B \cdot \underbrace{\left(\sum_{i=1}^N m_i [(\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G}) \mathbb{U} - (\mathbf{r}_{i/G} \otimes \mathbf{r}_{i/G})] \right)}_{\triangleq \mathbb{I}_G} \cdot \mathcal{I} \boldsymbol{\omega}^B
 \end{aligned}$$

Work and Energy of Rigid Bodies

$$T_G = \frac{1}{2} \mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} \cdot \mathbb{I}_G \cdot \mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} = \frac{1}{2} [\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}}]_{\mathcal{B}}^T [\mathbb{I}_G]_{\mathcal{B}} [\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}}]_{\mathcal{B}}$$

- When working in a principal axis frame:

$$T_G = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$

- Every other one of our previous work/energy derivations related to rigid bodies did not make any assumptions about planar motion, and so applies fully in the 3D case

Recall Euler's Equations

$$[\mathbb{I}_G]_{\mathcal{B}_p} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p}$$

$$[\mathbb{I}_G]_{\mathcal{B}_p} \left[\frac{{}^{\mathcal{B}} d}{dt} \mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} \right]_{\mathcal{B}_p} + [\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}} \times]_{\mathcal{B}_p} [\mathbb{I}_G]_{\mathcal{B}_p} [\mathcal{I} \boldsymbol{\omega}^{\mathcal{B}}]_{\mathcal{B}_p} = [\mathbf{M}_G]_{\mathcal{B}_p} \implies$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}_{\mathcal{B}_p} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}_{\mathcal{B}_p} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_{\mathcal{B}_p} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}_{\mathcal{B}_p}$$

Euler's Equations

$$I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = M_1$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_1 \omega_3 = M_2$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 = M_3$$

Torque-Free Motion

- **Torque-free** motion

occurs when

$$\mathbf{M}_G = \mathbf{0}$$

- Consider two cases:

$$\begin{array}{ll} \text{Case 1: } I_1 = I_2 & \text{Case 2: } \omega_3 \gg \omega_1, \omega_2 \\ I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = 0 & I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = 0 \\ I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_1 \omega_3 = 0 & I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_1 \omega_3 = 0 \\ I_3 \dot{\omega}_3 = 0 & I_3 \dot{\omega}_3 \approx 0 \end{array}$$

$$\left. \begin{array}{l} \dot{\omega}_1 = -\frac{I_3 - I_2}{I_1} \omega_2 \omega_3 \\ \dot{\omega}_2 = -\frac{I_1 - I_3}{I_2} \omega_1 \omega_3 \end{array} \right\} \left. \begin{array}{l} \ddot{\omega}_1 = -\frac{I_3 - I_2}{I_1} \dot{\omega}_2 \omega_3 \\ \ddot{\omega}_2 = -\frac{I_1 - I_3}{I_2} \dot{\omega}_1 \omega_3 \end{array} \right\} \boxed{\begin{array}{l} \ddot{\omega}_1 + \omega_n^2 \omega_1 = 0 \\ \ddot{\omega}_2 + \omega_n^2 \omega_2 = 0 \\ \omega_n^2 \triangleq \frac{(I_3 - I_2)(I_3 - I_1) \omega_3^2}{I_1 I_2} \end{array}}$$

- This is simple harmonic oscillation!
- The spin axis is slightly offset from the body axis of the maximum component of the angular velocity, and oscillates about it

Torque-Free Motion Conserved Quantities

- In the case where there are no net external forces or moments on a rigid body, both the angular momentum and kinetic energy about the COM are conserved:

$$\left. \begin{array}{l} {}^I \mathbf{h}_G = \mathbb{I}_G \cdot {}^I \boldsymbol{\omega}^B = I_1 \omega_1 \hat{\mathbf{b}}_1 + I_2 \omega_2 \hat{\mathbf{b}}_2 + I_3 \omega_3 \hat{\mathbf{b}}_3 \\ T_G = \frac{1}{2} {}^I \boldsymbol{\omega}^B \cdot \mathbb{I}_G \cdot {}^I \boldsymbol{\omega}^B = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \end{array} \right\} \text{Constant}$$

- If a body is spinning about a principal axis $\hat{\mathbf{b}}_i$: $T_G = \frac{\|{}^I \mathbf{h}_G\|^2}{2I_i}$

- For $I_3 > I_2 > I_1$ $\begin{cases} \max T = \text{spin about minor axis } (I_1) \\ \min T = \text{spin about major axis } (I_3) \end{cases}$

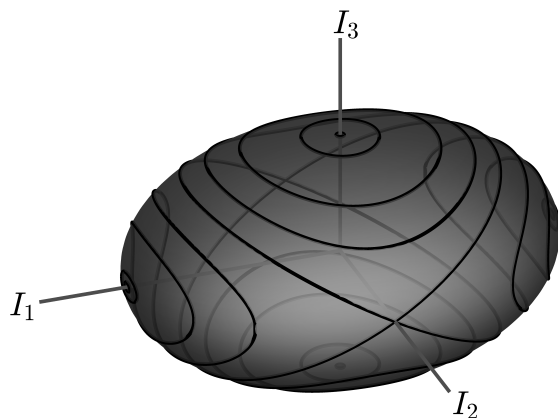
$$\sqrt{2TI_1} < \|{}^I \mathbf{h}_G\| < \sqrt{2TI_3}$$

The Inertia Ellipsoid

- The conserved quantities for torque-free motion produce two ellipsoids:

$$T_G = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \implies \frac{\omega_1^2}{2T_G/I_1} + \frac{\omega_2^2}{2T_G/I_2} + \frac{\omega_3^2}{2T_G/I_3} = 1$$

$$\|\mathcal{I} \mathbf{h}_G\|^2 \triangleq h_G^2 = \|\mathbb{I}_G \cdot \mathcal{I} \boldsymbol{\omega}^{\mathcal{B}}\|^2 \implies \frac{\omega_1^2}{h_G^2/I_1^2} + \frac{\omega_2^2}{h_G^2/I_2^2} + \frac{\omega_3^2}{h_G^2/I_3^2} = 1$$



- The path of the angular velocity vector in torque-free rigid body motion must follow the intersection of these ellipsoids (called **polhodes**)
- Major and minor axis spinners are stable. Intermediate axis spinners are unstable